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A special approach to derive new formulas for some special numbers and polynomials

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Abstract: By applying Laplace differential operator to harmonic conjugate components of the analytic functions and using Wirtinger derivatives, some identities and relations including Bernoulli and Euler polynomials and numbers are obtained. Next, using the Legendre identity, trigonometric functions and the Dirichlet kernel, some formulas and relations involving Bernoulli and Euler numbers, cosine-type Bernoulli and Euler polynomials, and sine-type Bernoulli and Euler polynomials are driven. Then, by using the generating functions method and the well-known Euler identity, many new identities, formulas, and combinatorial sums among the Fibonacci numbers and polynomials, the Lucas numbers and polynomials, the Chebyshev polynomials, and Bernoulli and Euler type polynomials are given. Finally, some infinite series representations for these special numbers and polynomials and their numerical examples are presented.

Key words: Bernoulli and Euler type numbers and polynomials, Chebyshev polynomials, Fibonacci and Lucas polynomials, Generating functions, Harmonic and trigonometric functions, Laplace operator

1. Introduction

There are various kinds of useful applications for Laplace differential operator, Wirtinger derivatives, Legendre identity, trigonometric functions and the Dirichlet kernel, such as in the theory of complex analysis, harmonic analysis, differential equations, partial differential equations, mathematical physics and variety fields of engineering. In this article, we use the Laplace differential operator to harmonic functions related to the generating functions, which resulted new relations and formulas related to Bernoulli and Euler polynomials and numbers. We then apply the Wirtinger derivatives to the Bernoulli and Euler polynomials. Also, by using generating functions for special numbers and polynomials including the Fibonacci numbers, Bernoulli and Euler numbers and polynomials, the Lucas numbers, the Chebyshev polynomials and other special polynomials such as the cosine-Bernoulli polynomials, the cosine-Euler polynomials, the sine-Bernoulli polynomials and the sine-Euler polynomials, many new formulas, relations and combinatorial sums are obtained. Lastly, we present remarks and observations for inequalities of special numbers and polynomials.

Let us briefly give the following definitions and notations.

Let $\mathbb{N}=\{1,2,3,\ldots\}$ and $\mathbb{N}_0=\{0,1,2,3,\ldots\}$. Let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. For $x,y\in\mathbb{R}$, set $z=x+iy\in\mathbb{C}$, $\overline{z}=x-iy\in\mathbb{C}$ and $i^2=-1$.

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The Bernoulli polynomials $B_v(x)$ are defined by

$$H_B(x,t) = \frac{te^{xt}}{e^t - 1} = \sum_{v=0}^{\infty} \frac{B_v(x)t^v}{v!}, \tag{1.1}$$

where $|t| < 2\pi$ (*cf.* [2–36] ; and references therein).

Substituting $x = 0$ into (1.1), Bernoulli numbers: $B_v = B_v(0)$ are derived (*cf.* [2–36] ; and references therein). By using (1.1), we have

$$\sum_{v=0}^{\infty} \frac{B_v(x)t^v}{v!} = e^{xt} \sum_{v=0}^{\infty} \frac{B_v t^v}{v!}.$$

As a result of the product of the two series on the right side of the above equation, the following formula is very easily achieved:

$$B_v(x) = \sum_{j=0}^{v} \binom{v}{j} B_j x^{v-j}, \tag{1.2}$$

where $B_0 = 1$ and for $v \geq 2$, we have

$$\binom{v}{v-1} B_{v-1} = -\sum_{j=0}^{v-2} \binom{v}{j} B_j$$

and finally for $v > 0$

$$B_{2v+1} = 0$$

(see for detail [2–36]).

The Euler polynomials $E_v(x)$ are defined by

$$\frac{2e^{xt}}{e^t + 1} = \sum_{v=0}^{\infty} \frac{E_v(x)t^v}{v!}, \tag{1.3}$$

where $|t| < \pi$ (*cf.* [2–36]; and references therein).

Substituting $x = 0$ into (1.3), Euler numbers $E_v = E_v(0)$ are derived (*cf.* [2–36] ; and references therein). By using (1.3), we have

$$\sum_{v=0}^{\infty} \frac{E_v(x)t^v}{v!} = e^{xt} \sum_{v=0}^{\infty} \frac{E_v t^v}{v!}.$$

Once again, as a result of the product of the two series on the right side of the above equation, the following formula is given:

$$E_v(x) = \sum_{j=0}^{v} \binom{v}{j} x^j E_{v-j}, \tag{1.4}$$

where $E_0 = 1$ and for $v \geq 1$, we have

$$2E_v = -\sum_{j=0}^{v-1} \binom{v}{j} E_j$$

and finally for $v \geq 1$

$$E_{2v} = 0$$

($cf.$ [2–36] ; and references therein).

The Fibonacci polynomials $F_d(x)$ are defined by

$$\frac{t}{1-xt-t^2} = \sum_{d=0}^{\infty} F_d(x)\, t^d, \tag{1.5}$$

where $|t| < 1$ ($cf.$ [4, 5, 16, 17]).

Substituting $x = 1$ into (1.5), Fibonacci numbers $F_d = F_d(1)$ are derived. These numbers are computed by the following recursive formula:

$$F_d = F_{d-1} + F_{d-2},$$

where $F_0 = 0$ and $F_1 = 1$ ($cf.$ [4, 5, 16, 17, 38]).

The Lucas polynomials $L_k(x)$ are defined by

$$\frac{2-xt}{1-xt-t^2} = \sum_{k=0}^{\infty} L_k(x)\, t^k, \tag{1.6}$$

($cf.$ [4, 5, 16, 17]).

Substituting $x = 1$ into (1.6), Lucas numbers $L_k = L_k(1)$ are derived. These numbers are computed by the following recursive formula:

$$L_k = L_{k-1} + L_{k-2},$$

where $L_0 = 2$ and $L_1 = 1$ ($cf.$ [4, 5, 16, 17, 38]).

The first kind of Chebyshev polynomials $T_l(x)$ are defined by

$$\frac{1-xt}{1-2xt+t^2} = \sum_{l=0}^{\infty} T_l(x)\, t^l, \tag{1.7}$$

while the second kind of Chebyshev polynomials $U_l(x)$ are defined by

$$\frac{1}{1-2xt+t^2} = \sum_{l=0}^{\infty} U_l(x)\, t^l, \tag{1.8}$$

see [1–3, 7, 16, 18].

Combining (1.5) and (1.6) with (1.8) and (1.7), respectively, we have the following well-known relations:

$$U_l\left(\frac{i}{2}\right) = i^l F_{l+1}, \tag{1.9}$$

and

$$T_l\left(\frac{i}{2}\right) = \frac{i^l}{2} L_l, \tag{1.10}$$

see [16, 17, 38].

The Fibonacci-type polynomials in two variables $\mathcal{G}_v(x,y;k,m,n)$ are defined by

$$\frac{1}{1-x^k t-y^m t^{m+n}}=\sum_{v=0}^{\infty}\mathcal{G}_v(x,y;k,m,n)\,t^v, \tag{1.11}$$

where $k,m,n\in\mathbb{N}_0$ [26]. Using (1.11), we have

$$\mathcal{G}_v(x,y;k,m,n)=\sum_{j=0}^{\left[\frac{v}{m+n}\right]}\binom{v-j(m+n-1)}{j}y^{mj}x^{vk-mjk-njk}, \tag{1.12}$$

where $[\alpha]$ is the largest integer $\leq\alpha$ [26, 27].

Substituting $x=y=1$ and $m=n=1$ into (1.11) and (1.12) and using (1.5), we get the following well-known Lucas formula, which was found by Lucas in 1876, for the Fibonacci numbers:

$$F_v=\mathcal{G}_{v-1}(1,1;1,1,1)=\sum_{j=0}^{\left[\frac{v-1}{2}\right]}\binom{v-j-1}{j},$$

where $v\in\mathbb{N}$ (*cf.* [16, 17, 26, 32]).

The polynomials $C_l(x,y)$ and $S_l(x,y)$ are given, respectively, as follows:

$$e^{xt}\cos(yt)=\sum_{l=0}^{\infty}C_l(x,y)\frac{t^l}{l!}, \tag{1.13}$$

and

$$e^{xt}\sin(yt)=\sum_{l=0}^{\infty}S_l(x,y)\frac{t^l}{l!}, \tag{1.14}$$

(*cf.* [10–14, 20, 21, 35, 36]).

Using (1.13) and (1.14), we directly obtain

$$C_l(x,y)=\sum_{j=0}^{\left[\frac{l}{2}\right]}(-1)^j\binom{l}{2j}x^{l-2j}y^{2j}, \tag{1.15}$$

and

$$S_l(x,y)=\sum_{j=0}^{\left[\frac{l-1}{2}\right]}(-1)^j\binom{l}{2j+1}x^{l-2j-1}y^{2j+1}, \tag{1.16}$$

(*cf.* for detail, see [10–14, 20, 21, 35, 36]).

By using (1.7), (1.8), (1.13) and (1.14), the following relations are derived [12, 13]

$$T_l(x)=C_l\left(x,\sqrt{1-x^2}\right)\quad(l\in\mathbb{N}_0), \tag{1.17}$$

and

$$U_{l-1}(x) = \frac{S_l\left(x, \sqrt{1-x^2}\right)}{\sqrt{1-x^2}} \quad (l \in \mathbb{N}). \tag{1.18}$$

Substituting $y = -1$, $k = m = n = 1$ into (1.11) and replacing x by $2x$, we obtain the following identities for the polynomials $C_l(x, y)$ and $S_l(x, y)$ and the Fibonacci-type polynomials [14]:

$$C_l\left(x, \sqrt{1-x^2}\right) = \mathcal{G}_l(2x, -1; 1, 1, 1) - x\mathcal{G}_{l-1}(2x, -1; 1, 1, 1) \quad (l \in \mathbb{N}), \tag{1.19}$$

and

$$S_{l+1}\left(x, \sqrt{1-x^2}\right) = \sqrt{1-x^2}\mathcal{G}_l(2x, -1; 1, 1, 1) \quad (l \in \mathbb{N}_0). \tag{1.20}$$

The cosine-Bernoulli polynomials $B_v^{(C)}(x, y)$ and the sine-Bernoulli polynomials $B_v^{(S)}(x, y)$ are defined as follows:

$$\frac{te^{xt}\cos(yt)}{e^t - 1} = \sum_{v=0}^{\infty} B_v^{(C)}(x, y)\frac{t^v}{v!}, \tag{1.21}$$

and

$$\frac{te^{xt}\sin(yt)}{e^t - 1} = \sum_{v=0}^{\infty} B_v^{(S)}(x, y)\frac{t^v}{v!}, \tag{1.22}$$

for detail, see [22].

By using (1.13), (1.14), (1.21) and (1.22), we have the following well-known identities:

$$B_v^{(C)}(x, y) = \sum_{m=0}^{v} \binom{v}{m} B_{v-m}C_m(x, y),$$

and

$$B_v^{(S)}(x, y) = \sum_{m=0}^{v} \binom{v}{m} B_{v-m}S_m(x, y),$$

for detail, see [22].

By using (1.13), (1.14), (1.21) and (1.22), we also have the following well-known identities: for $v \in \mathbb{N}$,

$$\frac{B_v^{(C)}(x+1, y) - B_v^{(C)}(x, y)}{v} = C_{v-1}(x, y), \tag{1.23}$$

and

$$\frac{B_v^{(S)}(x+1, y) - B_v^{(S)}(x, y)}{v} = S_{v-1}(x, y), \tag{1.24}$$

see [22].

By combining (1.21) and (1.22) with (1.2), we get [14]:

$$B_v^{(C)}(x, y) + iB_v^{(S)}(x, y) = \sum_{j=0}^{v}\sum_{d=0}^{j} \binom{v}{j}\binom{j}{d} B_{v-j}x^d(iy)^{j-d}, \tag{1.25}$$

and

$$B_v^{(C)}(x,y) - iB_v^{(S)}(x,y) = \sum_{j=0}^{v}\sum_{d=0}^{j}(-1)^{j-d}\binom{v}{j}\binom{j}{d}B_{v-j}x^d(iy)^{j-d}. \tag{1.26}$$

The cosine-Euler polynomials $E_v^{(C)}(x,y)$ and the sine-Euler polynomials $E_v^{(S)}(x,y)$ are defined as follows [21]:

$$\frac{2e^{xt}\cos(yt)}{e^t+1} = \sum_{v=0}^{\infty} E_v^{(C)}(x,y)\frac{t^v}{v!}, \tag{1.27}$$

and

$$\frac{2e^{xt}\sin(yt)}{e^t+1} = \sum_{v=0}^{\infty} E_v^{(S)}(x,y)\frac{t^v}{v!}. \tag{1.28}$$

By using (1.13), (1.14), (1.27) and (1.28), we have the following well-known identities [21]:

$$E_v^{(C)}(x,y) = \sum_{m=0}^{v}\binom{v}{m}E_{v-m}C_m(x,y),$$

and

$$E_v^{(S)}(x,y) = \sum_{m=0}^{v}\binom{v}{m}E_{v-m}S_m(x,y).$$

By using (1.13), (1.14), (1.27) and (1.28), we also have the following well-known identities [21]:

$$\frac{E_v^{(C)}(x+1,y) + E_v^{(C)}(x,y)}{2} = C_v(x,y), \tag{1.29}$$

and

$$\frac{E_v^{(S)}(x+1,y) + E_v^{(S)}(x,y)}{2} = S_v(x,y). \tag{1.30}$$

By combining (1.27) and (1.28) with (1.4), we have [14]:

$$E_v^{(C)}(x,y) + iE_v^{(S)}(x,y) = \sum_{j=0}^{v}\sum_{d=0}^{j}\binom{v}{j}\binom{j}{d}E_{v-j}x^d(iy)^{j-d}, \tag{1.31}$$

and

$$E_v^{(C)}(x,y) - iE_v^{(S)}(x,y) = \sum_{j=0}^{v}\sum_{d=0}^{j}(-1)^{j-d}\binom{v}{j}\binom{j}{d}E_{v-j}x^d(iy)^{j-d}. \tag{1.32}$$

The results of this paper are summarized as follows:

In Section 2, by applying the Laplace differential operator to the analytic and harmonic functions including generating functions for special polynomials, we give many identities for Bernoulli numbers and polynomials and for Euler numbers and polynomials. Applying partial derivative operators to the generating functions for Bernoulli and Euler polynomials, we also give derivative formulas for these polynomials.

In Section 3, by using the Legendre identity and trigonometric functions, and the Dirichlet kernel, we present many new formulas, identities and relations including the Bernoulli numbers, the cosine-Bernoulli polynomials, the sine-Bernoulli polynomials, the cosine-Euler polynomials and the sine-Euler polynomials. Moreover, by using Umbral calculus methods, we obtain some combinatorial identities involving the Bernoulli and Euler numbers and polynomials.

In Section 4, we give some new identities and relations among the Fibonacci numbers, the Lucas numbers, the cosine-Bernoulli polynomials, the sine-Bernoulli polynomials, the cosine-Euler polynomials and the sine-Euler polynomials.

In Section 5, with the help of series representations for Fibonacci-type polynomials, we give some infinite series representations for the cosine-Euler polynomials, the cosine-Bernoulli polynomials and the sine-Bernoulli polynomials.

Finally, in Section 6, we give some remarks and observations on inequalities for special polynomials and numbers.

2. Identities for Bernoulli and Euler polynomials

In this section, by applying the Laplace differential operator to the special functions, we give some identities and formulas for Bernoulli and Euler numbers and polynomials. Moreover, by using the Cauchy–Riemann equations, we obtain some relations including the cosine- and sine-Bernoulli polynomials and the cosine- and sine-Euler polynomials.

Theorem 2.1 *Assume that*

$$U(x,y\,|t\,) = \frac{te^{xt}}{e^t-1}\cos\left(yt\right), \tag{2.1}$$

is analytic in the open domain $\mathcal{R} \subseteq \mathbb{R}^2$ *and* $|t| < 2\pi$*. Then a harmonic conjugate of this function is given by*

$$V(x,y\,|t\,) = \frac{te^{xt}}{e^t-1}\sin\left(yt\right).$$

Proof Let

$$H_B(z,t) = U(x,y\,|t\,) + iV(x,y\,|t\,).$$

The function $H_B(z,t)$ is to be analytic on an open subset $\mathcal{R} \subseteq \mathbb{C}$. By applying the well-known Cauchy–Riemann equations to the functions $U(x,y\,|t\,)$ and $V(x,y\,|t\,)$, we get

$$\frac{\partial}{\partial x}U(x,y\,|t\,) = \frac{\partial}{\partial y}V(x,y\,|t\,), \tag{2.2}$$

and

$$\frac{\partial}{\partial y}U(x,y\,|t\,) = -\frac{\partial}{\partial x}V(x,y\,|t\,). \tag{2.3}$$

We now show that $U(x,y\,|t\,)$ is a harmonic function. By applying the following Laplace differential operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \overline{z}},$$

to Equation (2.1), we obtain

$$\Delta\left(U(x,y\,|t\,)\right)=\frac{\partial^2}{\partial x^2}U(x,y\,|t\,)+\frac{\partial^2}{\partial y^2}U(x,y\,|t\,)=0.$$

Hence $U(x,y\,|t\,)$ is harmonic. In order to find a harmonic conjugate of the function $U(x,y\,|t\,)$, using Cauchy–Riemann equations, we should solve the following differential equation:

$$\begin{aligned} V(x,y\,|t\,) &= \int\left\{-\frac{\partial}{\partial y}U(x,y\,|t\,)dx+\frac{\partial}{\partial x}U(x,y\,|t\,)dy\right\} \\ &= \frac{te^{xt}}{e^t-1}\sin\left(yt\right)+c, \end{aligned}$$

where c is an arbitrary constant. Since c is arbitrary, it can be ignored and therefore, we obtain components of the analytic function $H_B(z,t)$ as follows:

$$\begin{aligned} H_B(z,t) &= U(x,y\,|t\,)+iV(x,y\,|t\,) \\ &= \frac{te^{xt}}{e^t-1}\cos\left(yt\right)+i\frac{te^{xt}}{e^t-1}\sin\left(yt\right). \end{aligned} \tag{2.4}$$

□

By combining (2.4) with (1.21), (1.22) and (2.1), we have the following results:

The functions $U(x,y\,|t\,)$ and $V(x,y\,|t\,)$ are harmonic conjugate to eachother in the open domain $\mathcal{R}\subseteq\mathbb{R}^2$:

$$U(x,y\,|t\,)=\frac{te^{xt}}{e^t-1}\cos\left(yt\right)=\sum_{n=0}^{\infty}B_n^{(C)}\left(x,y\right)\frac{t^n}{n!},$$

and

$$V(x,y\,|t\,)=\frac{te^{xt}}{e^t-1}\sin\left(yt\right)=\sum_{n=0}^{\infty}B_n^{(S)}\left(x,y\right)\frac{t^n}{n!},$$

where $x,y\in\mathbb{R}$ and $|t|<2\pi$.

The functions $U(x,y\,|t\,)$ and $V(x,y\,|t\,)$ are also C^2-functions in the open domain $\mathcal{R}\subseteq\mathbb{R}^2$ and $|t|<2\pi$.

With the help of the Euler's formula, we have

$$H_B(z,t)=\frac{te^{zt}}{e^t-1}=\sum_{n=0}^{\infty}B_n\left(z\right)\frac{t^n}{n!},$$

where $z\in\mathbb{C}$ and $|t|<2\pi$.

Similar to the above method, the following functions are harmonic conjugate to eachother in the open domain $\mathcal{R}\subseteq\mathbb{R}^2$:

$$\psi(x,y\,|t\,)=\frac{2e^{xt}}{e^t+1}\cos\left(yt\right)=\sum_{n=0}^{\infty}E_n^{(C)}\left(x,y\right)\frac{t^n}{n!},$$

and

$$\varphi(x,y\,|t\,)=\frac{2e^{xt}}{e^t+1}\sin\left(yt\right)=\sum_{n=0}^{\infty}E_n^{(S)}\left(x,y\right)\frac{t^n}{n!},$$

where $|t| < \pi$. Thus, with the help of the above harmonic conjugate components, the analytic function

$$G_E(z,t) = \psi(x, y\,|t\,) + i\varphi(x, y\,|t\,)$$

is given by the generating function of the Euler polynomials $E_n(z)$ as

$$G_E(z,t) = \frac{2e^{zt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(z)\frac{t^n}{n!},$$

where $z \in \mathbb{C}$ and $|t| < \pi$.

Theorem 2.2 *Let Δ be the Laplace differential operator. Assume that*

$$H_B(z,t) = \frac{t}{e^t-1}e^{zt},$$

is analytic with respect to the variable z in the open domain $\mathcal{R} \subseteq \mathbb{C}$ with $|t| < 2\pi$. For any $z = x + iy \in \mathcal{R}$, we have

$$\Delta\left(|H_B(z,t)|^2\right) = 4\sum_{n=2}^{\infty} n(n-1)\sum_{j=0}^{n-2}\binom{n-2}{j}B_j(x)B_{n-j-2}(x)\frac{t^n}{n!}. \tag{2.5}$$

Proof Applying the Laplace differential operator to the following equation

$$\begin{aligned} |H_B(z,t)|^2 &= \left|\frac{te^{xt}}{e^t-1}(\cos(yt) + i\sin(yt))\right|^2 \\ &= |U(x, y\,|t\,) + iV(x, y\,|t\,)|^2 \end{aligned}$$

first yields

$$\begin{aligned} \Delta\left(|H_B(z,t)|^2\right) &= \Delta\left(U^2(x, y\,|t\,) + V^2(x, y\,|t\,)\right) \\ &= \frac{\partial^2}{\partial x^2}\left\{U^2(x, y\,|t\,) + V^2(x, y\,|t\,)\right\} + \frac{\partial^2}{\partial y^2}\left\{U^2(x, y\,|t\,) + V^2(x, y\,|t\,)\right\}. \end{aligned}$$

Since

$$\frac{\partial^2}{\partial x^2}\left\{U^2(x, y\,|t\,) + V^2(x, y\,|t\,)\right\} = 2\frac{\partial}{\partial x}\left\{U(x, y\,|t\,)\frac{\partial}{\partial x}U(x, y\,|t\,) + V(x, y\,|t\,)\frac{\partial}{\partial x}V(x, y\,|t\,)\right\}$$

and

$$\frac{\partial^2}{\partial y^2}\left\{U^2(x, y\,|t\,) + V^2(x, y\,|t\,)\right\} = 2\frac{\partial}{\partial y}\left\{U(x, y\,|t\,)\frac{\partial}{\partial y}U(x, y\,|t\,) + V(x, y\,|t\,)\frac{\partial}{\partial y}V(x, y\,|t\,)\right\},$$

after some calculations, we obtain

$$\Delta\left(|H_B(z,t)|^2\right) = 4t^2H_B^2(x,t)$$

or

$$\Delta\left(|H_B(z,t)|^2\right) = 4t^2 H_B(2x,t)H_B(0,t)$$

where

$$H_B^2(x,t) = H_B(x,t)H_B(x,t).$$

Combining the above equation with (1.1), we arrive at the desired result. □

Corollary 2.3 *Under the restrictions of Theorem 2.2, we have*

$$\Delta\left(|H_B(z,t)|^2\right) = 4\sum_{n=2}^{\infty} n(n-1)B_{n-2}^{(2)}(2x)\frac{t^n}{n!}. \tag{2.6}$$

Corollary 2.4 *Under the restrictions of Theorem 2.2, we have*

$$\Delta\left(|H_B(z,t)|^2\right) = 4\sum_{n=2}^{\infty} n(n-1)\sum_{j=0}^{n-2}\binom{n-2}{j}B_j B_{n-j-2}(2x)\frac{t^n}{n!}. \tag{2.7}$$

Combining (2.5) with (2.6), we have the following well-known relations:

$$\begin{aligned} B_n^{(2)}(2x) &= \sum_{j=0}^{n}\binom{n}{j}B_j(x)B_{n-j}(x) \\ &= \sum_{j=0}^{n}\binom{n}{j}B_j B_{n-j}(2x). \end{aligned}$$

Combining the above relation with (2.7), we arrive at the following corollary:

Corollary 2.5 *Under the restrictions of Theorem 2.2, we have*

$$\sum_{n=2}^{\infty} n(n-1)B_{n-2}^{(2)}(x)\frac{t^n}{n!} = \left|\frac{d}{dz}H_B(z,t)\right|^2. \tag{2.8}$$

Theorem 2.6 *Let Δ be the Laplace differential operator. Assume that*

$$G_E(z,t) = \frac{2}{e^t+1}e^{zt},$$

is analytic with respect to the variable z in the open domain $\mathcal{R} \subseteq \mathbb{C}$ with $|t| < \pi$. For any $z = x+iy \in \mathcal{R}$, we have

$$\Delta\left(|G_E(z,t)|^2\right) = 4\sum_{n=2}^{\infty} n(n-1)\sum_{j=0}^{n-2}\binom{n-2}{j}E_j(x)E_{n-j-2}(x)\frac{t^n}{n!}. \tag{2.9}$$

The proof is the same as that of Theorem 2.2, and so we omit it.

Corollary 2.7 *Under the restrictions of Theorem 2.6, we have*

$$\Delta\left(|G_E(z,t)|^2\right)=4\sum_{n=2}^{\infty}n(n-1)\sum_{j=0}^{n-2}\binom{n-2}{j}E_jE_{n-j-2}(2x)\frac{t^n}{n!}. \tag{2.10}$$

Corollary 2.8 *Under the restrictions of Theorem 2.6, we have*

$$\Delta\left(|G_E(z,t)|^2\right)=4\sum_{n=2}^{\infty}n(n-1)E_{n-2}^{(2)}(2x)\frac{t^n}{n!}. \tag{2.11}$$

Combining (2.11) with (2.9) and (2.10), we have the following well-known relations:

$$\begin{aligned} E_n^{(2)}(2x) &= \sum_{j=0}^{n}\binom{n}{j}E_jE_{n-j}(2x) \\ &= \sum_{j=0}^{n}\binom{n}{j}E_j(x)E_{n-j}(x). \end{aligned}$$

Remark 2.9 *In the work of Mejlbro [25], we see that if $h(z)$ is any analytic function for $z=x+iy\in\mathcal{R}$, then the following result holds true*

$$\Delta\left(|h(z)|^2\right)=4\left|\frac{d}{dz}h(z)\right|^2. \tag{2.12}$$

Theorem 2.10 *Let Δ be the Laplace differential operator. Assume that*

$$H_B(z,t)=\frac{t}{e^t-1}e^{zt}$$

is analytic with respect to the variable z in the open domain $\mathcal{R}\subseteq\mathbb{C}$ with $|t|<2\pi$. For any $z=x+iy\in\mathcal{R}$, we have

$$\Delta\left(U^2(x,y\,|t)\right)=2\sum_{n=2}^{\infty}n(n-1)\sum_{j=0}^{n-2}\binom{n-2}{j}B_j\,(x)\,B_{n-j-2}(x)\frac{t^n}{n!}.$$

Proof By applying the chain rule in partial derivatives to the function $U(x,y\,|t)V(x,y\,|t)$, we first have

$$\frac{\partial}{\partial x}\{U(x,y\,|t)V(x,y\,|t)\}=V(x,y\,|t)\frac{\partial}{\partial x}U(x,y\,|t)+U(x,y\,|t)\frac{\partial}{\partial x}V(x,y\,|t),$$

and

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\{U(x,y\,|t)V(x,y\,|t)\} &= V(x,y\,|t)\frac{\partial^2}{\partial x^2}U(x,y\,|t) \\ &+U(x,y\,|t)\frac{\partial^2}{\partial x^2}V(x,y\,|t)+2\frac{\partial}{\partial x}U(x,y\,|t)\frac{\partial}{\partial x}V(x,y\,|t). \end{aligned}$$

So, by applying the above formulas to the harmonic C^2-function in $\mathcal{R}\subseteq\mathbb{R}^2$:

$$U(x,y\,|t)=\frac{t}{e^t-1}e^{xt}\cos(yt),$$

after some elementary calculations, we obtain

$$\begin{aligned}\Delta\left(U^2(x,y\,|t\,)\right) &= 2\left(\frac{\partial}{\partial x}U(x,y\,|t\,)\right)^2+2\left(\frac{\partial}{\partial y}U(x,y\,|t\,)\right)^2\\ &= 2t^2H_B^2(x,t).\end{aligned}$$

Combining the above equation with (1.1), we arrive at the desired result. □

Corollary 2.11 *Under the restrictions of Theorem 2.10, we have*

$$\Delta\left(U^2(x,y\,|t\,)\right)=2\sum_{n=2}^{\infty}n(n-1)B_{n-2}^{(2)}(2x)\frac{t^n}{n!}.$$

Corollary 2.12 *Under the restrictions of Theorem 2.10, we have*

$$\left|\text{grad } U(x,y\,|t\,)\right|^2=\sum_{n=2}^{\infty}n(n-1)B_{n-2}^{(2)}(2x)\frac{t^n}{n!},$$

where

$$\text{grad } H_B(z,t)=\frac{\partial}{\partial x}U(x,y\,|t\,)-\frac{\partial}{\partial y}V(x,y\,|t\,)+i\left(\frac{\partial}{\partial y}U(x,y\,|t\,)+\frac{\partial}{\partial x}V(x,y\,|t\,)\right).$$

Theorem 2.13 *Let* Δ *be the Laplace differential operator. Assume that*

$$G_E(z,t)=\frac{2}{e^t+1}e^{zt},$$

is analytic with respect to the variable z *in the open domain* $\mathcal{R}\subseteq\mathbb{C}$ *with* $|t|<\pi$. *For any* $z=x+iy\in\mathcal{R}$, *we have*

$$\Delta\left(\psi^2(x,y\,|t\,)\right)=2\sum_{n=2}^{\infty}n(n-1)E_{n-2}^{(2)}(2x)\frac{t^n}{n!}.$$

The proof is the same as that of Theorem 2.10, and so we omit it.

Remark 2.14 *Since*

$$H_B(z,t)=U(x,y\,|t\,)+iV(x,y\,|t\,)$$

is an analytic function for $z=x+iy\in\mathcal{R}\subseteq\mathbb{C}$, *the functions* $U(x,y\,|t\,)V(x,y\,|t\,)$ *and* $U^2(x,y\,|t\,)$ *are harmonic in* $\mathcal{R}\subseteq\mathbb{R}^2$. *Therefore, we have [25]*

$$\Delta\left(U^2(x,y\,|t\,)\right)=2\left|\text{grad } U(x,y\,|t\,)\right|^2.$$

The derivative formulas for Bernoulli and Euler polynomials can be represented by the derivative of their generating functions, which belong to the family of Appell polynomials. However, with the help of the following partial derivative operators

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right), \tag{2.13}$$

and

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{2.14}$$

we provide a proof for these derivative formulas as follows:

Here we add that the operators (2.13) and (2.14) are also known as the Wirtinger derivatives (*cf.* [33–37]).

By combining (1.25) with (1.2) and applying the operator $\frac{\partial}{\partial z}$ to the final Bernoulli polynomials equation, we get

$$\frac{\partial}{\partial z}B_n(z) = \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}B_{n-j}\left(\frac{\partial}{\partial x}(x+iy)^j - i\frac{\partial}{\partial y}(x+iy)^j\right).$$

After some elementary calculations for the above equation, we obtain

$$\frac{\partial}{\partial z}B_n(z) = \sum_{j=0}^{n}\binom{n}{j}jz^{j-1}B_{n-j}.$$

Thus, we have the following result:

Corollary 2.15 *Let $n \in \mathbb{N}$. Then*

$$\frac{\partial}{\partial z}B_n(z) = nB_{n-1}(z).$$

By combining (1.26) with (1.2) and applying the operator $\frac{\partial}{\partial \overline{z}}$ to the final Bernoulli polynomials equation, we have

$$\frac{\partial}{\partial \overline{z}}B_n(\overline{z}) = \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}B_{n-j}\left(\frac{\partial}{\partial x}(x-iy)^j + i\frac{\partial}{\partial y}(x-iy)^j\right).$$

After some elementary calculations for the above equation, we obtain

$$\frac{\partial}{\partial \overline{z}}B_n(\overline{z}) = \sum_{j=0}^{n}\binom{n}{j}j(\overline{z})^{j-1}B_{n-j}.$$

Thus, we have the following result:

Corollary 2.16 *Let $n \in \mathbb{N}$. Then*

$$\frac{\partial}{\partial \overline{z}}B_n(\overline{z}) = nB_{n-1}(\overline{z}).$$

By combining (1.31) with (1.4) and applying the operator $\frac{\partial}{\partial z}$ to the final Euler polynomials equation, we have

$$\frac{\partial}{\partial z}E_n(z) = \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}E_{n-j}\left(\frac{\partial}{\partial x}(x+iy)^j - i\frac{\partial}{\partial y}(x+iy)^j\right).$$

After some elementary calculations for the above equation, we obtain

$$\frac{\partial}{\partial z}E_n(z) = \sum_{j=0}^{n}\binom{n}{j}jz^{j-1}E_{n-j}.$$

Thus, we have the following result:

Corollary 2.17 *Let* $n \in \mathbb{N}$*. Then we have*

$$\frac{\partial}{\partial z} E_n(z) = n E_{n-1}(z).$$

Finally, by combining (1.32) with (1.4) and applying the operator $\frac{\partial}{\partial \overline{z}}$ to the final Euler polynomials equation, we have

$$\frac{\partial}{\partial \overline{z}} E_n(\overline{z}) = \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} E_{n-j} \left(\frac{\partial}{\partial x}(x-iy)^j + i\frac{\partial}{\partial y}(x-iy)^j \right).$$

After some elementary calculations for the above equation, we obtain

$$\frac{\partial}{\partial \overline{z}} E_n(\overline{z}) = \sum_{j=0}^{n} \binom{n}{j} j(\overline{z})^{j-1} E_{n-j}.$$

Thus, we have the following result:

Corollary 2.18 *Let* $n \in \mathbb{N}$*. Then we have*

$$\frac{\partial}{\partial \overline{z}} E_n(\overline{z}) = n E_{n-1}(\overline{z}).$$

3. Another kind of identities for Bernoulli and Euler numbers and polynomials

In this section, by using the Legendre identity and some trigonometric functions including the Dirichlet kernel, and also using the Umbral calculus methods, we give some new formulas, identities and relations including the Bernoulli and Euler numbers and Bernoulli and Euler type polynomials.

3.1. Identities for Bernoulli numbers, cosine- and sine-Bernoulli polynomials and cosine- and sine-Euler polynomials

With the help of the following Legendre identity and trigonometric functions associated with the Dirichlet kernel, some new formulas and relations will be proved. We have first

$$\sin\left(t\left(n+\frac{1}{2}\right)\right) = \sin\left(\frac{t}{2}\right) + 2\sin\left(\frac{t}{2}\right)\sum_{k=1}^{n}\cos(kt), \tag{3.1}$$

as the Dirichlet kernel and

$$\cos\left(t\left(n+\frac{1}{2}\right)\right) = \cos\left(\frac{t}{2}\right) - 2\sin\left(\frac{t}{2}\right)\sum_{k=1}^{n}\sin(kt), \tag{3.2}$$

(cf. [9, p. 129], [28, p. 342]).

Theorem 3.1 *Let $n, v \in \mathbb{N}$. Then we have*

$$B_v^{(C)}\left(x, \frac{1}{2}+n\right) = \frac{2}{v+1} B_{v+1}^{(S)}\left(x, \frac{1}{2}\right)$$
$$+\sum_{j=0}^{\left[\frac{v-1}{2}\right]} (-1)^{j+1} \binom{v}{2j+1} \frac{1}{j+1} B_{2j+2} B_{v-1-2j}^{(S)}\left(x, \frac{1}{2}\right)$$
$$-2\sum_{k=1}^{n} \sum_{j=0}^{\left[\frac{v-1}{2}\right]} (-1)^{j} \binom{v}{2j+1} 2^{-1-2j} B_{v-1-2j}^{(S)}(x,k).$$

Proof Let us modify (3.2) as follows:

$$\cos\left(t\left(n+\frac{1}{2}\right)\right) = \sin\left(\frac{t}{2}\right)\cot\left(\frac{t}{2}\right) - 2\sin\left(\frac{t}{2}\right)\sum_{k=1}^{n}\sin(kt). \tag{3.3}$$

After multiplying the function $\frac{te^{xt}}{e^t-1}$ on both sides of Equation (3.3), combining the resulting equation with Equations (1.21)–(1.22) and the following well-known equation

$$\cot(t) - \frac{1}{t} = \sum_{l=1}^{\infty} (-1)^l \frac{4^l B_{2l}}{(2l)!} t^{2l-1},$$

where $|t| < \pi$ (cf. [9, p. 44], [34]), we reach the following equation:

$$\sum_{v=0}^{\infty} B_v^{(C)}\left(x, \frac{1}{2}+n\right)\frac{t^v}{v!}$$
$$= \frac{2}{t}\sum_{v=0}^{\infty} B_v^{(S)}\left(x, \frac{1}{2}\right)\frac{t^v}{v!} + \sum_{v=0}^{\infty} B_v^{(S)}\left(x, \frac{1}{2}\right)\frac{t^v}{v!}\sum_{v=0}^{\infty}(-1)^{v+1}\frac{4^{v+1}B_{2v+2}}{(2v+2)!}\left(\frac{t}{2}\right)^{2v+1}$$
$$-2\sum_{k=1}^{n}\sum_{v=0}^{\infty} B_v^{(S)}(x,k)\frac{t^v}{v!}\sum_{v=0}^{\infty}\frac{(-1)^v}{(2v+1)!}\left(\frac{t}{2}\right)^{2v+1}.$$

Therefore

$$\sum_{v=0}^{\infty} B_v^{(C)}\left(x, \frac{1}{2}+n\right)\frac{t^v}{v!} = \sum_{v=0}^{\infty}\frac{2}{v+1}B_{v+1}^{(S)}\left(x, \frac{1}{2}\right)\frac{t^v}{v!}$$
$$+\sum_{v=0}^{\infty}\sum_{j=0}^{\left[\frac{v-1}{2}\right]}(-1)^{j+1}\binom{v}{2j+1}\frac{4^{j+1}}{2^{2j+1}(2j+2)}B_{2j+2}B_{v-1-2j}^{(S)}\left(x, \frac{1}{2}\right)\frac{t^v}{v!}$$
$$-2\sum_{k=1}^{n}\sum_{v=0}^{\infty}\sum_{j=0}^{\left[\frac{v-1}{2}\right]}(-1)^j\binom{v}{2j+1}2^{-1-2j}B_{v-1-2j}^{(S)}(x,k)\frac{t^v}{v!}.$$

Comparing the coefficients of $\frac{t^v}{v!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 3.2 *Let $n \in \mathbb{N}$ and $v \in \mathbb{N}_0$. Then we have*

$$B_v^{(S)}\left(x, \frac{1}{2}+n\right) = B_v^{(S)}\left(x, \frac{1}{2}\right) + 2\sum_{k=1}^{n}\sum_{j=0}^{\left[\frac{v}{2}\right]}(-1)^j \binom{v}{2j} k^{2j} B_{v-2j}^{(S)}\left(x, \frac{1}{2}\right).$$

Proof After multiplying the function $\frac{te^{xt}}{e^t-1}$ on both sides of Equation (3.1), combining the resulting equation with Equation (1.22), we reach the following equation:

$$\sum_{v=0}^{\infty} B_v^{(S)}\left(x, \frac{1}{2}+n\right)\frac{t^v}{v!} = \sum_{v=0}^{\infty} B_v^{(S)}\left(x, \frac{1}{2}\right)\frac{t^v}{v!} + 2\sum_{k=1}^{n}\sum_{v=0}^{\infty} B_v^{(S)}\left(x, \frac{1}{2}\right)\frac{t^v}{v!}\sum_{v=0}^{\infty}(-1)^v k^{2v}\frac{t^{2v}}{(2v)!}.$$

Therefore

$$\sum_{v=0}^{\infty} B_v^{(S)}\left(x, \frac{1}{2}+n\right)\frac{t^v}{v!} = \sum_{v=0}^{\infty} B_v^{(S)}\left(x, \frac{1}{2}\right)\frac{t^v}{v!} + 2\sum_{k=1}^{n}\sum_{v=0}^{\infty}\sum_{j=0}^{\left[\frac{v}{2}\right]}(-1)^j\binom{v}{2j}k^{2j}B_{v-2j}^{(S)}\left(x, \frac{1}{2}\right)\frac{t^v}{v!}.$$

Comparing the coefficients of $\frac{t^v}{v!}$ on both sides of the above equation, we arrive at the desired result. □

Similarly, after multiplying the function $\frac{2e^{xt}}{e^t+1}$ on both sides of Equations (3.3) and (3.1), combining the resulting equations with Equations (1.27) and (1.28), and after some necessary operations as the same form as in Theorem 3.1 and Theorem 3.2, we arrive at the following theorems:

Theorem 3.3 *Let $n \in \mathbb{N}$ and $v \in \mathbb{N}_0$. Then we have*

$$E_v^{(S)}\left(x, \frac{1}{2}+n\right) = E_v^{(S)}\left(x, \frac{1}{2}\right) + 2\sum_{k=1}^{n}\sum_{j=0}^{\left[\frac{v}{2}\right]}(-1)^j \binom{v}{2j} k^{2j} E_{v-2j}^{(S)}\left(x, \frac{1}{2}\right).$$

Theorem 3.4 *Let $n, v \in \mathbb{N}$. Then we have*

$$\begin{aligned} E_v^{(C)}\left(x, \frac{1}{2}+n\right) &= \frac{2}{v+1}E_{v+1}^{(S)}\left(x, \frac{1}{2}\right) \\ &+ \sum_{j=0}^{\left[\frac{v-1}{2}\right]}(-1)^{j+1}\binom{v}{2j+1}\frac{1}{j+1}B_{2j+2}E_{v-1-2j}^{(S)}\left(x, \frac{1}{2}\right) \\ &-2\sum_{k=1}^{n}\sum_{j=0}^{\left[\frac{v-1}{2}\right]}(-1)^j\binom{v}{2j+1}2^{-1-2j}E_{v-1-2j}^{(S)}(x,k). \end{aligned}$$

Theorem 3.5 *Let $v \in \mathbb{N}_0$. Then we have*

$$B_{v+1}^{(S)}(x,2) = 2\sum_{j=0}^{\left[\frac{v}{2}\right]}(-1)^j\binom{v+1}{2j+1}B_{v-2j}^{(C)}(x,1).$$

Proof Substituting $y = 2$ into (1.22), we get

$$\frac{2te^{xt}\cos(t)}{e^t - 1}\sin(t) = \sum_{v=0}^{\infty} B_v^{(S)}(x,2)\frac{t^v}{v!}. \tag{3.4}$$

From the above equation, we get

$$2\sum_{v=0}^{\infty}(-1)^v \frac{t^{2v+1}}{(2v+1)!}\sum_{v=0}^{\infty} B_v^{(C)}(x,1)\frac{t^v}{v!} = \sum_{v=0}^{\infty} B_v^{(S)}(x,2)\frac{t^v}{v!}.$$

Therefore

$$2\sum_{v=0}^{\infty}\sum_{j=0}^{\left[\frac{v}{2}\right]}(-1)^j \frac{B_{v-2j}^{(C)}(x,1)}{(2j+1)!(v-2j)!}t^v = \sum_{v=0}^{\infty} B_{v+1}^{(S)}(x,2)\frac{t^v}{(v+1)!}.$$

Comparing the coefficients of t^v on both sides of the above equation, we arrive at the desired result. □

Theorem 3.6 *Let $v \in \mathbb{N}_0$. Then we have*

$$B_v^{(S)}(x,2) = 2\sum_{j=0}^{\left[\frac{v}{2}\right]}(-1)^j \binom{v}{2j} B_{v-2j}^{(S)}(x,1).$$

Proof Using (3.4), we also get

$$2\sum_{v=0}^{\infty}(-1)^v \frac{t^{2v}}{(2v)!}\sum_{v=0}^{\infty} B_v^{(S)}(x,1)\frac{t^v}{v!} = \sum_{v=0}^{\infty} B_v^{(S)}(x,2)\frac{t^v}{v!}.$$

Therefore

$$2\sum_{v=0}^{\infty}\sum_{j=0}^{\left[\frac{v}{2}\right]}\frac{(-1)^j}{(2j)!}\frac{B_{v-2j}^{(S)}(x,1)}{(v-2j)!}t^v = \sum_{v=0}^{\infty} B_v^{(S)}(x,2)\frac{t^v}{v!}.$$

Comparing the coefficients of t^v on both sides of the above equation, we arrive at the desired result. □

3.2. Identities for Bernoulli and Euler polynomials

Here, we give some identities and formulas for the Euler and Bernoulli numbers and polynomials.

Theorem 3.7 *Let $n \in \mathbb{N}_0$. Then we have*

$$\left(\sqrt{2}e^{\frac{\pi i}{4}}x + B\right)^n = \sum_{j=0}^{n}\sum_{k=0}^{\left[\frac{j}{2}\right]}(-1)^k \binom{n}{j}\binom{j}{2k} x^j B_{n-j} - i\sum_{j=0}^{n}\sum_{k=1}^{\left[\frac{j+1}{2}\right]}(-1)^k \binom{n}{j}\binom{j}{2k-1} x^j B_{n-j},$$

where we use the usual Umbral calculus convention of symbolically replacing $(B)^n$ by B_n.

Proof Combining (1.25) with (1.2), we have

$$B_n(x+iy) = \sum_{j=0}^{n} \binom{n}{j} (x+iy)^j B_{n-j}. \tag{3.5}$$

By substituting $x = y$ into (3.5), we get

$$B_n(x+ix) = \sum_{j=0}^{n} \binom{n}{j} x^j (1+i)^j B_{n-j}. \tag{3.6}$$

Therefore, by combining the above equation with the well-known identity * Eq. (2.24):

$$(1+i)^n = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n}{2k} - i \sum_{k=1}^{\left[\frac{n+1}{2}\right]} (-1)^k \binom{n}{2k-1} \tag{3.7}$$

and

$$1+i = \sqrt{2} e^{\frac{\pi i}{4}},$$

we eventually arrive at the desired result. □

Theorem 3.8 *Let $n \in \mathbb{N}_0$. Then we have*

$$\left(\sqrt{2} e^{\frac{7\pi i}{4}} x + B\right)^n = \sum_{j=0}^{n} \sum_{k=0}^{\left[\frac{j}{2}\right]} (-1)^k \binom{n}{j} \binom{j}{2k} x^j B_{n-j} + i \sum_{j=0}^{n} \sum_{k=1}^{\left[\frac{j+1}{2}\right]} (-1)^k \binom{n}{j} \binom{j}{2k-1} x^j B_{n-j},$$

where we use the usual considered as convention of symbolically replacing $(B)^n$ by B_n.

The proof can be the same method as we used for Theorem 3.8. However, let us briefly summarize it. Combining (1.26) with (1.2), we have

$$B_n(x-iy) = \sum_{j=0}^{n} \binom{n}{j} (x-iy)^j B_{n-j}. \tag{3.8}$$

By substituting $x = y$ into (3.8), we get

$$B_n(x-ix) = \sum_{j=0}^{n} \binom{n}{j} x^j (1-i)^j B_{n-j}. \tag{3.9}$$

Therefore, by combining the above equation with the well-known identity * Eq. (2.25):

$$(1-i)^n = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n}{2k} + i \sum_{k=1}^{\left[\frac{n+1}{2}\right]} (-1)^k \binom{n}{2k-1} \tag{3.10}$$

*Gould HW (2011). Table for Fundamentals of Series: Part I: Basic properties of series and products, Vol. 1 [online]. Website https://math.wvu.edu/~hgould/Vol.1.PDF [accessed 00 Month Year].

and

$$1-i=\sqrt{2}e^{\frac{7\pi i}{4}},$$

we eventually arrive at the desired result.

Combining (1.31) and (1.32) with (1.4), we also have

$$E_n\left(x+iy\right)=\sum_{j=0}^{n}\binom{n}{j}(x+iy)^j E_{n-j} \tag{3.11}$$

and

$$E_n\left(x-iy\right)=\sum_{j=0}^{n}\binom{n}{j}(x-iy)^j E_{n-j}. \tag{3.12}$$

By replacing $x=y$ in (3.11) and (3.12) and combining the final equations with (3.7) and (3.10), we arrive at the following theorems:

Theorem 3.9 *Let $n\in\mathbb{N}_0$. Then we have*

$$\left(\sqrt{2}e^{\frac{\pi i}{4}}x+E\right)^n=\sum_{j=0}^{n}\sum_{k=0}^{\left[\frac{j}{2}\right]}(-1)^k\binom{n}{j}\binom{j}{2k}x^jE_{n-j}-i\sum_{j=0}^{n}\sum_{k=1}^{\left[\frac{j+1}{2}\right]}(-1)^k\binom{n}{j}\binom{j}{2k-1}x^jE_{n-j},$$

where we use the usual convention of symbolically replacing $(E)^n$ by E_n.

Theorem 3.10 *Let $n\in\mathbb{N}_0$. Then we have*

$$\left(\sqrt{2}e^{\frac{7\pi i}{4}}x+E\right)^n=\sum_{j=0}^{n}\sum_{k=0}^{\left[\frac{j}{2}\right]}(-1)^k\binom{n}{j}\binom{j}{2k}x^jE_{n-j}+i\sum_{j=0}^{n}\sum_{k=1}^{\left[\frac{j+1}{2}\right]}(-1)^k\binom{n}{j}\binom{j}{2k-1}x^jE_{n-j}.$$

4. Relations among Fibonacci numbers, Lucas numbers and some special polynomials

In this section, we present some novel identities and relations including Fibonacci numbers, Lucas numbers, the cosine and sine Bernoulli polynomials and the cosine and sine Euler polynomials.

By combining (1.17) with (1.10), we obtain the following relationship between the Lucas numbers and polynomials $C_n\left(x,y\right)$:

$$L_n=\frac{2}{i^n}C_n\left(\frac{i}{2},\frac{\sqrt{5}}{2}\right). \tag{4.1}$$

By combining (1.18) with (1.9), we also obtain the following relationship between the Fibonacci numbers and polynomials $S_n\left(x,y\right)$:

$$F_n=\frac{2\sqrt{5}}{5i^{n-1}}S_n\left(\frac{i}{2},\frac{\sqrt{5}}{2}\right). \tag{4.2}$$

Therefore, by using (4.1) and (4.2), we obtain

$$C_n\left(\frac{i}{2},\frac{\sqrt{5}}{2}\right)+S_n\left(\frac{i}{2},\frac{\sqrt{5}}{2}\right)=\frac{i^n}{2}\left(L_n-i\sqrt{5}F_n\right).$$

Combining (4.1) with (1.23), we can also get the following theorems:

Theorem 4.1 *Let* $n \in \mathbb{N}_0$. *Then*

$$L_n = \frac{2B_{n+1}^{(C)}\left(\frac{i+2}{2}, \frac{\sqrt{5}}{2}\right) - 2B_{n+1}^{(C)}\left(\frac{i}{2}, \frac{\sqrt{5}}{2}\right)}{i^n (n+1)}.$$

Combining (4.2) with (1.24), we have

Theorem 4.2 *Let* $n \in \mathbb{N}_0$. *Then*

$$F_n = \frac{2B_{n+1}^{(S)}\left(\frac{i+2}{2}, \frac{\sqrt{5}}{2}\right) - 2B_{n+1}^{(S)}\left(\frac{i}{2}, \frac{\sqrt{5}}{2}\right)}{\sqrt{5}(n+1) i^{n-1}}.$$

Combining (4.1) with (1.29), we have

Theorem 4.3 *Let* $n \in \mathbb{N}_0$. *Then*

$$L_n = i^{-n}\left(E_n^{(C)}\left(\frac{i+2}{2}, \frac{\sqrt{5}}{2}\right) + E_n^{(C)}\left(\frac{i}{2}, \frac{\sqrt{5}}{2}\right)\right).$$

Combining (4.2) with (1.30), we have

Theorem 4.4 *Let* $n \in \mathbb{N}_0$. *Then*

$$F_n = \frac{E_n^{(S)}\left(\frac{i+2}{2}, \frac{\sqrt{5}}{2}\right) + E_n^{(S)}\left(\frac{i}{2}, \frac{\sqrt{5}}{2}\right)}{i^{n-1}\sqrt{5}}.$$

5. Infinite series representations for some special polynomials

In this section, by using series representations for the Fibonacci-type polynomials, we present some interesting infinite series representations including the cosine-Euler polynomials and the cosine- and sine-Bernoulli polynomials. Some numerical examples for each of these series representations are also given.

For $q > 1$, Ozdemir and Simsek [26] gave the following infinite series representation for the polynomials $\mathcal{G}_j(x, y; k, m, n)$:

$$\sum_{j=0}^{\infty} \frac{W_j(x, y; k, m, n)}{q^j} = \frac{q^m}{q^{m+n} - x^k q^{n+m-1} - y^m}, \tag{5.1}$$

where

$$W_j(x, y; k, m, n) = \mathcal{G}_{j-n}(x, y; k, m, n),$$

where $j \geq n$. In light of the previous equation, we modify Equation (5.1) as follows:

$$\sum_{j=0}^{\infty} \frac{\mathcal{G}_j(x, y; k, m, n)}{q^j} = \frac{q^{n+m}}{q^{n+m} - x^k q^{n+m-1} - y^m}, \tag{5.2}$$

where $q>1$ (*cf.* [14, 26]).

If we substitute $x=2\alpha$, $y=-1$, $k=m=n=1$ into (5.2), then we get

$$\sum_{j=0}^{\infty}\frac{\mathcal{G}_j\left(2\alpha,-1;1,1,1\right)}{q^j}=\frac{q^2}{q^2-2\alpha q+1}. \tag{5.3}$$

By using (1.19), (1.23) and (5.3) and after some elementary calculations, we obtain:

$$\begin{aligned}&\sum_{j=1}^{\infty}\frac{\mathcal{G}_j\left(2\alpha,-1;1,1,1\right)-\alpha\mathcal{G}_{j-1}\left(2\alpha,-1;1,1,1\right)}{q^j}\\ =&\sum_{j=1}^{\infty}\frac{B_{j+1}^{(C)}\left(\alpha+1,\sqrt{1-\alpha^2}\right)-B_{j+1}^{(C)}\left(\alpha,\sqrt{1-\alpha^2}\right)}{(j+1)\,q^j}.\end{aligned}$$

From the above equation, the following theorem is concluded.

Theorem 5.1 *Let* $|\alpha|<1$ *and* $q>1$*. If* $q|\alpha|\neq 1$*, then*

$$\sum_{j=1}^{\infty}\frac{B_{j+1}^{(C)}\left(\alpha+1,\sqrt{1-\alpha^2}\right)-B_{j+1}^{(C)}\left(\alpha,\sqrt{1-\alpha^2}\right)}{(j+1)\,q^j}=\frac{\alpha q-1}{q^2-2\alpha q+1}. \tag{5.4}$$

Let us consider a particular example for the series representation (5.4). Substituting $q=5$ and $\alpha=\frac{1}{10}$ into (5.4) gives

$$\sum_{j=1}^{\infty}\frac{B_{j+1}^{(C)}\left(\frac{11}{10},\frac{3\sqrt{11}}{10}\right)-B_{j+1}^{(C)}\left(\frac{1}{10},\frac{3\sqrt{11}}{10}\right)}{(j+1)\,5^{j-2}}=\frac{-1}{2}.$$

By using (1.20), (1.24) and (5.3), the following infinite series is concluded

$$\sum_{j=0}^{\infty}\frac{\mathcal{G}_j\left(2\alpha,-1;1,1,1\right)}{q^j}=\sum_{j=0}^{\infty}\frac{B_{j+2}^{(S)}\left(\alpha+1,\sqrt{1-\alpha^2}\right)-B_{j+2}^{(S)}\left(\alpha,\sqrt{1-\alpha^2}\right)}{(j+2)\,q^j\sqrt{1-\alpha^2}}.$$

Therefore:

Theorem 5.2 *Let* $|\alpha|<1$ *and* $q>1$*. If* $q|\alpha|\neq 1$*, then*

$$\sum_{j=0}^{\infty}\frac{B_{j+2}^{(S)}\left(\alpha+1,\sqrt{1-\alpha^2}\right)-B_{j+2}^{(S)}\left(\alpha,\sqrt{1-\alpha^2}\right)}{(j+2)\,q^j}=\frac{q^2\sqrt{1-\alpha^2}}{q^2-2\alpha q+1}. \tag{5.5}$$

For example, substituting $q=5$ and $\alpha=\frac{1}{10}$ into (5.5) gives

$$\sum_{j=0}^{\infty}\frac{B_{j+2}^{(S)}\left(\frac{11}{10},\frac{3\sqrt{11}}{10}\right)-B_{j+2}^{(S)}\left(\frac{1}{10},\frac{3\sqrt{11}}{10}\right)}{(j+2)\,5^{j-1}}=\frac{3\sqrt{11}}{2}.$$

By using (1.19), (1.29) and (5.3) and after some elementary calculations, we can obtain another infinite series:

$$\sum_{j=1}^{\infty} \frac{\mathcal{G}_j\left(2\alpha,-1;1,1,1\right)-\alpha\mathcal{G}_{j-1}\left(2\alpha,-1;1,1,1\right)}{q^j}$$

$$= \sum_{j=1}^{\infty} \frac{E_j^{(C)}\left(\alpha+1,\sqrt{1-\alpha^2}\right)+E_j^{(C)}\left(\alpha,\sqrt{1-\alpha^2}\right)}{2q^j}.$$

Therefore:

Theorem 5.3 *Let* $|\alpha|<1$ *and* $q>1$. *If* $q\left|\alpha\right|\neq 1$, *then*

$$\sum_{j=1}^{\infty} \frac{E_j^{(C)}\left(\alpha+1,\sqrt{1-\alpha^2}\right)+E_j^{(C)}\left(\alpha,\sqrt{1-\alpha^2}\right)}{q^j}=\frac{2\alpha q-2}{q^2-2\alpha q+1}. \tag{5.6}$$

For example, substituting $q=5$ and $\alpha=\frac{1}{10}$ into (5.6) gives

$$\sum_{j=1}^{\infty} \frac{E_j^{(C)}\left(\frac{11}{10},\frac{3\sqrt{11}}{10}\right)+E_j^{(C)}\left(\frac{1}{10},\frac{3\sqrt{11}}{10}\right)}{5^{j-2}}=-1.$$

6. Remarks on inequalities for special polynomials and numbers

Polynomials are frequently used in various branches of mathematics and many applied sciences due to their straightfoward usage. One of most commonly used polynomials is known as polynomial inequalities. The polynomial inequalities plays significant role in mathematical inequalities especially in theory of approximation and many applied sciences since the time of Cauchy, Chebysev, Gauss, Hardy and other researchers.

In this section, we briefly study some well-known inequalities of some particular classes of polynomials. In fact, their applications in approximation theory is not discussed in this paper, but it can be investigated in future studies.

Lehmer [19] gave the following inequality for the Bernoulli numbers with help the Fourier expansion of the Bernoulli polynomials:

$$\left|B_{2k}\right|<\frac{(2k)!2^{1-2k}\pi^{-2k}}{1-2^{-2k+1}},$$

where $k\in\mathbb{N}$.

In [21], Masjed-Jamei et al. gave the following inequality for the cosine-Euler polynomials $E_v^{(C)}\left(x,y\right)$:

$$\sup_{x\in[0,1]}\left|E_{2v+1}^{(C)}\left(x,y\right)\right|\leq\frac{2v+1}{2}\max\left\{\left|E_{2v}^{(C)}\left(0,y\right)\right|,\left|E_{2v}^{(C)}\left(\frac{1}{2},y\right)\right|\right\},$$

where $v\in\mathbb{N}$ and $y\in\mathbb{R}$. They also gave many inequalities for the cosine-Bernoulli polynomials $B_v^{(C)}\left(x,y\right)$, the sine-Bernoulli polynomials $B_v^{(S)}\left(x,y\right)$, the cosine-Euler polynomials $E_v^{(C)}\left(x,y\right)$ and the sine-Euler polynomials $E_v^{(S)}\left(x,y\right)$ (*cf.* [20–24]).

In [8], by using inequalities including binomial coefficients, Gun and Simsek have studied the upper bound and the lower bound of many special polynomials and numbers.

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