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Representation numbers of seven quaternary quadratic forms each in a genus consisting of only two classes

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Abstract: The purpose of this paper is to present some examples of positive-definite integral nondiagonal quaternary quadratic forms whose representation numbers can be determined explicitly using the theory of modular forms. Very few such examples appear in the literature. The seven forms presented were selected because they each belong to a genus containing exactly two form classes for which the single genus mate is a diagonal form whose representation number has been determined recently.

Key words: Representaion number, genus of quaternary quadratic forms, Dedekind eta function, eta quotients, theta functions, modular forms

1. Introduction

The number of representations of a positive integer n by a positive-definite integral diagonal quaternary quadratic form $ax^2 + by^2 + cz^2 + dt^2$ has been determined explicitly for many such forms, see for example [5, 6, 8, 10, 11]. However, for positive-definite integral nondiagonal quaternary quadratic forms

$$ax^2 + by^2 + cz^2 + dt^2 + exy + fxz + gxt + hyz + jyt + kzt$$

very few have been determined explicitly. The reader can find some in [12]. Even rarer are explicit representation numbers for such forms belonging to a genus containing two or more form classes. In this paper, we determine explicitly the representation numbers of the seven positive-definite integral nondiagonal quaternary quadratic forms:

$$\begin{aligned} x^2 + 2y^2 + 2z^2 + 2t^2 + 2yt, &\qquad x^2 + y^2 + 2z^2 + 4t^2 + 2zt, \\ 2x^2 + 2y^2 + 2z^2 + 3t^2 + 2xy + 2xz + 2yt, &\qquad 2x^2 + 2y^2 + 2z^2 + 3t^2 + 2xy + 2xz, \\ x^2 + 2y^2 + 2z^2 + 4t^2 + 2yt + 2zt, &\qquad x^2 + y^2 + 2z^2 + 8t^2 + 2zt, \\ x^2 + 2y^2 + 2z^2 + 5t^2 + 2yt + 2zt. & \end{aligned}$$

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These forms were chosen as each of them has a single genus mate whose representation number has recently been determined explicitly. The genus mates are respectively

$$\begin{aligned} x^2+y^2+z^2+6t, &\qquad x^2+y^2+z^2+7t^2,\\ x^2+y^2+z^2+9t^2, &\qquad x^2+y^2+z^2+12t^2,\\ x^2+y^2+3z^2+4t^2, &\qquad x^2+y^2+3z^2+5t^2,\\ x^2+y^2+z^2+16t^2. & \end{aligned}$$

Before stating our results we give some notation as well as stating some results that we need. We let $\mathbb{N}$ denote the set of positive integers, $\mathbb{Z}$ the set of all integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We denote the field of complex numbers by $\mathcal{C}$ and the Poincaré upper half-plane $\{z \in \mathcal{C} \mid \mathrm{Im}(z) > 0\}$ by $\mathbb{H}$. For $z \in \mathbb{H}$ we define $q = e^{2\pi i z}$ so that $|q| < 1$. If $f := f(x_1, \ldots, x_k)$ is a positive-definite integral quadratic form in k variables the representation number of f is the nonnegative integer defined for all $n \in \mathbb{N}_0$ by

$$N(f(x_1,\ldots,x_k)=n) := \mathrm{card}\{(x_1,\ldots,x_k) \in \mathbb{Z}^k \mid f(x_1,\ldots,x_k)=n\}.$$

As f is positive-definite we have $N(f(x_1,\ldots,x_k)=0)=1$. The theta function of f is defined by

$$\theta_f(z) := \sum_{(x_1,\ldots,x_k)\in\mathbb{Z}^k} q^{f(x_1,\ldots,x_k)} = 1 + \sum_{n=1}^{\infty} N(f(x_1,\ldots,x_k)=n)q^n, \quad z \in \mathbb{H}. \tag{1.1}$$

The Dedekind eta function $\eta(z)$ is defined by

$$\eta(z) := e^{\pi i z/12} \prod_{m=1}^{\infty} (1 - e^{2\pi i m z}), \quad z \in \mathbb{H}. \tag{1.2}$$

An eta quotient $f(z)$ is a function of the form

$$f(z) := \eta^{a(m_1)}(m_1 z) \cdots \eta^{a(m_r)}(m_r z), \quad z \in \mathbb{H},$$

where $r \in \mathbb{N}$, $m_1, \ldots, m_r \in \mathbb{N}$ satisfy $m_1 < \cdots < m_r$, and $a(m_1), \ldots, a(m_r)$ are nonzero integers. Sometimes we write $f(z)$ in the form

$$f(z) = \prod_{d \mid N} \eta^{r(d)}(dz),$$

where $N \in \mathbb{N}$ is a multiple of $m_1, \ldots, m_r$, d runs through the positive integers dividing N and

$$r(d) = \begin{cases} a(m_j) & \text{if } d = m_j, \ \ j = 1, \ldots, r, \\ 0 & \text{otherwise.} \end{cases}$$

Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$ are defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \quad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{1.3}$$

The modular group $\mathrm{SL}_2(\mathbb{Z})$ is the group

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\}.$$

For a positive integer N, the Hecke congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ is defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

The index of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ is

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p \mid N} \big(1 + \frac{1}{p}\big),$$

where p runs through the primes dividing N, see [15, Proposition 2.12]. The space of modular forms of weight k for $\Gamma_0(N)$ with character χ is denoted by $M_k(\Gamma_0(N),\chi)$ and the space of cusp forms of weight k for $\Gamma_0(N)$ with character χ is denoted by $S_k(\Gamma_0(N),\chi)$. If χ is the trivial character we write $M_k(\Gamma_0(N))$ for the former space and $S_k(\Gamma_0(N))$ for the latter space.

For $n \in \mathbb{N}$ the sum of divisors function $\sigma(n)$ is defined by

$$\sigma(n) := \sum_{d \mid n} d.$$

Further, for nonzero integers r and s with $r,s \equiv 0$ or $1 \pmod 4$, it is convenient to define the arithmetic sum

$$A_{r,s}(n) := \sum_{d \mid n} \Big(\frac{r}{d}\Big)\Big(\frac{s}{n/d}\Big) d,$$

where $\left(\frac{r}{*}\right)$ is the Legendre-Jacobi-Kronecker symbol. Clearly $A_{1,1}(n) = \sigma(n)$.

In this paper we give explicit formulas for the representation numbers of the seven positive-definite primitive integral quaternary quadratic forms listed at the beginning of this introduction. Each of these forms belongs to a genus consisting of only two form classes. We now state our main results, which we prove in Sections 3–9. Preliminary results that we need are stated in Section 2.

Theorem 1.1 *Let $n \in \mathbb{N}$. Then*

$$N(x^2+2y^2+2z^2+2t^2+2yt=n) = 4A_{1,24}(n) - \frac{1}{3}A_{24,1}(n) - \frac{4}{3}A_{-3,-8}(n) + A_{-8,-3}(n) - \frac{4}{3}a_1(n),$$

where the integers $a_1(n)$ $(n \in \mathbb{N})$ are given by

$$\frac{\eta^4(2z)\eta(3z)\eta^2(8z)\eta(12z)}{\eta(z)\eta^2(4z)\eta(6z)} = \sum_{n=1}^{\infty} a_1(n) q^n.$$

Theorem 1.2 *Let $n \in \mathbb{N}$. Then*

$$N(x^2+y^2+2z^2+4t^2+2zt=n)=\frac{7}{2}A_{1,28}(n)-\frac{1}{4}A_{28,1}(n)+\frac{7}{4}A_{-4,-7}(n)-\frac{1}{2}A_{-7,-4}(n)-\frac{1}{2}a_2(n),$$

where the integers $a_2(n)$ $(n \in \mathbb{N})$ are given by

$$\frac{\eta^9(2z)\eta(7z)\eta(28z)}{\eta^3(z)\eta^3(4z)\eta(14z)}=\sum_{n=1}^{\infty}a_2(n)q^n.$$

Theorem 1.3 *Let $n \in \mathbb{N}$. Then*

$$N(2x^2+2y^2+2z^2+3t^2+2xy+2xz+2yt=n)=\begin{cases} 2\sigma(n)-2a_3(n) & \text{if } n \equiv 1 \pmod{6},\\ 12\sigma(n)-24\sigma(n/2) & \text{if } n \equiv 2 \pmod{6},\\ 8\sigma(n/3) & \text{if } n \equiv 3 \pmod{6},\\ 6\sigma(n)-12\sigma(n/2) & \text{if } n \equiv 4 \pmod{6},\\ 4\sigma(n) & \text{if } n \equiv 5 \pmod{6},\\ 24\sigma(n/3)-48\sigma(n/6) & \text{if } n \equiv 0 \pmod{6}, \end{cases}$$

where

$$\eta^4(6z)=\sum_{n=1}^{\infty}a_3(n)q^n \quad \text{and} \quad a_3(n)=\frac{1}{3}\sum_{\substack{(x,y)\in\mathbb{Z}^2\\ n=x^2+3xy+3y^2\\ x\equiv 2 \pmod{3}\\ y\equiv 1 \pmod{2}}}(-1)^x x.$$

Theorem 1.4 *Let $n \in \mathbb{N}$. Then*

$$N(2x^2+2y^2+2z^2+3t^2+2xy+2xz=n)$$

$$=\begin{cases} 3A_{1,12}(n)-A_{-3,-4}(n)+\frac{3}{2}A_{-4,-3}(n)-\frac{1}{2}A_{12,1}(n)-3a_4(n) & \text{if } n \equiv 1 \pmod{2},\\ \frac{9}{2}A_{1,12}(n)-\frac{3}{2}A_{-3,-4}(n) & \text{if } n \equiv 2 \pmod{4},\\ \frac{3}{2}A_{1,12}(n)-\frac{1}{2}A_{-3,-4}(n)+3A_{-4,-3}(n)-A_{12,1}(n) & \text{if } n \equiv 0 \pmod{4}, \end{cases}$$

where

$$\frac{\eta^2(4z)\eta^3(6z)}{\eta(2z)}=\sum_{n=1}^{\infty}a_4(n)q^n \quad \text{and} \quad a_4(n)=\sum_{\substack{(i,j)\in\mathbb{N}^2\\ i\equiv j\equiv 1 \pmod{2}\\ 4n=i^2+3j^2}}(-1)^{(j-1)/2}j.$$

Theorem 1.5 *Let* $n \in \mathbb{N}$. *Then*

$$N(x^2+2y^2+2z^2+4t^2+2yt+2zt=n)$$
$$=\begin{cases} 3A_{1,12}(n)-A_{-3,-4}(n)+\frac{3}{2}A_{-4,-3}(n)-\frac{1}{2}A_{12,1}(n)-a_5(n) & \text{if } n\equiv 1 \pmod 2,\\ \frac{3}{2}A_{1,12}(n)-\frac{3}{2}A_{-3,-4}(n) & \text{if } n\equiv 2 \pmod 4,\\ \frac{9}{2}A_{1,12}(n)-\frac{3}{2}A_{-3,-4}(n)+3A_{-4,-3}(n)-A_{12,1}(n) & \text{if } n\equiv 0 \pmod 4,\end{cases}$$

where

$$\frac{\eta^3(2z)\eta^2(12z)}{\eta(6z)}=\sum_{n=1}^{\infty}a_5(n)q^n \quad \text{and} \quad a_5(n)=\sum_{\substack{(i,j)\in\mathbb{N}^2\\ i\equiv j\equiv 1 \pmod 2\\ 4n=i^2+3j^2}}(-1)^{(i-1)/2}i.$$

Theorem 1.6 *Let* $n \in \mathbb{N}$. *Then*

$$N(x^2+y^2+2z^2+8t^2+2zt=n)=\frac{1}{12}\big(30A_{1,60}(n)-A_{60,1}(n)-6A_{5,12}(n)+5A_{12,5}(n)-3A_{-20,-3}(n)$$
$$+10A_{-3,-20}(n)-2A_{-15,-4}(n)+15A_{-4,-15}(n)\big)+a_6(n),$$

where

$$\frac{\eta^9(2z)\eta(15z)\eta(60z)}{\eta^3(z)\eta^3(4z)\eta(30z)}=\sum_{n=1}^{\infty}a_6(n)q^n.$$

Theorem 1.7 *Let* $n \in \mathbb{N}$. *Then*

$$N(x^2+2y^2+2z^2+5t^2+2yt+2zt=n)=\begin{cases} 3\sigma(n)-a_7(n) & \text{if } n\equiv 1 \pmod 4,\\ 2\sigma(n)-a_7(n) & \text{if } n\equiv 2 \pmod 8,\\ 2\sigma(n) & \text{if } n\equiv 3,6 \pmod 8,\\ 6\sigma(n/4) & \text{if } n\equiv 4 \pmod{16},\\ 0 & \text{if } n\equiv 7 \pmod 8,\\ 4\sigma(n/4) & \text{if } n\equiv 8 \pmod{16},\\ 2\sigma(n/4) & \text{if } n\equiv 12 \pmod{16},\\ 8\sigma(n/16)-32\sigma(n/64) & \text{if } n\equiv 0 \pmod{16},\end{cases}$$

where

$$\frac{\eta^5(2z)\eta^3(8z)}{\eta^2(z)\eta^2(4z)}=\sum_{n=1}^{\infty}a_7(n)q^n$$

and

$$a_7(n) = \begin{cases} \displaystyle\sum_{\substack{(r,s)\in\mathbb{N}\times\mathbb{Z}\\ r\equiv 1 \pmod 2\\ n=r^2+4s^2}} (-1)^{(r-1)/2} r & \text{if } n \equiv 1 \,(\text{mod } 4),\\ \displaystyle 2\Big(\frac{2}{n/2}\Big) \sum_{\substack{(r,s)\in\mathbb{N}\times\mathbb{Z}\\ r\equiv 1 \pmod 2\\ n/2=r^2+4s^2}} (-1)^{(r-1)/2} r & \text{if } n \equiv 2 \,(\text{mod } 8),\\ 0 & \text{if } n \equiv 0,3,4,6,7 \,(\text{mod } 8).\end{cases}$$

2. Preliminary results

We use the bound given in Theorem 2.1 to show the equality of two modular forms in the same modular space, see [15, Section 3.3] for a proof of this result. This bound goes back to Hecke and is called either the Hecke bound or the Sturm bound in the literature.

Theorem 2.1 *Let* $N \in \mathbb{N}$. *If*

$$f = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma_0(N), \chi)$$

and

$$g = \sum_{n=0}^{\infty} b_n q^n \in M_k(\Gamma_0(N), \chi)$$

and

$$a_n = b_n \text{ for } n = 0,1,2,\dots,\Big[\frac{k}{12} N \prod_{p\mid N} \big(1+\frac{1}{p}\big)\Big]$$

then $f = g$.

We use the following theorem to determine if a given eta quotient $f(z)$ is in a certain modular space, see [15, Theorem 5.7] and [16, Corollary 2.3].

Theorem 2.2 *Let* $N \in \mathbb{N}$. *Let*

$$f(z) = \prod_{d\mid N} \eta^{r(d)}(dz)$$

be an eta quotient, where d *runs through the positive divisors of* N *and* $r(d) \in \mathbb{Z}$ *for each* $d \mid N$. *Suppose that*

$$k = \frac{1}{2}\sum_{d\mid N} r(d) \in \mathbb{N}.$$

If $f(z)$ *satisfies the following conditions*

(i) $\sum_{d|N} d \cdot r(d) \equiv 0 \,(\text{mod } 24)$,

(ii) $\sum_{d|N} \frac{N}{d} \cdot r(d) \equiv 0 \,(\text{mod } 24)$,

(iii) $\sum_{d|N} \frac{\gcd(d,e)^2 \cdot r(d)}{d} \geq 0$ *for each positive integer* e *dividing* N,

then

$$f(z) \in M_k(\Gamma_0(N), \chi),$$

where χ *is the Dirichlet character* $\left(\frac{M}{*}\right)$ *and* M *is the integer defined by*

$$M := (-1)^k \prod_{d|N} d^{|r(d)|}.$$

If $f(z)$ *satisfies* (i), (ii) *and*

(iv) $\sum_{d|N} \frac{\gcd(d,e)^2 \cdot r(d)}{d} > 0$ *for each positive integer* e *dividing* N,

then

$$f(z) \in S_k(\Gamma_0(N), \chi).$$

The next theorem identifies the modular space to which the theta function of a positive-definite integral quadratic form belongs, see [18, Theorem 10.1].

Theorem 2.3 *Let* $f := f(x_1, \ldots, x_k)$ *be a positive-definite integral quadratic form in* k *variables and let* $A(f)$ *be the matrix of* $f(x_1, \ldots, x_k)$. *Let* N *be the level of* $f(x_1, \ldots, x_k)$, *that is, the least positive integer such that* $NA(f)^{-1}$ *is an integral matrix with even diagonal entries. Then*

$$\theta_f(z) \in M_{k/2}(\Gamma_0(N), \chi),$$

where the character χ *is given by*

$$\chi = \begin{cases} \left(\frac{2\det A(f)}{*}\right) & \text{if } k \text{ is odd}, \\ \left(\frac{(-1)^{k/2}\det A(f)}{*}\right) & \text{if } k \text{ is even.} \end{cases}$$

The following theorem identifies a difference of theta functions as a cusp form, see [18, p. 365].

Theorem 2.4 *Let* $f := f(x_1, \ldots, x_k)$ *and* $g := g(x_1, \ldots, x_k)$ *be two positive-definite integral quadratic forms in* k *variables which belong to the same genus. Let* $\theta_f(z)$ *and* $\theta_g(z)$ *be the theta functions of* f *and* g *respectively. Then* $\theta_f(z) - \theta_g(z)$ *is a cusp form.*

In two of our proofs we make use of properties of Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$ defined in (1.3). Basic properties of these functions include

$$\varphi(q)+\varphi(-q)=2\varphi(q^4), \quad \varphi(q)-\varphi(-q)=4q\psi(q^8), \tag{2.1}$$

$$\varphi(q)=\frac{E^5(q^2)}{E^2(q)E^2(q^4)}, \quad \varphi(-q)=\frac{E^2(q)}{E(q^2)}, \tag{2.2}$$

$$\psi(q)=\frac{E^2(q^2)}{E(q)}, \quad \psi(-q)=\frac{E(q)E(q^4)}{E(q^2)}, \tag{2.3}$$

where

$$E(q):=\prod_{n=1}^{\infty}(1-q^n). \tag{2.4}$$

Proofs of these results can be found in Berndt's book [14]. It follows from (1.2) and (2.4) that

$$\eta(z)=q^{1/24}E(q). \tag{2.5}$$

3. Proof of Theorem 1.1

Let

$$Q_1:=Q_1(x,y,z,t)=x^2+y^2+z^2+6t^2 \tag{3.1}$$

and

$$Q_2:=Q_2(x,y,z,t)=x^2+2y^2+2z^2+2t^2+2yt. \tag{3.2}$$

The classes of the forms in (3.1) and (3.2) belong to the same genus of discriminant 96 and this genus contains no other classes [17].

The matrix $A(Q_1)$ of Q_1 is $\mathrm{diag}(2,2,2,12)$ so $\det A(Q_1)=96$ and $A(Q_1)^{-1}=\mathrm{diag}(1/2,1/2,1/2,1/12)$. Thus $24A(Q_1)^{-1}=\mathrm{diag}(12,12,12,2)$ showing that the level of Q_1 is 24. The character associated with Q_1 is

$$\Big(\frac{\det A(Q_1)}{*}\Big)=\Big(\frac{96}{*}\Big)=\Big(\frac{24}{*}\Big).$$

Hence by Theorem 2.3 we have

$$\theta_{Q_1}(z)\in M_2(\Gamma_0(24),\Big(\frac{24}{*}\Big)). \tag{3.3}$$

The matrix $A(Q_2)$ of Q_2 is

$$A(Q_2)=\begin{bmatrix} 2 & 0 & 0 & 0\\ 0 & 4 & 0 & 2\\ 0 & 0 & 4 & 0\\ 0 & 2 & 0 & 4 \end{bmatrix}$$

so $\det A(Q_2) = 96$ and

$$24A(Q_2)^{-1} = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 0 & 8 & 0 & -4 \\ 0 & 0 & 6 & 0 \\ 0 & -4 & 0 & 8 \end{bmatrix}.$$

Hence the level of Q_2 is 24 and the character associated with Q_2 is

$$\left(\frac{\det A(Q_2)}{*}\right) = \left(\frac{96}{*}\right) = \left(\frac{24}{*}\right).$$

Thus by Theorem 2.3 we have

$$\theta_{Q_2}(z) \in M_2(\Gamma_0(24), \left(\frac{24}{*}\right)). \tag{3.4}$$

From (3.3), (3.4) and Theorem 2.4 we deduce

$$\theta_{Q_1}(z) - \theta_{Q_2}(z) \in S_2(\Gamma_0(24), \left(\frac{24}{*}\right)). \tag{3.5}$$

The bound of Theorem 2.1 for $M_2(\Gamma_0(24), \left(\frac{24}{*}\right))$ is

$$\left[\frac{2 \cdot 24}{12} \prod_{p|24} \left(1 + \frac{1}{p}\right)\right] = \left[4 \cdot \frac{3}{2} \cdot \frac{4}{3}\right] = 8. \tag{3.6}$$

The first nine terms of $\theta_{Q_1}(z)$ and $\theta_{Q_2}(z)$ are

$$\theta_{Q_1}(z) = 1 + 6q + 12q^2 + 8q^3 + 6q^4 + 24q^5 + 26q^6 + 12q^7 + 36q^8 + \cdots$$

and

$$\theta_{Q_2}(z) = 1 + 2q + 8q^2 + 16q^3 + 14q^4 + 24q^5 + 22q^6 + 12q^7 + 44q^8 + \cdots$$

so that the first nine terms of $\theta_{Q_1}(z) - Q_{Q_2}(z)$ are

$$\theta_{Q_1}(z) - \theta_{Q_2}(z) = 4q + 4q^2 - 8q^3 - 8q^4 + 4q^6 - 8q^8 + \cdots . \tag{3.7}$$

By Theorem 2.2 we have

$$A_1(z) := \frac{\eta^4(2z)\eta(3z)\eta^2(8z)\eta(12z)}{\eta(z)\eta^2(4z)\eta(6z)} \in S_2(\Gamma_0(24), \left(\frac{24}{*}\right)), \tag{3.8}$$

$$C_1(z) := \frac{\eta(z)\eta(4z)\eta^4(6z)\eta^2(24z)}{\eta(2z)\eta(3z)\eta^2(12z)} \in S_2(\Gamma_0(24), \left(\frac{24}{*}\right)), \tag{3.9}$$

$$C_2(z) := \frac{\eta^2(z)\eta^4(4z)\eta(6z)\eta(24z)}{\eta^2(2z)\eta(8z)\eta(12z)} \in S_2(\Gamma_0(24), \left(\frac{24}{*}\right)). \tag{3.10}$$

We define $a_1(n)$, $c_1(n)$ and $c_2(n)$ for $n \in \mathbb{N}$ by

$$A_1(z) = \sum_{n=1}^{\infty} a_1(n)q^n,\ C_1(z) = \sum_{n=1}^{\infty} c_1(n)q^n,\ C_2(z) = \sum_{n=1}^{\infty} c_2(n)q^n.$$

The first nine terms of each of $A_1(z)$, $C_1(z)$ and $C_2(z)$ are

$$A_1(z) = q + q^2 - 2q^3 - 2q^4 + q^6 - 2q^8 + \cdots, \tag{3.11}$$

$$C_1(z) = q^2 - q^3 - q^6 - 2q^8 + \cdots, \tag{3.12}$$

$$C_2(z) = q - 2q^2 + q^3 - 2q^4 + 4q^6 + 4q^8 - \cdots. \tag{3.13}$$

Appealing to (3.6), (3.7) and (3.11)–(3.13), we see that $\theta_{Q_1}(z) - \theta_{Q_2}(z)$ and $4A_1(z)$, as well as $A_1(z)$ and $3C_1(z) + C_2(z)$, agree up to the Hecke bound. Hence, by (3.5), (3.8)–(3.10) and Theorem 2.1, we see that

$$\theta_{Q_1}(z) - \theta_{Q_2}(z) = 4A_1(z) \tag{3.14}$$

and

$$A_1(z) = 3C_1(z) + C_2(z). \tag{3.15}$$

Equating coefficients of q^n $(n \in \mathbb{N})$ in (3.14) and (3.15) we obtain

$$N(x^2 + y^2 + z^2 + 6t^2 = n) - N(x^2 + 2y^2 + 2z^2 + 2t^2 + 2yt = n) = 4a_1(n) \tag{3.16}$$

and

$$a_1(n) = 3c_1(n) + c_2(n). \tag{3.17}$$

Alaca, Alaca and Aygin [3, Theorem 4.1(xii)] have recently evaluated $N(x^2 + y^2 + z^2 + 6t^2 = n)$. They proved that

$$N(x^2 + y^2 + z^2 + 6t^2 = n) = 4A_{1,24}(n) - \frac{1}{3}A_{24,1}(n) - \frac{4}{3}A_{-3,-8}(n) + A_{-8,-3}(n) + 8c_1(n) + \frac{8}{3}c_2(n). \tag{3.18}$$

The formula for $N(x^2 + 2y^2 + 2z^2 + 2t^2 + 2yt = n)$ now follows from (3.16)–(3.18). □

4. Proof of Theorem 1.2

Let

$$Q_1 := Q_1(x, y, z, t) = x^2 + y^2 + z^2 + 7t^2 \tag{4.1}$$

and

$$Q_2 := Q_2(x, y, z, t) = x^2 + y^2 + 2z^2 + 4t^2 + 2zt. \tag{4.2}$$

The classes of the forms in (4.1) and (4.2) belong to the same genus of discriminant 112 and this genus contains no other classes [17].

The matrix $A(Q_1)$ of Q_1 is $\text{diag}(2,2,2,14)$ so $\det A(Q_1) = 112$ and $A(Q_1)^{-1} = \text{diag}(1/2,1/2,1/2,1/14)$. Thus $28A(Q_1)^{-1} = \text{diag}(14,14,14,2)$. Hence the level of Q_1 is 28. The character associated with Q_1 is

$$\Big(\frac{\det A(Q_1)}{*}\Big) = \Big(\frac{112}{*}\Big) = \Big(\frac{28}{*}\Big).$$

Therefore by Theorem 2.3 we have

$$\theta_{Q_1}(z) \in M_2(\Gamma_0(28), \Big(\frac{28}{*}\Big)). \tag{4.3}$$

The matrix $A(Q_2)$ of Q_2 is

$$A(Q_2) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 8 \end{bmatrix}$$

so $\det A(Q_2) = 112$ and

$$28A(Q_2)^{-1} = \begin{bmatrix} 14 & 0 & 0 & 0 \\ 0 & 14 & 0 & 0 \\ 0 & 0 & 8 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix}.$$

Hence the level of Q_2 is 28 and the character associated with Q_2 is

$$\Big(\frac{\det A(Q_2)}{*}\Big) = \Big(\frac{112}{*}\Big) = \Big(\frac{28}{*}\Big).$$

Thus by Theorem 2.3 we have

$$\theta_{Q_2}(z) \in M_2(\Gamma_0(28), \Big(\frac{28}{*}\Big)). \tag{4.4}$$

From (4.3), (4.4) and Theorem 2.4, we deduce

$$\theta_{Q_1}(z) - \theta_{Q_2}(z) \in S_2(\Gamma_0(28), \Big(\frac{28}{*}\Big)). \tag{4.5}$$

The bound of Theorem 2.1 for $M_2(\Gamma_0(28), \Big(\frac{28}{*}\Big))$ is

$$\Big[\frac{2\cdot 28}{12}\prod_{p\mid 28}\big(1+\frac{1}{p}\big)\Big] = \Big[\frac{14}{3}\cdot\frac{3}{2}\cdot\frac{8}{7}\Big] = 8. \tag{4.6}$$

Define

$$A_2(z) := \sum_{n=1}^{\infty} a_2(n)q^n = \frac{\eta^9(2z)\eta(7z)\eta(28z)}{\eta^3(z)\eta^3(4z)\eta(14z)}. \tag{4.7}$$

Applying Theorem 2.2 to the eta quotient in (4.7), we deduce that

$$A_2(z) \in S_2(\Gamma_0(28), \Big(\frac{28}{*}\Big)). \tag{4.8}$$

The first nine terms of $\theta_{Q_1}(z)$, $\theta_{Q_2}(z)$, $\theta_{Q_1}(z) - \theta_{Q_2}(z)$ and $A_2(z)$ are

$$\begin{aligned}
\theta_{Q_1}(z) &= 1 + 6q + 12q^2 + 8q^3 + 6q^4 + 24q^5 + 24q^6 + 2q^7 + 24q^8 + \cdots, \\
\theta_{Q_2}(z) &= 1 + 4q + 6q^2 + 8q^3 + 16q^4 + 24q^5 + 24q^6 + 16q^7 + 26q^8 + \cdots, \\
\theta_{Q_1}(z) - \theta_{Q_2}(z) &= 2q + 6q^2 - 10q^4 - 14q^7 - 2q^8 - \cdots, \\
A_2(z) &= q + 3q^2 - 5q^4 - 7q^7 - q^8 - \cdots,
\end{aligned}$$

so that by (4.6) we see that $\theta_{Q_1}(z) - \theta_{Q_2}(z)$ and $2A_2(z)$ agree up to the Hecke bound. Hence, by (4.5), (4.8) and Theorem 2.1, we have

$$\theta_{Q_1}(z) - \theta_{Q_2}(z) = 2A_2(z). \tag{4.9}$$

By Theorem 2.2 or [10, Theorem 3.2(b)] we have

$$B_2(z) := \sum_{n=1}^{\infty} b_2(n)q^n = \frac{\eta^3(8z)\eta^2(28z)}{\eta(56z)} \in S_2(\Gamma_0(56), \Big(\frac{28}{*}\Big)), \tag{4.10}$$

$$B_3(z) := \sum_{n=1}^{\infty} b_3(n)q^n = \frac{\eta^2(4z)\eta^3(56z)}{\eta(8z)} \in S_2(\Gamma_0(56), \Big(\frac{28}{*}\Big)), \tag{4.11}$$

$$B_4(z) := \sum_{n=1}^{\infty} b_4(n)q^n = \frac{\eta^3(z)\eta^2(14z)}{\eta(7z)} \in S_2(\Gamma_0(56), \Big(\frac{28}{*}\Big)). \tag{4.12}$$

By Theorem 2.2 we also have

$$A_2(z) \in S_2(\Gamma_0(56), \Big(\frac{28}{*}\Big)). \tag{4.13}$$

The bound of Theorem 2.1 for $M_2(\Gamma_0(56), \Big(\frac{28}{*}\Big))$ is

$$\Big[\frac{2 \cdot 56}{12} \prod_{p \mid 56} \big(1 + \frac{1}{p}\big)\Big] = \Big[\frac{28}{3} \cdot \frac{3}{2} \cdot \frac{8}{7}\Big] = 16. \tag{4.14}$$

The first 17 terms of $A_2(z)$, $B_2(z)$, $B_3(z)$ and $B_4(z)$ are

$$\begin{aligned}
A_2(z) &= q + 3q^2 - 5q^4 - 7q^7 - q^8 - 3q^9 + 14q^{11} + 7q^{14} + 11q^{16} - 9q^{18} - \cdots, \\
B_2(z) &= q - 3q^9 + \cdots, \\
B_3(z) &= q^7 - 2q^{11} + \cdots, \\
B_4(z) &= q - 3q^2 + 5q^4 - 7q^7 + q^8 - 3q^9 + 14q^{11} - 7q^{14} - 11q^{16} + \cdots.
\end{aligned}$$

Thus the first 17 terms of $2B_2(z) - 14B_3(z) - B_4(z)$ are

$$2B_2(z) - 14B_3(z) - B_4(z) = q + 3q^2 - 5q^4 - 7q^7 - q^8 - 3q^9 + 14q^{11} + 7q^{14} + 11q^{16} - 9q^{18} - \cdots.$$

Hence by (4.14) we see that $2B_2(z) - 14B_3(z) - B_4(z)$ and $A_2(z)$ agree up to the Hecke bound. Then, by (4.10)–(4.13) and Theorem 2.1, we deduce

$$2B_2(z) - 14B_3(z) - B_4(z) = A_2(z). \tag{4.15}$$

Equating coefficients of q^n $(n \in \mathbb{N})$ in (4.9) and (4.15), we obtain

$$N(x^2 + y^2 + z^2 + 7t^2 = n) - N(x^2 + 2y^2 + 2z^2 + 4t^2 + 2zt = n) = 2a_2(n) \tag{4.16}$$

and

$$2b_2(n) - 14b_3(n) - b_4(n) = a_2(n). \tag{4.17}$$

Recently A. Alaca and J. Alanazi [10, Theorem 2.2(a)] established for $n \in \mathbb{N}$ that

$$\begin{aligned} N(x^2 + y^2 + z^2 + 7t^2 = n) &= \frac{7}{2}A_{1,28}(n) - \frac{1}{4}A_{28,1}(n) + \frac{7}{4}A_{-4,-7}(n) - \frac{1}{2}A_{-7,-4}(n) \\ &\quad +3b_2(n) - 21b_3(n) - \frac{3}{2}b_4(n). \end{aligned} \tag{4.18}$$

The formula for $N(x^2 + 2y^2 + 2z^2 + 4t^2 + 2zt = n)$ now follows from (4.16)–(4.18). □

5. Proof of Theorem 1.3

Let

$$Q_1 := Q_1(x, y, z, t) = x^2 + y^2 + z^2 + 9t^2 \tag{5.1}$$

and

$$Q_2 := Q_2(x, y, z, t) = 2x^2 + 2y^2 + 2z^2 + 3t^2 + 2xy + 2xz + 2yt. \tag{5.2}$$

The classes of the forms in (5.1) and (5.2) belong to the same genus of discriminant 144 and this genus contains no other classes [17].

The matrix $A(Q_1)$ of Q_1 is $\mathrm{diag}(2, 2, 2, 18)$ so $\det A(Q_1) = 144$ and $A(Q_1)^{-1} = \mathrm{diag}(1/2, 1/2, 1/2, 1/18)$. Thus $36A(Q_1)^{-1} = \mathrm{diag}(18, 18, 18, 2)$. Hence the level of Q_1 is 36. The character associated with Q_1 is the trivial character modulo 6 as $\det A(Q_1) = 2^4 \cdot 3^2$ is a perfect square. Therefore by Theorem 2.3 we have

$$\theta_{Q_1}(z) \in M_2(\Gamma_0(36)). \tag{5.3}$$

The matrix $A(Q_2)$ of Q_2 is

$$A(Q_2) = \begin{bmatrix} 4 & 2 & 2 & 0 \\ 2 & 4 & 0 & 2 \\ 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 6 \end{bmatrix}$$

so $\det A(Q_2) = 144$ and

$$36A(Q_2)^{-1} = \begin{bmatrix} 20 & -12 & -10 & 4 \\ -12 & 18 & 6 & -6 \\ -10 & 6 & 14 & -2 \\ 4 & -6 & -2 & 8 \end{bmatrix}.$$

Hence the level of Q_2 is 36 and the character associated with Q_2 is the trivial character modulo 6 as $\det A(Q_2) = 2^4 \cdot 3^2$ is a perfect square. Thus by Theorem 2.3 we have

$$\theta_{Q_2}(z) \in M_2(\Gamma_0(36)). \tag{5.4}$$

From (5.3), (5.4) and Theorem 2.4, we deduce

$$\theta_{Q_1}(z) - \theta_{Q_2}(z) \in S_2(\Gamma_0(36)). \tag{5.5}$$

The bound of Theorem 2.1 for $M_2(\Gamma_0(36))$ is

$$\Big[\frac{2\cdot 36}{12}\prod_{p|36}\big(1+\frac{1}{p}\big)\Big] = \Big[6\cdot\frac{3}{2}\cdot\frac{4}{3}\Big] = 12. \tag{5.6}$$

The first 13 terms of $\theta_{Q_1}(z)$ and $\theta_{Q_2}(z)$ are

$$\theta_{Q_1}(z) = 1 + 6q + 12q^2 + 8q^3 + 6q^4 + 24q^5 + 24q^6 + 12q^8 + 32q^9 + 36q^{10} + 48q^{11} + 24q^{12} + \cdots$$

and

$$\theta_{Q_2}(z) = 1 + 12q^2 + 8q^3 + 6q^4 + 24q^5 + 24q^6 + 24q^7 + 12q^8 + 32q^9 + 36q^{10} + 48q^{11} + 24q^{12} + \cdots$$

so that the first 13 terms of $\theta_{Q_1}(z) - \theta_{Q_2}(z)$ are

$$\theta_{Q_1}(z) - \theta_{Q_2}(z) = 6q - 24q^7 + \cdots . \tag{5.7}$$

Now by Theorem 2.2 we have

$$\eta^4(6z) \in S_2(\Gamma_0(36)). \tag{5.8}$$

The first 13 terms of $\eta^4(6z)$ are

$$\eta^4(6z) = q - 4q^7 + \cdots . \tag{5.9}$$

From (5.6), (5.7) and (5.9) we see that $\theta_{Q_1}(z) - \theta_{Q_2}(z)$ and $6\eta^4(6z)$ agree up to the Hecke bound. Hence, by (5.5), (5.8) and Theorem 2.1, we have

$$\theta_{Q_1}(z) - \theta_{Q_2}(z) = 6\eta^4(6z). \tag{5.10}$$

Equating coefficients of q^n $(n \in \mathbb{N})$ in (5.10), we deduce

$$N(x^2+y^2+z^2+9t^2=n) - N(2x^2+2y^2+2z^2+3t^2+2xy+2xz+2yt=n) = 6a_3(n). \tag{5.11}$$

A. Alaca [1, Theorem 1.4] has shown for $n \in \mathbb{N}$ that

$$N(x^2+y^2+z^2+9t^2=n) = \begin{cases} 2\sigma(n) + 4a_3(n) & \text{if } n \equiv 1 \pmod 6, \\ 12\sigma(n) - 24\sigma(n/2) & \text{if } n \equiv 2 \pmod 6, \\ 8\sigma(n/3) & \text{if } n \equiv 3 \pmod 6, \\ 6\sigma(n) - 12\sigma(n/2) & \text{if } n \equiv 4 \pmod 6, \\ 4\sigma(n) & \text{if } n \equiv 5 \pmod 6, \\ 24\sigma(n/3) - 48\sigma(n/6) & \text{if } n \equiv 0 \pmod 6. \end{cases} \tag{5.12}$$

From the definition of $a_3(n)$ we have $a_3(n) = 0$ if $n \not\equiv 1 \pmod 6$. The formula for $N(2x^2 + 2y^2 + 2z^2 + 3t^2 + 2xy + 2xz + 2yt = n)$ now follows from (5.11) and (5.12). The origins of the classical formula for $a_3(n)$ stated in the theorem are given in [1, pp. 152, 153]. □

6. Proof of Theorem 1.4

Let

$$Q_1 := Q_1(x, y, z, t) = x^2 + y^2 + z^2 + 12t^2 \tag{6.1}$$

and

$$Q_2 := Q_2(x, y, z, t) = 2x^2 + 2y^2 + 2z^2 + 3t^2 + 2xy + 2xz. \tag{6.2}$$

The classes of the forms in (6.1) and (6.2) belong to the same genus of discriminant 192 and this genus contains no other classes [17].

The matrix $A(Q_1)$ of Q_1 is $\text{diag}(2, 2, 2, 24)$ so $\det A(Q_1) = 192$ and $A(Q_1)^{-1} = \text{diag}(1/2, 1/2, 1/2, 1/24)$. Thus $48A(Q_1)^{-1} = \text{diag}(24, 24, 24, 2)$. Hence the level of Q_1 is 48. The character associated with Q_1 is $\left(\frac{48}{*}\right) = \left(\frac{12}{*}\right)$. Therefore by Theorem 2.3 we have

$$\theta_{Q_1}(z) \in M_2(\Gamma_0(48), \left(\frac{12}{*}\right)). \tag{6.3}$$

The matrix $A(Q_2)$ of Q_2 is

$$A(Q_2) = \begin{bmatrix} 4 & 2 & 2 & 0 \\ 2 & 4 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

so $\det A(Q_2) = 192$ and

$$48A(Q_2)^{-1} = \begin{bmatrix} 24 & -12 & -12 & 0 \\ -12 & 18 & 6 & 0 \\ -12 & 6 & 18 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}.$$

Hence the level of Q_2 is 48. The character associated with Q_2 is $\left(\frac{192}{*}\right) = \left(\frac{12}{*}\right)$. By Theorem 2.3 we have

$$\theta_{Q_2}(z) \in M_2(\Gamma_0(48), \left(\frac{12}{*}\right)). \tag{6.4}$$

Thus, by (6.3), (6.4) and Theorem 2.4, we have

$$\theta_{Q_1}(z) - \theta_{Q_2}(z) \in S_2(\Gamma_0(48), \left(\frac{12}{*}\right)). \tag{6.5}$$

The bound of Theorem 2.1 for $M_2(\Gamma_0(48), \left(\frac{12}{*}\right))$ is

$$\left[\frac{2\cdot 48}{12}\prod_{p\mid 48}\left(1+\frac{1}{p}\right)\right] = \left[8\cdot\frac{3}{2}\cdot\frac{4}{3}\right] = 16. \tag{6.6}$$

The first 17 terms of $\theta_{Q_1}(z)$ and $\theta_{Q_2}(z)$ are

$$\begin{aligned}\theta_{Q_1}(z) = \quad & 1+6q+12q^2+8q^3+6q^4+24q^5+24q^6+12q^8+30q^9+24q^{10}\\ & +24q^{11}+10q^{12}+36q^{13}+72q^{14}+16q^{15}+18q^{16}+\cdots\end{aligned}$$

and

$$\begin{aligned}\theta_{Q_2}(z) = \quad & 1+12q^2+2q^3+6q^4+24q^5+24q^6+12q^7+12q^8+48q^9+24q^{10}\\ & +24q^{11}+10q^{12}+48q^{13}+72q^{14}+16q^{15}+18q^{16}+\cdots\end{aligned}$$

so that the first 17 terms of $\theta_{Q_1}(z)-\theta_{Q_2}(z)$ are

$$\theta_{Q_1}(z)-\theta_{Q_2}(z) = 6q+6q^3-12q^7-18q^9-12q^{13}+\cdots. \tag{6.7}$$

By Theorem 2.2 we have

$$\frac{\eta^2(4z)\eta^3(6z)}{\eta(2z)} \in S_2(\Gamma_0(48), \left(\frac{12}{*}\right)). \tag{6.8}$$

The first 17 terms of $\frac{\eta^2(4z)\eta^3(6z)}{\eta(2z)}$ are

$$\frac{\eta^2(4z)\eta^3(6z)}{\eta(2z)} = q+q^3-2q^7-3q^9-2q^{13}+\cdots. \tag{6.9}$$

Thus, by (6.6), (6.7) and (6.9), $\theta_{Q_1}(z)-\theta_{Q_2}(z)$ and $6\frac{\eta^2(4z)\eta^3(6z)}{\eta(2z)}$ agree up to the Hecke bound. Hence, by (6.5), (6.8) and Theorem 2.1, we have

$$\theta_{Q_1}(z)-\theta_{Q_2}(z) = 6\frac{\eta^2(4z)\eta^3(6z)}{\eta(2z)}. \tag{6.10}$$

Equating coefficients of q^n $(n\in\mathbb{N})$ in (6.10), we obtain

$$N(x^2+y^2+z^2+12t^2=n) - N(2x^2+2y^2+2z^2+3t^2+2xy+2xz=n) = 6a_4(n). \tag{6.11}$$

As in [7, pp. 32, 33] we define

$$p = p(q) := \frac{\varphi^2(q)-\varphi^2(q^3)}{2\varphi^2(q^3)}$$

and

$$k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}.$$

From [6, pp. 541, 542] or [13, pp. 48, 49] we have

$$\varphi^3(q)\varphi(-q^3) = (1+p)^{3/4}(1-p)^{1/4}(1+2p)^{9/4}k^2$$

and

$$\varphi^3(-q)\varphi(q^3) = (1+p)^{3/4}(1-p)^{9/4}(1+2p)^{1/4}k^2$$

so that

$$\varphi^3(q)\varphi(-q^3) - \varphi^3(-q)\varphi(q^3) = 3p(1+p)^{3/4}(1-p)^{1/4}(1+2p)^{1/4}(2+p)k^2.$$

By [6, Theorem 2.3(b), Proposition 2.8] we have

$$\begin{aligned}\sum_{n=1}^{\infty}\Big(\sum_{\substack{(i,j)\in\mathbb{N}^2\\ i\equiv j\equiv 1 \pmod 2\\ 4n=i^2+3j^2}}(-1)^{(j-1)/2}j\Big)q^n &= \frac{1}{12}\big(\varphi^3(q)\varphi(-q^3)-\varphi^3(-q)\varphi(q^3)\big)\\ &= \frac{1}{4}p(1+p)^{3/4}(1-p)^{1/4}(1+2p)^{1/4}(2+p)k^2.\end{aligned} \tag{6.12}$$

From [9, p. 92] or [13, pp. 48, 49] we have

$$\prod_{n=1}^{\infty}(1-q^{2n}) = q^{-1/12}2^{-1/3}p^{1/12}(1+p)^{1/12}(1-p)^{1/4}(1+2p)^{1/4}(2+p)^{1/4}k^{1/2},$$

$$\prod_{n=1}^{\infty}(1-q^{4n}) = q^{-1/6}2^{-2/3}p^{1/6}(1+p)^{1/24}(1-p)^{1/8}(1+2p)^{1/8}(2+p)^{1/2}k^{1/2},$$

$$\prod_{n=1}^{\infty}(1-q^{6n}) = q^{-1/4}2^{-1/3}p^{1/4}(1+p)^{1/4}(1-p)^{1/12}(1+2p)^{1/12}(2+p)^{1/12}k^{1/2},$$

so that

$$\begin{aligned}\sum_{n=1}^{\infty}a_4(n)q^n &= \frac{\eta^2(4z)\eta^3(6z)}{\eta(2z)} = q\prod_{n=1}^{\infty}\frac{\big(1-q^{4n}\big)^2\big(1-q^{6n}\big)^3}{1-q^{2n}}\\ &= \frac{1}{4}p(1+p)^{3/4}(1-p)^{1/4}(1+2p)^{1/4}(2+p)k^2.\end{aligned} \tag{6.13}$$

Equating coefficients of q^n $(n \in \mathbb{N})$ in (6.12) and (6.13), we deduce

$$a_4(n) = \sum_{\substack{(i,j)\in\mathbb{N}^2\\ i\equiv j\equiv 1 \pmod 2\\ 4n=i^2+3j^2}}(-1)^{(j-1)/2}j, \tag{6.14}$$

which is the formula for $a_4(n)$ stated in the theorem. From the definition of $a_4(n)$ or (6.14) we see that $a_4(n)=0$ for $n\equiv 0 \pmod 2$. Finally, from [6, Theorem 7.2(a), p. 571], we have for $n\in\mathbb{N}$

$$N(x^2+y^2+z^2+12t^2=n)=\begin{cases} 3A_{1,12}(n)-A_{-3,-4}(n)+\frac{3}{2}A_{-4,-3}(n)-\frac{1}{2}A_{12,1}(n) & \\ +3a_4(n) & \text{if } n\equiv 1 \pmod 2, \\ \frac{9}{2}A_{1,12}(n)-\frac{3}{2}A_{-3,-4}(n) & \text{if } n\equiv 2 \pmod 4, \\ \frac{3}{2}A_{1,12}(n)-\frac{1}{2}A_{-3,-4}(n)+3A_{-4,-3}(n)-A_{12,1}(n) & \text{if } n\equiv 0 \pmod 4, \end{cases}$$

and the formula for $N(2x^2+2y^2+2z^2+3t^2+2xy+2xz=n)$ follows from this formula and (6.11). □

7. Proof of Theorem 1.5

The proof follows in a similar fashion to that of Theorem 1.4. Let

$$Q_1 := Q_1(x,y,z,t) = x^2+y^2+3z^2+4t^2 \tag{7.1}$$

and

$$Q_2 := Q_2(x,y,z,t) = x^2+2y^2+2z^2+4t^2+2yt+2zt. \tag{7.2}$$

In this case the classes of the forms in (7.1) and (7.2) belong to the same genus of discriminant 192 and this genus contains no other classes [17]. As in Section 6 we can show that

$$\theta_{Q_1}(z)-\theta_{Q_2}(z) = 2\frac{\eta^3(2z)\eta^2(12z)}{\eta(6z)}. \tag{7.3}$$

Equating coefficients of q^n $(n\in\mathbb{N})$ in (7.3) we obtain

$$N(x^2+y^2+3z^2+4t^2=n)-N(x^2+2y^2+2z^2+4t^2+2yt+2zt=n)=2a_5(n). \tag{7.4}$$

From [9, p. 92] or [13, pp. 48, 49] we have

$$\prod_{n=1}^{\infty}(1-q^{12n}) = q^{-1/2}2^{-2/3}p^{1/2}(1+p)^{1/8}(1-p)^{1/24}(1+2p)^{1/24}(2+p)^{1/6}k^{1/2}.$$

Appealing to this formula and the formulas for $\prod_{n=1}^{\infty}(1-q^{2n})$ and $\prod_{n=1}^{\infty}(1-q^{6n})$ given in Section 6, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}a_5(n)q^n &= \frac{\eta^3(2z)\eta^2(12z)}{\eta(6z)} = q\prod_{n=1}^{\infty}\frac{\left(1-q^{2n}\right)^3\left(1-q^{12n}\right)^2}{1-q^{6n}} \\ &= \frac{1}{4}p(1+p)^{1/4}(1-p)^{3/4}(1+2p)^{3/4}(2+p)k^2.\end{aligned} \tag{7.5}$$

From [6, pp. 541, 542] we have

$$\varphi(q)\varphi^3(-q^3) = (1+p)^{9/4}(1-p)^{3/4}(1+2p)^{3/4}k^2$$

and

$$\varphi(-q)\varphi^3(q^3) = (1+p)^{1/4}(1-p)^{3/4}(1+2p)^{3/4}k^2.$$

Thus

$$\varphi(q)\varphi^3(-q^3) - \varphi(-q)\varphi^3(q^3) = p(1+p)^{1/4}(1-p)^{3/4}(1+2p)^{3/4}(2+p)k^2. \tag{7.6}$$

Hence, by [6, Theorem 2.3(a), Proposition 2.8], (7.6) and (7.5), we deduce

$$\begin{aligned}\sum_{n=1}^{\infty}\Big(\sum_{\substack{(i,j)\in\mathbb{N}^2\\ i\equiv j\equiv 1 \pmod 2\\ 4n=i^2+3j^2}}(-1)^{(i-1)/2}i\Big)q^n &= \frac{1}{4}\big(\varphi(q)\varphi^3(-q^3)-\varphi(-q)\varphi^3(q^3)\big)\\ &= \frac{1}{4}p(1+p)^{1/4}(1-p)^{3/4}(1+2p)^{3/4}(2+p)k^2 \qquad (7.7)\\ &= \sum_{n=1}^{\infty}a_5(n)q^n.\end{aligned}$$

Equating coefficients of q^n $(n\in\mathbb{N})$ in (7.7), we obtain

$$a_5(n) = \sum_{\substack{(i,j)\in\mathbb{N}^2\\ i\equiv j\equiv 1 \pmod 2\\ 4n=i^2+3j^2}}(-1)^{(i-1)/2}i, \tag{7.8}$$

which is the formula for $a_5(n)$ stated in the theorem. From the definition of $a_5(n)$ or (7.8) we see that $a_5(n)=0$ for $n\equiv 0 \pmod 2$. Finally from [6, Theorem 7.2(b)] we have for $n\in\mathbb{N}$

$$N(x^2+y^2+3z^2+4t^2=n) = \begin{cases} 3A_{1,12}(n) - A_{-3,-4}(n) + \frac{3}{2}A_{-4,-3}(n) - \frac{1}{2}A_{12,1}(n) & \\ +a_5(n) & \text{if } n\equiv 1 \pmod 2,\\ \frac{3}{2}A_{1,12}(n) - \frac{1}{2}A_{-3,-4}(n) & \text{if } n\equiv 2 \pmod 4,\\ \frac{9}{2}A_{1,12}(n) - \frac{3}{2}A_{-3,-4}(n) + 3A_{-4,-3}(n) - A_{12,1}(n) & \text{if } n\equiv 0 \pmod 4,\end{cases}$$

and the formula for $N(x^2+2y^2+2z^2+4t^2+2yt+2zt=n)$ follows from this formula and (7.4). □

8. Proof of Theorem 1.6

Let

$$Q_1 := Q_1(x,y,z,t) = x^2+y^2+3z^2+5t^2 \tag{8.1}$$

and

$$Q_2 := Q_2(x,y,z,t) = x^2+y^2+2z^2+8t^2+2zt. \tag{8.2}$$

The classes of the forms in (8.1) and (8.2) belong to the same genus of discriminant 240 and this genus contains no other classes [17].

The matrix $A(Q_1)$ of Q_1 is $\text{diag}(2,2,6,10)$ so $\det A(Q_1) = 240$ and $A(Q_1)^{-1} = \text{diag}(1/2, 1/2, 1/6, 1/10)$. Thus $60A(Q_1)^{-1} = \text{diag}(30, 30, 10, 6)$ showing that the level of Q_1 is 60. The character associated with Q_1 is

$$\Big(\frac{\det A(Q_1)}{*}\Big) = \Big(\frac{240}{*}\Big) = \Big(\frac{60}{*}\Big).$$

Hence by Theorem 2.3 we have

$$\theta_{Q_1}(z) \in M_2(\Gamma_0(60), \Big(\frac{60}{*}\Big)). \tag{8.3}$$

The matrix $A(Q_2)$ of Q_2 is

$$A(Q_2) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 16 \end{bmatrix}$$

so $\det A(Q_2) = 240$ and

$$60A(Q_2)^{-1} = \begin{bmatrix} 30 & 0 & 0 & 0 \\ 0 & 30 & 0 & 0 \\ 0 & 0 & 16 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix}.$$

Hence the level of Q_2 is 60 and the character associated with Q_2 is

$$\Big(\frac{\det A(Q_2)}{*}\Big) = \Big(\frac{240}{*}\Big) = \Big(\frac{60}{*}\Big).$$

Thus by Theorem 2.3 we have

$$\theta_{Q_2}(z) \in M_2(\Gamma_0(60), \Big(\frac{60}{*}\Big)). \tag{8.4}$$

From (8.3), (8.4) and Theorem 2.4, we deduce

$$\theta_{Q_1}(z) - \theta_{Q_2}(z) \in S_2(\Gamma_0(60), \Big(\frac{60}{*}\Big)). \tag{8.5}$$

The bound of Theorem 2.1 for $M_2(\Gamma_0(60), \Big(\frac{60}{*}\Big))$ is

$$\Big[\frac{2 \cdot 60}{12} \prod_{p|60} \Big(1 + \frac{1}{p}\Big)\Big] = \Big[10 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{6}{5}\Big] = 24. \tag{8.6}$$

The first 26 terms of $\theta_{Q_1}(z)$ and $\theta_{Q_2}(z)$ are

$$\begin{aligned} \theta_{Q_1}(z) = \quad & 1 + 4q + 4q^2 + 2q^3 + 12q^4 + 18q^5 + 8q^6 + 16q^7 + 24q^8 + 28q^9 + 40q^{10} \\ & + 8q^{11} + 26q^{12} + 72q^{13} + 16q^{14} + 16q^{15} + 44q^{16} + 44q^{17} + 68q^{18} \\ & + 24q^{19} + 34q^{20} + 80q^{21} + 72q^{22} + 28q^{23} + 40q^{24} + 124q^{25} + \cdots \end{aligned}$$

and

$$\begin{aligned}\theta_{Q_2}(z) = \quad & 1 + 4q + 6q^2 + 8q^3 + 12q^4 + 8q^5 + 8q^6 + 16q^7 + 10q^8 + 28q^9 + 40q^{10} \\ & +8q^{11} + 44q^{12} + 72q^{13} + 16q^{14} + 16q^{15} + 44q^{16} + 64q^{17} + 62q^{18} \\ & +24q^{19} + 44q^{20} + 80q^{21} + 72q^{22} + 16q^{23} + 40q^{24} + 124q^{25} + \cdots\end{aligned}$$

so that the first 26 terms of $\theta_{Q_1}(z) - \theta_{Q_2}(z)$ are

$$\theta_{Q_1}(z) - \theta_{Q_2}(z) = -2q^2 - 6q^3 + 10q^5 + 14q^8 - 18q^{12} - 20q^{17} + 6q^{18} - 10q^{20} + 12q^{23} + \cdots . \tag{8.7}$$

From Theorem 2.2 we deduce that

$$\frac{\eta^9(2z)\eta(15z)\eta(60z)}{\eta^3(z)\eta^3(4z)\eta(30z)} \in S_2(\Gamma_0(60), \Big(\frac{60}{*}\Big)). \tag{8.8}$$

The first 26 terms of $\dfrac{\eta^9(2z)\eta(15z)\eta(60z)}{\eta^3(z)\eta^3(4z)\eta(30z)}$ are

$$\frac{\eta^9(2z)\eta(15z)\eta(60z)}{\eta^3(z)\eta^3(4z)\eta(30z)} = q^2 + 3q^3 - 5q^5 - 7q^8 + 9q^{12} + 10q^{17} - 3q^{18} + 5q^{20} - 6q^{23} + \cdots . \tag{8.9}$$

By (8.6), (8.7) and (8.9) $\theta_{Q_1}(z) - \theta_{Q_2}(z)$ and $-2\dfrac{\eta^9(2z)\eta(15z)\eta(60z)}{\eta^3(z)\eta^3(4z)\eta(30z)}$ agree up to the Hecke bound. Hence, by (8.5), (8.8) and Theorem 2.1, we have

$$\theta_{Q_1}(z) - \theta_{Q_2}(z) = -2\frac{\eta^9(2z)\eta(15z)\eta(60z)}{\eta^3(z)\eta^3(4z)\eta(30z)}. \tag{8.10}$$

Equating the coefficients of q^n $(n \in \mathbb{N})$ in (8.10) we obtain

$$N(x^2 + y^2 + 3z^2 + 5t^2 = n) - N(x^2 + y^2 + 2z^2 + 8t^2 + 2zt = n) = -2a_6(n). \tag{8.11}$$

A. Alaca [2] has shown that

$$\begin{aligned}N(x^2 + y^2 + 3z^2 + 5t^2 = n) = \quad & \frac{1}{12}\Big(30A_{1,60}(n) - A_{60,1}(n) - 6A_{5,12}(n) + 5A_{12,5}(n) \\ & -3A_{-20,-3}(n) + 10A_{-3,-20}(n) - 2A_{-15,-4}(n) + 15A_{-4,-15}(n)\Big) - a_6(n).\end{aligned} \tag{8.12}$$

The formula for $N(x^2 + y^2 + 2z^2 + 8t^2 + 2zt = n)$ now follows from (8.11) and (8.12). □

9. Proof of Theorem 1.7

Let

$$Q_1 := Q_1(x, y, z, t) = x^2 + y^2 + z^2 + 16t^2 \tag{9.1}$$

and

$$Q_2 := Q_2(x, y, z, t) = x^2 + 2y^2 + 2z^2 + 5t^2 + 2yt + 2zt. \tag{9.2}$$

The classes of the forms in (9.1) and (9.2) belong to the same genus of discriminant 256 and this genus contains no other classes [17].

The matrix $A(Q_1)$ of Q_1 is $\operatorname{diag}(2,2,2,32)$ so $\det A(Q_1) = 256$ and $A(Q_1)^{-1} = \operatorname{diag}(1/2,1/2,1/2,1/32)$. Thus $64A(Q_1)^{-1} = \operatorname{diag}(32,32,32,2)$ showing that the level of Q_1 is 64. The character associated with Q_1 is the trivial character modulo 2 as $\det A(Q_1) = 2^8$ is a perfect square. Therefore by Theorem 2.3 we have

$$\theta_{Q_1}(z) \in M_2(\Gamma_0(64)). \tag{9.3}$$

The matrix $A(Q_2)$ of Q_2 is

$$A(Q_2) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 2 \\ 0 & 0 & 4 & 2 \\ 0 & 2 & 2 & 10 \end{bmatrix}$$

so $\det A(Q_2) = 256$ and

$$64A(Q_2)^{-1} = \begin{bmatrix} 32 & 0 & 0 & 0 \\ 0 & 18 & 2 & -4 \\ 0 & 2 & 18 & -4 \\ 0 & -4 & -4 & 8 \end{bmatrix}.$$

Hence the level of Q_2 is 64. The character associated with Q_2 is the trivial character modulo 2 as $\det A(Q_2) = 2^8$ is a perfect square. Thus by Theorem 2.3 we have

$$\theta_{Q_2}(z) \in M_2(\Gamma_0(64)). \tag{9.4}$$

From (9.3), (9.4) and Theorem 2.4, we deduce

$$\theta_{Q_1}(z) - \theta_{Q_2}(z) \in S_2(\Gamma_0(64)). \tag{9.5}$$

The bound of Theorem 2.1 for $M_2(\Gamma_0(64))$ is

$$\Big[\frac{2\cdot 64}{12}\prod_{p|64}\big(1+\frac{1}{p}\big)\Big] = 16. \tag{9.6}$$

The first 18 terms of $\theta_{Q_1}(z)$ and $\theta_{Q_2}(z)$ are

$$\begin{aligned}\theta_{Q_1}(z) = \;& 1 + 6q + 12q^2 + 8q^3 + 6q^4 + 24q^5 + 24q^6 + 12q^8 + 30q^9 + 24q^{10} \\ & + 24q^{11} + 8q^{12} + 24q^{13} + 48q^{14} + 8q^{16} + 60q^{17} + \cdots\end{aligned}$$

and

$$\begin{aligned}\theta_{Q_2}(z) = \;& 1 + 2q + 4q^2 + 8q^3 + 6q^4 + 16q^5 + 24q^6 + 12q^8 + 42q^9 + 40q^{10} \\ & + 24q^{11} + 8q^{12} + 48q^{13} + 48q^{14} + 8q^{16} + 52q^{17} + \cdots\end{aligned}$$

so that the first 18 terms of $\theta_{Q_1}(z) - \theta_{Q_2}(z)$ are

$$\theta_{Q_1}(z) - \theta_{Q_2}(z) = \; 4q + 8q^2 + 8q^5 - 12q^9 - 16q^{10} - 24q^{13} + 8q^{17} + \cdots. \tag{9.7}$$

From Theorem 2.2 we deduce that

$$\frac{\eta^5(2z)\eta^3(8z)}{\eta^2(z)\eta^2(4z)} \in S_2(\Gamma_0(64)). \tag{9.8}$$

The first 18 terms of $\dfrac{\eta^5(2z)\eta^3(8z)}{\eta^2(z)\eta^2(4z)}$ are

$$\frac{\eta^5(2z)\eta^3(8z)}{\eta^2(z)\eta^2(4z)} = q + 2q^2 + 2q^5 - 3q^9 - 4q^{10} - 6q^{13} + 2q^{17} + \cdots . \tag{9.9}$$

From (9.6), (9.7) and (9.9) we see that $\theta_{Q_1}(z) - \theta_{Q_2}(z)$ and $4\dfrac{\eta^5(2z)\eta^3(8z)}{\eta^2(z)\eta^2(4z)}$ agree up to the Hecke bound. Hence, by (9.5), (9.8) and Theorem 2.1, we have

$$\theta_{Q_1}(z) - \theta_{Q_2}(z) = 4\frac{\eta^5(2z)\eta^3(8z)}{\eta^2(z)\eta^2(4z)}. \tag{9.10}$$

Equating the coefficients of q^n $(n \in \mathbb{N})$ in (9.10), we obtain

$$N(x^2 + y^2 + z^2 + 16t^2 = n) - N(x^2 + 2y^2 + 2z^2 + 5t^2 + 2yt + 2zt = n) = 4a_7(n). \tag{9.11}$$

In [5, Theorem 4.10] it was shown that for $n \in \mathbb{N}$ we have

$$N(x^2 + y^2 + z^2 + 16t^2 = n) = \begin{cases} 3\sigma(n) + 3K_2(n) & \text{if } n \equiv 1 \pmod 4, \\ 2\sigma(n) & \text{if } n \equiv 3 \pmod 8, \\ 0 & \text{if } n \equiv 7 \pmod 8, \\ 2\sigma(n) + 6\left(\dfrac{2}{n/2}\right)K_2(n/2) & \text{if } n \equiv 2 \pmod 8, \\ 2\sigma(n) & \text{if } n \equiv 6 \pmod 8, \\ 6\sigma(n/4) & \text{if } n \equiv 4 \pmod{16}, \\ 2\sigma(n/4) & \text{if } n \equiv 12 \pmod{16}, \\ 12\sigma(n/8) & \text{if } n \equiv 8 \pmod{16}, \\ 8\sigma(n/16) - 32\sigma(n/64) & \text{if } n \equiv 0 \pmod{16}, \end{cases} \tag{9.12}$$

where

$$K_2(n) := \sum_{\substack{(r,s) \in \mathbb{N} \times \mathbb{Z} \\ r \equiv 1 \ (\mathrm{mod}\ 2) \\ n = r^2 + 4s^2}} (-1)^{(r-1)/2} r, \quad n \equiv 1 \pmod 2. \tag{9.13}$$

It is clear from (9.13) that $K_2(n) = 0$ for $n \equiv 3 \pmod 4$.

From (9.11) and (9.12) we have

$$N(x^2+2y^2+2z^2+5t^2+2yt+2zt=n)$$
$$=\begin{cases} 3\sigma(n)+3K_2(n)-4a_7(n) & \text{if } n\equiv 1 \pmod 4,\\ 2\sigma(n)-4a_7(n) & \text{if } n\equiv 3 \pmod 8,\\ -4a_7(n) & \text{if } n\equiv 7 \pmod 8,\\ 2\sigma(n)+6\Big(\dfrac{2}{n/2}\Big)K_2(n/2)-4a_7(n) & \text{if } n\equiv 2 \pmod 8,\\ 2\sigma(n)-4a_7(n) & \text{if } n\equiv 6 \pmod 8,\\ 6\sigma(n/4)-4a_7(n) & \text{if } n\equiv 4 \pmod{16},\\ 2\sigma(n/4)-4a_7(n) & \text{if } n\equiv 12 \pmod{16},\\ 12\sigma(n/8)-4a_7(n) & \text{if } n\equiv 8 \pmod{16},\\ 8\sigma(n/16)-32\sigma(n/64)-4a_7(n) & \text{if } n\equiv 0 \pmod{16}. \end{cases} \tag{9.14}$$

Comparing the evaluation of $N(x^2+2y^2+2z^2+5t^2+2yt+2zt=n)$ in (9.14) with that stated in Theorem 1.7, we see that it remains to prove the following relationships between $a_7(n)$ and $K_2(n)$

$$\begin{aligned} a_7(n) &= K_2(n) && \text{if } n\equiv 1 \pmod 4,\\ a_7(2n) &= 2\Big(\frac{2}{n}\Big)K_2(n) && \text{if } n\equiv 1 \pmod 4,\\ a_7(n) &= 0 && \text{if } n\equiv 0,3,4,6,7 \pmod 8. \end{aligned}$$

First we prove that $a_7(n)=K_2(n)$ for $n\equiv 1 \pmod 4$. Using (2.5) in the sum of the series defining $a_7(n)$, we obtain

$$\sum_{n=1}^{\infty} a_7(n)q^n = q\frac{E^5(q^2)E^3(q^8)}{E^2(q)E^2(q^4)}. \tag{9.15}$$

Hence by (2.2) we have

$$\sum_{n=1}^{\infty} a_7(n)q^n = q\varphi(q)E^3(q^8). \tag{9.16}$$

Thus for $a\in\{0,1,2,3\}$ we have

$$\sum_{n=1}^{\infty} a_7(n)(i^aq)^n = i^aq\varphi(i^aq)E^3(q^8).$$

Hence

$$\begin{aligned} 4\sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty} a_7(n)q^n &= \sum_{n=1}^{\infty} a_7(n)q^n\sum_{a=0}^{3} i^{a(n-1)} = \sum_{a=0}^{3} i^{-a}\sum_{n=1}^{\infty} a_7(n)(i^aq)^n\\ &= \sum_{a=0}^{3} i^{-a}i^aq\varphi(i^aq)E^3(q^8) = qE^3(q^8)\sum_{a=0}^{3}\varphi(i^aq) = 4qE^3(q^8)\varphi(q^4), \end{aligned}$$

by (2.1), so

$$\sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty} a_7(n)q^n = q\varphi(q^4)E^3(q^8). \tag{9.17}$$

On the other hand, from [5, Theorem 2.3], we have appealing to (2.1)

$$\sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty} K_2(n)q^n = \frac{1}{4}(\varphi(q)-\varphi(-q))\varphi(q^4)\varphi^2(-q^8) = q\psi(q^8)\varphi^2(-q^8)\varphi(q^4) = q\frac{E^2(q^{16})}{E(q^8)}\frac{E^4(q^8)}{E^2(q^{16})}\varphi(q^4),$$

so

$$\sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty} K_2(n)q^n = q\varphi(q^4)E^3(q^8). \tag{9.18}$$

Thus from (9.17) and (9.18) we obtain

$$\sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty} a_7(n)q^n = \sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty} K_2(n)q^n,$$

and, on equating the coefficients of q^n, we obtain $a_7(n) = K_2(n)$ for $n \equiv 1 \pmod 4$.

Next we prove that $a_7(2n) = 2\left(\frac{2}{n}\right)K_2(n)$ for $n \equiv 1 \pmod 4$. Let ω be a primitive 8th root of unity. From (9.15) we obtain for $b \in \{0,1,2,3,4,5,6,7\}$

$$\sum_{n=1}^{\infty} a_7(n)(\omega^b q)^n = \omega^b q\varphi(\omega^b q)E^3(q^8).$$

Hence

$$\begin{aligned}
8\sum_{\substack{n=1\\ n\equiv 2 \pmod 8}}^{\infty} a_7(n)q^n &= \sum_{n=1}^{\infty} a_7(n)q^n \sum_{b=0}^{7}\omega^{b(n-2)} = \sum_{b=0}^{7}\omega^{-2b}\sum_{n=1}^{\infty} a_7(n)(\omega^b q)^n \\
&= \sum_{b=0}^{7}\omega^{-2b}\omega^b q\varphi(\omega^b q)E^3(q^8) = qE^3(q^8)\sum_{b=0}^{7}\omega^{-b}\varphi(\omega^b q) \\
&= qE^3(q^8)\sum_{b=0}^{7}\omega^{-b}\sum_{n=-\infty}^{\infty}(\omega^b q)^{n^2} = qE^3(q^8)\sum_{n=-\infty}^{\infty} q^{n^2}\sum_{b=0}^{7}\omega^{(n^2-1)b} \\
&= 8qE^3(q^8)\sum_{\substack{n=-\infty\\ n^2\equiv 1 \pmod 8}}^{\infty} q^{n^2} = 8qE^3(q^8)\sum_{\substack{n=-\infty\\ n\equiv 1 \pmod 2}}^{\infty} q^{n^2} \\
&= 4qE^3(q^8)(\varphi(q)-\varphi(-q)) = 16q^2E^3(q^8)\psi(q^8),
\end{aligned}$$

by (2.1), so

$$\sum_{\substack{n=1 \\ n \equiv 1 \ (\mathrm{mod}\ 4)}}^{\infty} a_7(2n)q^{2n} = 2q^2E^3(q^8)\psi(q^8),$$

and thus replacing q^2 by q we obtain

$$\sum_{\substack{n=1 \\ n \equiv 1 \ (\mathrm{mod}\ 4)}}^{\infty} a_7(2n)q^{n} = 2qE^3(q^4)\psi(q^4).$$

On the other hand, as $K_2(n) = 0$ for $n \equiv 3 \pmod 4$ and $\left(\frac{2}{n}\right) = 0$ for $n \equiv 0 \pmod 2$, we have appealing to [5, Theorem 2.4(iv)] and (2.1)–(2.3)

$$\begin{aligned}\sum_{\substack{n=1 \\ n \equiv 1 \ (\mathrm{mod}\ 4)}}^{\infty} \left(\frac{2}{n}\right)K_2(n)q^n &= \sum_{n=1}^{\infty} \left(\frac{2}{n}\right)K_2(n)q^n = \frac{1}{4}(\varphi(q)-\varphi(-q))\varphi(-q^4)\varphi^2(-q^8) \\ &= q\psi(q^8)\varphi(-q^4)\varphi^2(-q^8) = q\frac{E^2(q^{16})}{E(q^8)}\frac{E^2(q^4)}{E(q^8)}\frac{E^4(q^8)}{E^2(q^{16})} \\ &= qE^2(q^4)E^2(q^8) = qE^3(q^4)\frac{E^2(q^8)}{E(q^4)} = qE^3(q^4)\psi(q^4).\end{aligned}$$

Thus

$$\sum_{\substack{n=1 \\ n \equiv 1 \ (\mathrm{mod}\ 4)}}^{\infty} a_7(2n)q^{n} = 2\sum_{\substack{n=1 \\ n \equiv 1 \ (\mathrm{mod}\ 4)}}^{\infty} \left(\frac{2}{n}\right)K_2(n)q^n,$$

and equating the coefficients of q^n we obtain the asserted result.

Finally we prove that $a_7(n) = 0$ for $n \equiv 0, 3, 4, 6, 7 \pmod 8$. We have by (9.16)

$$\sum_{n=0}^{\infty} a_7(n+1)q^n = \frac{1}{q}\sum_{n=1}^{\infty} a_7(n)q^n = E^{-2}(q)E^5(q^2)E^{-2}(q^4)E^3(q^8). \tag{9.19}$$

Applying [4, Theorem 1.2(ii)(iii)(v)(vi)(vii)] to (9.19), we deduce

$$a_7(8k+3) = a_7(8k+4) = a_7(8k+6) = a_7(8k+7) = a_7(8k+8) = 0$$

for all $k \in \mathbb{N}_0$, which is the asserted result. □

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