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Surface pencil with a common adjoint curve

Ergin BAYRAM*
Department of Mathematics, Faculty of Arts and Sciences, Ondokuz Mayıs University, Samsun, Turkey

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Abstract: In the present paper, we construct surfaces possessing an adjoint curve of a given space curve as an asymptotic curve, geodesic or line of curvature. We obtain conditions for ruled surfaces and developable ones. Finally, we present illustrative examples to show the validity of the present method.

Key words: Surface pencil, asymptotic curve, geodesic, line of curvature, adjoint curve

1. Introduction

Curves and surfaces are at the core of differential geometry. We encounter them in almost every differential geometry book [3, 10, 14]. Most of the previous work about surface curves focus on how to find them on a given surface. However, the more relevant problem is to find surfaces passing through a given curve accepting it as a special curve such as geodesic, asymptotic curve or line of curvature. The first study about this type of problem was conducted by Wang et al. [13] in 2004. They proposed a method to find surfaces accepting a given curve as a common geodesic. In 2008, Kasap and Akyıldız [5] generalized the method of Wang and obtained a larger family of surfaces with a common geodesic. Li et al. [8] obtained conditions on marching scale functions such that the given curve is a line of curvature on each member of the surface pencil. Surface pencil with a common asymptotic curve defined by Bayram et al. [2] in 2012. Bayram and Bilici [1] presented constraints for an involute curve to be an asymptotic curve on a surface pencil. Güler et al. [4] parametrically expressed offset surface pencil with a common asymptotic curve.

Beside the Euclidean space, there exist researches about surface pencils in other spaces. Atalay and Kasap [12] expressed necessary and sufficient conditions for surfaces to have the same curve as a null asymptotic curve in Minkowski 3-space. Yüzbaşı [7] obtained constraints for family of surfaces through a given asymptotic curve in Galilean 3-space. Şaffak et al. [11] presented a method for surfaces with a common asymptotic curve in Minkowski 3-space. Yoon et al. [15] analyzed a new method for constructing a surface family accepting a given curve as an asymptotic curve in 3 dimensional Lie group.

In the present study, we obtain conditions for surface pencil possessing an adjoint curve as an asymptotic curve, geodesic or line of curvature. We give results about ruled surfaces and developable ones. Finally, we present examples to show the validity of the method.

*Correspondence: erginbayram@yahoo.com
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2. Preliminaries

This section is devoted to the some basics of differential geometry of curves and surfaces.

Let $\alpha : I \to \mathbb{R}^3,\ s \to \alpha(s)$ be a unit speed curve. The orthonormal frame $\{T(s),\ N(s),\ B(s)\}$ is called the Frenet frame of the curve α, where $T(s),\ N(s),\ B(s)$ are the unit tangent, the principal normal and the binormal vector fields of α, respectively, and they are defined by

$$T(s) = \alpha'(s),\ N(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|},\ B(s) = T(s) \times N(s).$$

The curvature $\kappa(s)$ and torsion $\tau(s)$ of α are given by $\kappa(s) = \|\alpha''(s)\|,\ \tau(s) = \langle N'(s),\ B(s)\rangle$. Then the famous Frenet formula is

$$T'(s) = \kappa(s) N(s),\ N'(s) = -\kappa(s) T(s) + \tau(s) B(s),\quad B'(s) = -\tau(s) N(s).$$

Let α be a unit speed curve in $\mathbb{R}^3$ with nonvanishing torsion and $\{T(s),\ N(s),\ B(s)\}$ be its Frenet frame. The adjoint curve of α is defined as [6]

$$\beta(s) = \int_{s_0}^{s} B(u)\, du.$$

Theorem 2.1 *If α is a curve with arc length parameter s then the arc length parameter of β is also s [9].*

Theorem 2.2 *Let α be a curve with arc length parameter s and β be the adjoint curve of α. If the Frenet vectors of α and β are $\{T(s),\ N(s),\ B(s)\}$ and $\left\{\overline{T}(s),\ \overline{N}(s),\ \overline{B}(s)\right\}$, and the curvature and the torsion of α and β are $\kappa(s),\ \tau(s)$ and $\overline{\kappa}(s),\ \overline{\tau}(s)$, respectively, then the following relations hold [9]*

$$\overline{T}(s) = B(s),\ \overline{N}(s) = -N(s),\ \overline{B}(s) = T(s),\ \overline{\kappa}(s) = \tau(s),\quad \overline{\tau}(s) = \kappa(s).$$

Remark 2.3 *Since curvature of a curve is nonnegative, $\overline{\kappa}(s)$ should be equal to $|\tau(s)|$ in Theorem 2.2.*

A curve α in $M \subset \mathbb{R}^3$ is a geodesic of M provided its acceleration α'' is always normal to M [10]. A regular curve α in $M \subset \mathbb{R}^3$ is an asymptotic curve provided its velocity α' always points in an asymptotic direction, that is, the direction in which the normal curvature is zero [10]. The simplest criterion for a curve in M to be asymptotic is that its acceleration α'' is always be tangent to M. A regular curve α in $M \subset \mathbb{R}^3$ is a line of curvature provided its velocity α' always points in a principal direction [10].

Theorem 2.4 *A surface curve is a line of curvature if and only if the surface normals along the curve forms a developable surface [3].*

A ruled surface M is one which is swept out by a straight line moving in $\mathbb{R}^3$. Thus M has a parametrization as

$$M(s,t) = \beta(s) + t\gamma(s),$$

where β and γ are curves in $\mathbb{R}^3$ with γ never zero. A ruled surface is said to be noncylindrical provided $\gamma \times \gamma'$ never vanishes. The normal and binormal surface of a space curve $\beta(s)$ are defined as

$$M(s,t) = \beta(s) + t\overline{N}(s)$$

and

$$M(s,t) = \beta(s) + t\overline{B}(s),$$

respectively.

3. Surface pencil with a common adjoint curve

Let α be a unit speed curve in $\mathbb{R}^3$ with nonvanishing torsion and β be its adjoint curve. Surfaces passing through the adjoint curve are given by parametrically as

$$\begin{cases} M(s,t) = \beta(s) + f(s,t)\overline{T}(s) + g(s,t)\overline{N}(s) + h(s,t)\overline{B}(s), \\ S_1 \leq s \leq S_2,\ T_1 \leq t \leq T_2, \end{cases} \tag{3.1}$$

where $f(s,t)$, $g(s,t)$ and $h(s,t)$ are C^1 functions. Choosing $f(s,t_0) = g(s,t_0) = h(s,t_0) \equiv 0,\ \forall s$ we have $\beta(s)$ as a parameter curve on M, that is, $M(s,t_0) = \beta(s)$.

The normal vector field of the surface M is given by

$$n(s,t) = \frac{\partial M}{\partial s}(s,t) \times \frac{\partial M}{\partial t}(s,t),$$

where "$\times$" is the vector product. We calculate

$$\begin{aligned}
\frac{\partial M}{\partial s}(s,t) &= \beta'(s) + \frac{\partial f}{\partial s}(s,t)\overline{T}(s) + f(s,t)\overline{T}'(s) + \frac{\partial g}{\partial s}(s,t)\overline{N}(s) \\
&\quad + g(s,t)\overline{N}'(s) + \frac{\partial h}{\partial s}(s,t)\overline{B}(s) + h(s,t)\overline{B}'(s) \\
&= \overline{T}(s) + \frac{\partial f}{\partial s}(s,t)\overline{T}(s) + f(s,t)\overline{\kappa}(s)\overline{N}(s) + \frac{\partial g}{\partial s}(s,t)\overline{N}(s) \\
&\quad + g(s,t)\left(-\overline{\kappa}(s)\overline{T}(s) + \overline{\tau}(s)\overline{B}(s)\right) + \frac{\partial h}{\partial s}(s,t)\overline{B}(s) \\
&\quad - h(s,t)\overline{\tau}(s)\overline{N}(s) \\
&= \left(1 + \frac{\partial f}{\partial s}(s,t) - g(s,t)\overline{\kappa}(s)\right)\overline{T}(s) \\
&\quad + \left(f(s,t)\overline{\kappa}(s) + \frac{\partial g}{\partial s}(s,t) - h(s,t)\overline{\tau}(s)\right)\overline{N}(s) \\
&\quad + \left(g(s,t)\overline{\tau}(s) + \frac{\partial h}{\partial s}(s,t)\right)\overline{B}(s)
\end{aligned}$$

and

$$\frac{\partial M}{\partial t}(s,t) = \frac{\partial f}{\partial t}(s,t)\overline{T}(s) + \frac{\partial g}{\partial t}(s,t)\overline{N}(s) + \frac{\partial h}{\partial t}(s,t)\overline{B}(s)$$

and

$$\begin{aligned} n(s,t) &= \left[\left(f(s,t)\overline{\kappa}(s)+\frac{\partial g}{\partial s}(s,t)-h(s,t)\overline{\tau}(s)\right)\frac{\partial h}{\partial t}(s,t)\right.\\ &\quad \left.-\left(g(s,t)\overline{\tau}(s)+\frac{\partial h}{\partial s}(s,t)\right)\frac{\partial g}{\partial t}(s,t)\right]\overline{T}(s)\\ &\quad +\left[\left(g(s,t)\overline{\tau}(s)+\frac{\partial h}{\partial s}(s,t)\right)\frac{\partial f}{\partial t}(s,t)\right.\\ &\quad \left.-\left(1+\frac{\partial f}{\partial s}(s,t)-g(s,t)\overline{\kappa}(s)\right)\frac{\partial h}{\partial t}(s,t)\right]\overline{N}(s)\\ &\quad +\left[\left(1+\frac{\partial f}{\partial s}(s,t)-g(s,t)\overline{\kappa}(s)\right)\frac{\partial g}{\partial t}(s,t)\right.\\ &\quad \left.-\left(f(s,t)\overline{\kappa}(s)+\frac{\partial g}{\partial s}(s,t)-h(s,t)\overline{\tau}(s)\right)\frac{\partial f}{\partial t}(s,t)\right]\overline{B}(s). \end{aligned}$$

Along the adjoint curve $\beta(s)$ we have

$$n(s,t_0)=\frac{\partial g}{\partial t}(s,t_0)\,\overline{B}(s)-\frac{\partial h}{\partial t}(s,t_0)\,\overline{N}(s). \tag{3.2}$$

Theorem 3.1 *Let α be a unit speed curve with nonvanishing torsion and β be its adjoint curve. β is an asymptotic curve on the surface pencil in Eq.* (3.1) *if*

$$\begin{cases} f(s,t_0)=g(s,t_0)=h(s,t_0)=\frac{\partial h}{\partial t}(s,t_0)\equiv 0\neq \frac{\partial g}{\partial t}(s,t_0), \\ S_1\le s\le S_2,\ T_1\le t,\ t_0\le T_2,\ (t_0\ fixed). \end{cases} \tag{3.3}$$

Corollary 3.2 *The surface pencil given by*

$$\begin{cases} M(s,t)=\beta(s)+(t-t_0)\,\overline{T}(s)+(t-t_0)\,\overline{N}(s), \\ S_1\le s\le S_2,\ T_1\le t,\ t_0\le T_2,\ (t_0\ fixed) \end{cases} \tag{3.4}$$

is a ruled surface accepting β as a common asymptotic curve.

Corollary 3.3 *The ruled surface given by Eq.* (3.4) *is not developable.*

Corollary 3.4 *The ruled surface given by Eq.* (3.4) *is noncylindrical.*

Theorem 3.5 *Let α be a unit speed curve with nonvanishing torsion and β be its adjoint curve. Surface pencil given by Eq.* (3.1) *is a normal surface accepting β as a common asymptotic curve if*

$$\begin{cases} f(s,t)=h(s,t)\equiv 0,\quad g(s,t)=t-t_0, \\ S_1\le s\le S_2,\ T_1\le t,\ t_0\le T_2,\ (t_0\ fixed). \end{cases} \tag{3.5}$$

Corollary 3.6 *The normal surface given by Eq.* (3.5) *is not developable.*

Corollary 3.7 *The normal surface given by Eq.* (3.5) *is noncylindrical.*

Corollary 3.8 *The metric of the normal surface given by Eq.* (3.5) *depends only on the curvature of the adjoint curve* $\beta(s)$.

Theorem 3.9 *Let* α *be a unit speed curve with nonvanishing torsion and* β *be its adjoint curve.* β *is a geodesic on the surface pencil given by Eq.* (3.1) *if*

$$\begin{cases} f(s,t_0) = g(s,t_0) = h(s,t_0) = \frac{\partial g}{\partial t}(s,t_0) \equiv 0 \neq \frac{\partial h}{\partial t}(s,t_0), \\ S_1 \leq s \leq S_2,\ T_1 \leq t,\ t_0 \leq T_2,\ (t_0\ fixed). \end{cases} \tag{3.6}$$

Corollary 3.10 *The surface pencil given by*

$$\begin{cases} M(s,t) = \beta(s) + (t-t_0)\overline{T}(s) + (t-t_0)\overline{B}(s), \\ S_1 \leq s \leq S_2,\ T_1 \leq t,\ t_0 \leq T_2,\ (t_0\ fixed), \end{cases} \tag{3.7}$$

is a ruled surface accepting β *as a common geodesic.*

Corollary 3.11 *The ruled surface given by Eq.* (3.7) *is noncylindrical if and only if* $\overline{\kappa}(s) \neq \overline{\tau}(s)$, $S_1 \leq s \leq S_2$.

Corollary 3.12 *The necessary and sufficient condition that the ruled surface given by Eq.* (3.7) *to be a developable surface is that the curvature and the torsion of the adjoint curve are equal.*

Corollary 3.13 *The metric of the ruled surfaces given by Eq.* (3.4) *or* (3.7) *depends only on the curvature and torsion of the adjoint curve* $\beta(s)$.

Theorem 3.14 *Let* α *be a unit speed curve with nonvanishing torsion and* β *be its adjoint curve. Surface pencil given by Eq.* (3.1) *is a binormal surface accepting* β *as a common geodesic if*

$$\begin{cases} f(s,t) = g(s,t) \equiv 0, \quad h(s,t) = t - t_0, \\ S_1 \leq s \leq S_2,\ T_1 \leq t,\ t_0 \leq T_2,\ (t_0\ fixed). \end{cases} \tag{3.8}$$

Corollary 3.15 *The binormal surface given by Eq.* (3.8) *is not developable.*

Corollary 3.16 *The binormal surface given by Eq.* (3.8) *is noncylindrical.*

Corollary 3.17 *The metric of the binormal surface given by Eq.* (3.8) *depends only on the torsion of the adjoint curve* $\beta(s)$.

Theorem 3.18 *Let* α *be a unit speed curve with nonvanishing torsion and* β *be its adjoint curve.* β *is a line of curvature on the surface pencil given by Eq.* (3.1) *if*

$$\begin{cases} f(s,t_0) = g(s,t_0) = h(s,t_0) \equiv 0, \\ \theta(s) = -\int \kappa(s)\,ds,\ \mu(s) \neq 0, \\ \frac{\partial g}{\partial t}(s,t_0) = \mu(s)\sin\theta(s),\ \frac{\partial h}{\partial t}(s,t_0) = -\mu(s)\cos\theta(s) \\ S_1 \leq s \leq S_2,\ T_1 \leq t,\ t_0 \leq T_2,\ (t_0\ fixed). \end{cases} \tag{3.9}$$

Proof Let $\overline{n}(s)$ be an orthogonal vector field to the adjoint curve $\beta(s)$. Then we can write

$$\overline{n}(s)=\cos\theta(s)\,\overline{N}(s)+\sin\theta(s)\,\overline{B}(s),$$

where $\theta(s)$ is the angle between $\overline{n}(s)$ and the principal normal vector field $\overline{N}(s)$ of the adjoint curve $\beta(s)$. According to Theorem 2.4, $\beta(s)$ is a line of curvature on the surface given by Eqn. (3.1) if and only if $\overline{n}(s)\,\|n(s,t_0)$ and the ruled surface

$$X(s,t)=\beta(s)+t\overline{n}(s),S_1\leq s\leq S_2,$$

is developable. Using Eq. (3.2) we have

$$\overline{n}(s)\,\|n(s,t_0)\Leftrightarrow\frac{\partial g}{\partial t}(s,t_0)=\mu(s)\sin\theta(s),\ \frac{\partial h}{\partial t}(s,t_0)=-\mu(s)\cos\theta(s)$$

and $X(s,t)$ is developable if and only if $\det(\beta'(s),\overline{n}(s),\overline{n}'(s))=0$. Since

$$\det(\beta'(s),\overline{n}(s),\overline{n}'(s))=\theta'(s)+\overline{\tau}(s),$$

we obtain

$$\det(\beta'(s),\overline{n}(s),\overline{n}'(s))=0\Leftrightarrow\theta(s)=-\int\kappa(s)\,ds,$$

which completes the proof. □

4. Examples

Example 4.1 *Let us take the unit speed helix* $\alpha(s)=\left(a\cos\frac{s}{\sqrt{a^2+b^2}},a\sin\frac{s}{\sqrt{a^2+b^2}},\frac{bs}{\sqrt{a^2+b^2}}\right)$ *with radius* $a>0$. *The Frenet apparatus of* α *are*

$$\begin{cases} T(s)=\frac{1}{\sqrt{a^2+b^2}}\left(-a\sin\frac{s}{\sqrt{a^2+b^2}},\cos\frac{s}{\sqrt{a^2+b^2}},b\right), \\ N(s)=\left(-\cos\frac{s}{\sqrt{a^2+b^2}},-\sin\frac{s}{\sqrt{a^2+b^2}},0\right), \\ B(s)=\frac{1}{\sqrt{a^2+b^2}}\left(b\sin\frac{s}{\sqrt{a^2+b^2}},-b\cos\frac{s}{\sqrt{a^2+b^2}},a\right), \\ \kappa(s)=\frac{a}{a^2+b^2},\quad\tau(s)=\frac{b}{a^2+b^2}. \end{cases}$$

The adjoint of α *is* $\beta(s)=\left(-b\cos\frac{s}{\sqrt{a^2+b^2}},-b\sin\frac{s}{\sqrt{a^2+b^2}},\frac{as}{\sqrt{a^2+b^2}}\right)$ *and its Frenet frame is* $\left\{\overline{T}(s)=B(s),\ \overline{N}(s)=-N(s),\ \overline{B}(s)=T(s)\right\}$. *Choosing* $a=b=1,\ f(s,t)\equiv0,\ g(s,t)=st,h(s,t)=t^2,\ t_0=0$ *we obtain the surface*

$$\begin{aligned} M(s,t) &= \beta(s)+f(s,t)\overline{T}(s)+g(s,t)\overline{N}(s)+h(s,t)\overline{B}(s) \\ &= \Bigg((st-1)\cos\frac{\sqrt{2}}{2}s-t^2\frac{\sqrt{2}}{2}\sin\frac{\sqrt{2}}{2}s, \\ &\qquad (st-1)\sin\frac{\sqrt{2}}{2}s+t^2\frac{\sqrt{2}}{2}\cos\frac{\sqrt{2}}{2}s,\ \frac{\sqrt{2}}{2}(s+t^2)\Bigg), \end{aligned}$$

$0 < s \leq 2,\ -1 \leq t \leq 1,$ *satisfying Eq.* (3.3) *and accepting the adjoint curve* $\beta(s)$ *as an asymptotic curve (Figure 1).*

For the same curve choosing $f(s,t) \equiv 0,\ g(s,t) = t^2, h(s,t) = st,\ t_0 = 0$ *yields the surface*

$$\begin{aligned} M(s,t) &= \beta(s) + f(s,t)\overline{T}(s) + g(s,t)\overline{N}(s) + h(s,t)\overline{B}(s) \\ &= \Bigg(\left(t^2 - 1\right)\cos\frac{\sqrt{2}}{2}s - st\frac{\sqrt{2}}{2}\sin\frac{\sqrt{2}}{2}s, \\ &\quad \left(t^2 - 1\right)\sin\frac{\sqrt{2}}{2}s + st\frac{\sqrt{2}}{2}\cos\frac{\sqrt{2}}{2}s, \frac{\sqrt{2}}{2}s(s+t) \Bigg), \end{aligned}$$

$0 < s \leq 2,\ -1 \leq t \leq 1,$ *satisfying Eq.* (3.6) *and passing through the adjoint curve* $\beta(s)$ *as a geodesic (Figure 2).*

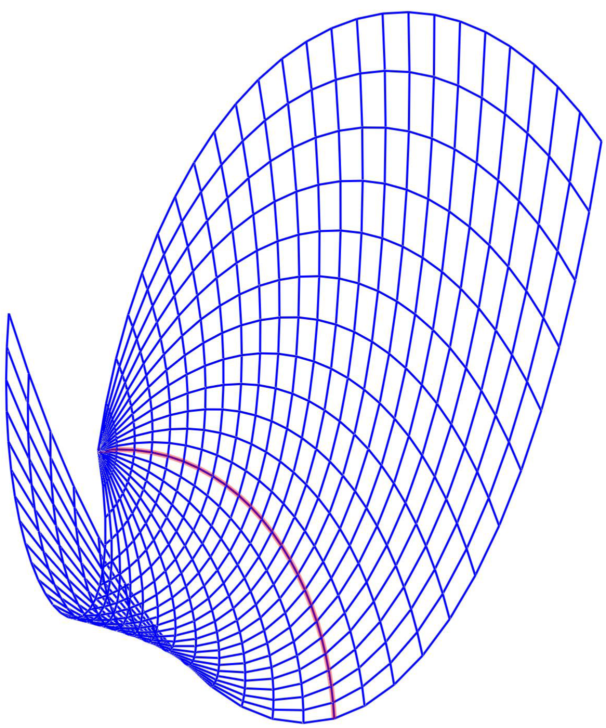

Figure 1. A member of the surface pencil accepting the adjoint curve $\beta(s)$ (red in colour) as an asymptotic curve.

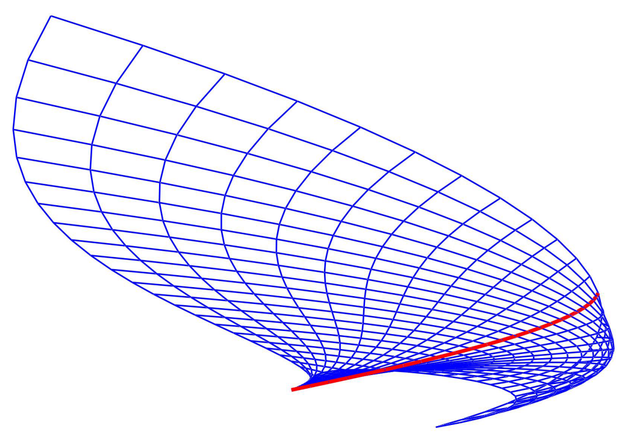

Figure 2. A member of the surface pencil accepting the adjoint curve $\beta(s)$ (red in colour) as a geodesic.

Choosing $\theta(s) = -\int \kappa(s)\,ds = -\frac{s}{2},\ \mu(s) \equiv 1,\ f(s,t) \equiv 0,\ g(s,t) = -t\sin\left(\frac{s}{2}\right), h(s,t) = -t\cos\left(\frac{s}{2}\right),\ t_0 =$

0 *Eq.* (3.9) *is satisfied and we get the surface*

$$\begin{aligned} M(s,t) &= \beta(s) + f(s,t)\overline{T}(s) + g(s,t)\overline{N}(s) + h(s,t)\overline{B}(s) \\ &= \left(\left(t\frac{\sqrt{2}}{2}\sin\frac{\sqrt{2}}{2}s - 1 - t\sin\frac{s}{2}\right)\cos\frac{\sqrt{2}}{2}s,\right. \\ &\quad \left(-1 - t\sin\frac{s}{2}\right)\sin\frac{\sqrt{2}}{2}s - t\frac{\sqrt{2}}{2}\cos\frac{s}{2}\cos\frac{\sqrt{2}}{2}s, \\ &\quad \left.\frac{\sqrt{2}}{2}(s - t\cos\frac{s}{2})\right), \end{aligned}$$

$-1 \leq s \leq 1,\ 0 \leq t \leq 1$ *possessing the adjoint curve* $\beta(s)$ *as a line of curvature (Figure 3).*

Taking $f(s,t) = h(s,t) = t,\ g(s,t) \equiv 0,\ t_0 = 0$ *we get the developable surface*

$$\begin{aligned} M(s,t) &= \beta(s) + f(s,t)\overline{T}(s) + g(s,t)\overline{N}(s) + h(s,t)\overline{B}(s) \\ &= \left(-\cos\frac{\sqrt{2}}{2}s, -\sin\frac{\sqrt{2}}{2}s, \frac{\sqrt{2}}{2}s + \sqrt{2}t\right), \end{aligned}$$

$-1 \leq s \leq 1,\ 0 \leq t \leq 1$ *satisfying Corollary 3.12 and accepting the adjoint curve* $\beta(s)$ *as a geodesic (Figure 4).*

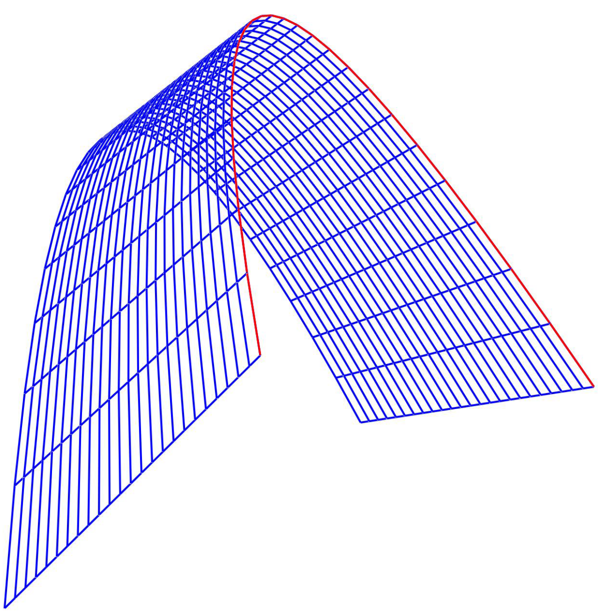

Figure 3. A member of the surface pencil accepting the adjoint curve $\beta(s)$ (red in colour) as a line of curvature.

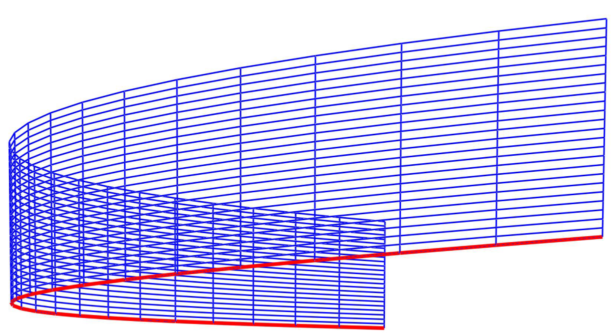

Figure 4. Developable surface accepting the adjoint curve $\beta(s)$ (red in colour) as a geodesic.

Letting $f(s,t) = g(s,t) = t,\ h(s,t) \equiv 0,\ t_0 = 0$ *Eq.* (3.4) *is satisfied and we obtain the ruled surface*

$$\begin{aligned} M(s,t) &= \beta(s) + f(s,t)\overline{T}(s) + g(s,t)\overline{N}(s) + h(s,t)\overline{B}(s) \\ &= \Bigg((t-1)\cos\frac{\sqrt{2}}{2}s + t\frac{\sqrt{2}}{2}\sin\frac{\sqrt{2}}{2}s, \\ &\qquad (t-1)\sin\frac{\sqrt{2}}{2}s - t\frac{\sqrt{2}}{2}\cos\frac{\sqrt{2}}{2}s, \frac{\sqrt{2}}{2}(t+s)\Bigg), \end{aligned}$$

$-1 \le s \le 1,\ 0 \le t \le 1$ *possessing the adjoint curve* $\beta(s)$ *as an asymptotic curve (Figure 5).*

Choosing $f(s,t) = h(s,t) \equiv 0,\ g(s,t) = t,\ t_0 = 0$ *Eq.* (3.5) *is satisfied and we obtain the normal surface*

$$\begin{aligned} M(s,t) &= \beta(s) + f(s,t)\overline{T}(s) + g(s,t)\overline{N}(s) + h(s,t)\overline{B}(s) \\ &= \left((t-1)\cos\frac{\sqrt{2}}{2}s, (t-1)\sin\frac{\sqrt{2}}{2}s, \frac{\sqrt{2}}{2}s\right) \end{aligned}$$

$-2 \le s \le 2,\ -2 \le t \le 2$ *possessing the adjoint curve* $\beta(s)$ *as an asymptotic curve (Figure 6).*

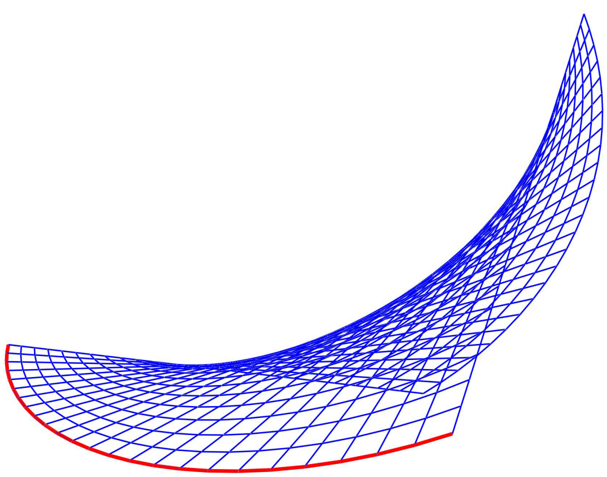

Figure 5. Ruled surface possessing the adjoint curve $\beta(s)$ (red in colour) as an asymptotic curve.

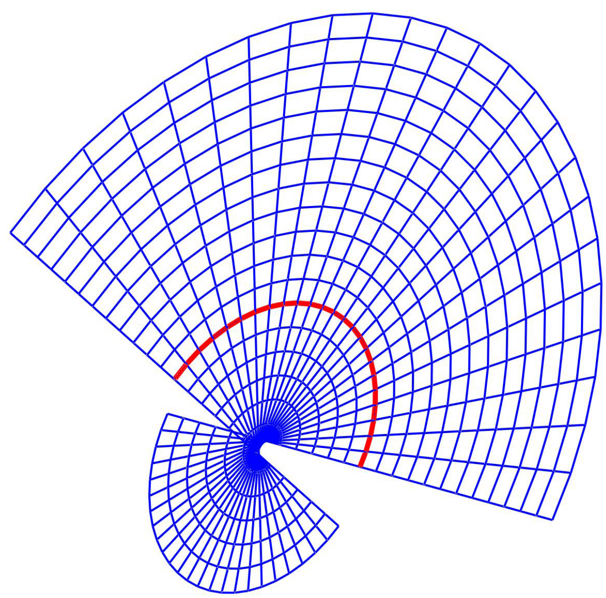

Figure 6. Normal surface accepting the adjoint curve $\beta(s)$ (red in colour) as an asymptotic curve.

If we take $f(s,t) = g(s,t) \equiv 0,\ h(s,t) = t,\ t_0 = 0$ *Eq.* (3.8) *is satisfied and we get*

$$\begin{aligned} M(s,t) &= \beta(s) + f(s,t)\overline{T}(s) + g(s,t)\overline{N}(s) + h(s,t)\overline{B}(s) \\ &= \Bigg(-\cos\frac{\sqrt{2}}{2}s - t\frac{\sqrt{2}}{2}\sin\frac{\sqrt{2}}{2}s, \\ &\qquad t\frac{\sqrt{2}}{2}\cos\frac{\sqrt{2}}{2}s - \sin\frac{\sqrt{2}}{2}s, \frac{\sqrt{2}}{2}(s+t)\Bigg), \end{aligned}$$

$-2 \leq s \leq 2,\ -2 \leq t \leq 2$ *as a binormal surface possessing the adjoint curve* $\beta(s)$ *as a geodesic (Figure 7).*

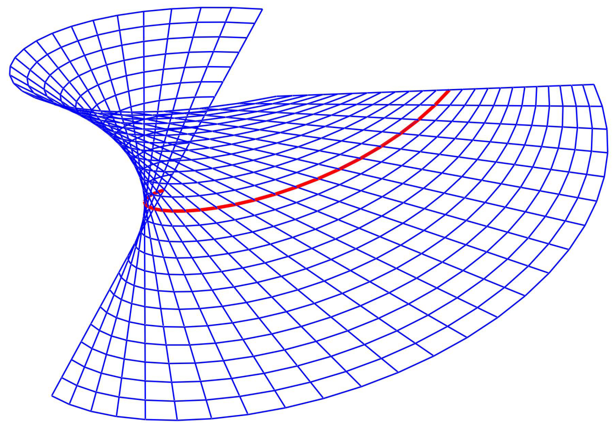

Figure 7. Binormal surface possessing the adjoint curve $\beta(s)$ (red in colour) as a geodesic.

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