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On certain subclasses of starlike and convex functions associated with Pascal distribution series

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Abstract: In this article, we introduced a new power series whose coefficients are probabilities of the Pascal distribution. We investigated new approaches between the Pascal distribution series and some subclasses of normalized analytic functions. Also, we defined some mappings containing these functions the Alexander type integral operator. Moreover, we obtained sufficient conditions such that these mappings belong to some subclass of univalent functions.

Key words: Univalent functions, Pascal distributions, coefficient bounds and coefficient estimates

1. Introduction

Let $\mathcal{A}$ stand for the standard class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}, \tag{1.1}$$

and let $\mathcal{S}$ the class of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ (see [4]).

It is well-known that a function $f \in \mathcal{A}$ starlike of order $\alpha (0 \leq \alpha < 1)$ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}^*(\alpha)$ the class of all functions which are starlike of order α. Furthermore, a function $f \in \mathcal{S}$ is convex of order $\alpha\,(0 \leq \alpha < 1)$ if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad (z \in \mathbb{U}).$$

We denote the class of convex functions of order α by $\mathcal{C}(\alpha)$. We note that $\mathcal{C}(0) = \mathcal{C}$, and $\mathcal{S}^*(0) = \mathcal{S}^*$, where $\mathcal{C}$ and $\mathcal{S}^*$ denote the classes of convex and starlike functions, respectively. Furthermore, for all $0 \leq \alpha < 1$, we have $\mathcal{C}(\alpha) \subset \mathcal{S}^*(\alpha)$.

For $0 \leq \alpha < 1$ and $0 \leq \beta < 1$, Thulasiram et al. [10] introduced the class $\mathcal{G}(\beta, \alpha)$ the subclass of functions $f \in \mathcal{A}$ which satisfy the condition:

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$$Re\left(\frac{zf'(z)+\beta z^2 f''(z)}{f(z)}\right)>\alpha, \qquad (z\in U).$$

Furthermore, let $\mathcal{K}(\beta,\alpha)$ the subclass of functions $f\in\mathcal{A}$ which satisfy the condition:

$$Re\left(\frac{(zf'(z)+\beta z^2 f''(z))'}{f'(z)}\right)>\alpha, \qquad (z\in U).$$

It is easy to verify that $f\in\mathcal{K}(\beta,\alpha)\Leftrightarrow zf'\in\mathcal{G}(\beta,\alpha)$. Clearly, for $\beta=0$ we have $\mathcal{G}(0,\alpha)=\mathcal{S}^*(\alpha)$ and $\mathcal{K}(0,\alpha)=\mathcal{C}(\alpha)$.

Although some studies in the literature stated that the class $\mathcal{K}(\beta,\alpha)$ was studied in the article given by [10], only the class $\mathcal{G}(\beta,\alpha)$ has given in the article.

Recently, there have been established some power series that their coefficients were probabilities of the elementary distributions such as Poisson, Pascal, Binomial, etc. Using these series, several researchers obtained necessary and sufficient conditions for these distribution series on certain subclasses of univalent functions (see, for example [1, 2, 5–9]).

A variable x is said to have the Pascal distribution if it takes the values $0,1,2,3,...$ with the probabilities $(1-q)^r$, $\frac{qr(1-q)^r}{1!}$, $\frac{q^2r(r+1)(1-q)^r}{2!}$, $\frac{q^3r(r+1)(r+2)(1-q)^r}{3!}$,..., respectively, where q, r are called the parameters, and thus

$$P(X=k)=\binom{k+r-1}{r-1}q^k(1-q)^r, \qquad k\in\{0,1,2,...\}.$$

Very recently, El-Deeb et al. [3] introduced the following power series whose coefficients are probabilities of the Pascal distribution and stated some sufficient conditions for the Pascal distribution series and other related series to be in some subclasses of analytic functions.

In this work we will consider the following power series whose coefficients are probabilities of the Pascal distribution:

$$\begin{gathered}\mathbf{\Phi}_q^r(z)=z+\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}q^{k-1}(1-q)^r z^k \\ (z\in\mathbb{U};\ r\geq 1;\ 0\leq q\leq 1)\end{gathered} \tag{1.2}$$

Note that, by using ratio test we deduce that the radius of convergence of $\mathbf{\Phi}_q^r(z)$ is infinite. Also, we consider the linear operator

$$\mathbf{\Theta}_q^r(z):\mathcal{A}\to\mathcal{A}$$

defined by

$$\begin{aligned}\mathbf{\Theta}_q^r(z)&:=\mathbf{\Phi}_q^r(z)*f(z)\\ &=z+\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}q^{k-1}(1-q)^r a_k z^k \\ &(z\in\mathbb{U};\ r\geq 1;\ 0\leq q\leq 1).\end{aligned} \tag{1.3}$$

Thulasiram et al. [10] obtained following coefficient inequalities for the function class $\mathcal{G}(\beta,\alpha)$:

Lemma 1 [10] A function $f \in \mathcal{A}$ belongs to the class $\mathcal{G}(\beta,\alpha)$ if

$$\sum_{k=2}^{\infty} (k + \beta k (k-1) - \alpha) |a_k| \leq 1 - \alpha.$$

Now, let us give the following coefficient inequalities for the function class $\mathcal{K}(\beta,\alpha)$:

Lemma 2 A function $f \in \mathcal{A}$ belongs to the class $\mathcal{K}(\beta,\alpha)$ if

$$\sum_{k=2}^{\infty} k (k + \beta k (k-1) - \alpha) |a_k| \leq 1 - \alpha.$$

Proof From $f \in \mathcal{K}(\beta,\alpha) \Leftrightarrow zf' \in \mathcal{G}(\beta,\alpha)$, replacing a_k by ka_k in Lemma 1, we obtain the desired result. □

In the present paper, we established sufficient conditions for the Pascal distribution series and other related series to be in $\mathcal{G}(\beta,\alpha)$ and $\mathcal{K}(\beta,\alpha)$. Also, we studied similar properties for an integral transform related to this series.

2. Main results

Theorem 2.1 *$\mathbf{\Phi}_q^r(z)$ given by (1.2) is in the class $\mathcal{G}(\beta,\alpha)$ if*

$$\frac{q^2 r(r+1)\beta}{(1-q)^2} + \frac{qr(1+2\beta)}{1-q} \leq (1-\alpha)(1-q)^r. \tag{2.1}$$

Proof To prove that $\mathbf{\Phi}_q^r \in \mathcal{G}(\beta,\alpha)$, according to Lemma 1, it is sufficient to show that

$$\sum_{k=2}^{\infty} [k + \beta k(k-1) - \alpha] \binom{k+r-2}{r-1} |q|^{k-1} |1-q|^r \leq 1 - \alpha. \tag{2.2}$$

We will use the following very known relation

$$\sum_{k=0}^{\infty} \binom{k+r-1}{r-1} q^k = \frac{1}{(1-q)^r}, \ 0 \leq q \leq 1.$$

and the corresponding ones obtained by replacing the value of r with $r-1$, $r+1$ and $r+2$ in our proofs.

By making calculations on the left hand side of the inequality (2.2) we obtain,

$$\sum_{k=2}^{\infty} [k + \beta k(k-1) - \alpha] \binom{k+r-2}{r-1} q^{k-1} (1-q)^r$$

$$= (1-q)^r \left[\sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1} \beta (k-1)(k-2) \right.$$

$$\left. + \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1} (k-1)(1+2\beta) + \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1} (1-\alpha) \right]$$

$$= (1-q)^r \left[q^2 \sum_{k=3}^{\infty} \binom{k+r-2}{r+1} q^{k-3} \beta r(r+1) + q \sum_{k=2}^{\infty} \binom{k+r-2}{r} q^{k-2} r(1+2\beta) + \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1}(1-\alpha) \right]$$

$$= (1-q)^r \left[q^2 \sum_{k=0}^{\infty} \binom{k+r+1}{r+1} q^{k} \beta r(r+1) + q \sum_{k=0}^{\infty} \binom{k+r}{r} q^{k} r(1+2\beta) + \sum_{k=0}^{\infty} \binom{k+r-1}{r-1} q^{k}(1-\alpha) - (1-\alpha) \right]$$

$$= \frac{q^2 r(r+1)\beta}{(1-q)^2} + \frac{qr(1+2\beta)}{1-q} + (1-\alpha)\left[1-(1-q)^r\right].$$

Therefore if the inequality (2.2) holds, we obtain

$$\frac{q^2 r(r+1)\beta}{(1-q)^2} + \frac{qr(1+2\beta)}{1-q} + (1-\alpha)\left[1-(1-q)^r\right] \leq (1-\alpha),$$

which is equivalent to (2.1). This completes the proof. □

Taking $\beta = 0$, we obtain the following corollary for the class of starlike functions of order α:

Corollary 1. $\mathbf{\Phi}_q^r(z)$ given by (1.2) is in the class $\mathcal{S}^*(\alpha)$ if

$$\frac{qr}{(1-q)^{r+1}} \leq (1-\alpha).$$

Theorem 2.2 *$\mathbf{\Phi}_q^r(z)$ given by (1.2) is in the class $\mathcal{K}(\beta,\alpha)$ if*

$$\frac{q^3 \beta r(r+1)(r+2)}{(1-q)^3} + \frac{q^2 r(r+1)(5\beta+1)}{(1-q)^2} + \frac{qr(4\beta+3-\alpha)}{1-q} \leq (1-\alpha)(1-q)^r. \tag{2.3}$$

Proof To prove that $\mathbf{\Phi}_q^r \in \mathcal{K}(\beta,\alpha)$, according to Lemma 2, it is sufficient to show that

$$\sum_{k=2}^{\infty} \binom{k+r-2}{r-1} |q|^{k-1} |1-q|^r k\left[k+\beta k(k-1)-\alpha\right] \leq 1-\alpha. \tag{2.4}$$

Now, using the same calculations as in the proof of Theorem 2.1, we obtain

$$\sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1} (1-q)^r k\left[k+\beta k(k-1)-\alpha\right]$$

$$= (1-q)^r \left[\sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1} \beta(k-1)(k-2)(k-3) \right.$$
$$+ \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1} (5\beta+1)(k-1)(k-2)$$
$$\left. + \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1} (k-1)(4\beta+3-\alpha) + \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1} (1-\alpha) \right]$$

$$= (1-q)^r \left[q^3 \sum_{k=4}^{\infty} \binom{k+r-2}{r+2} q^{k-4} \beta r(r+1)(r+2) \right.$$
$$+ q^2 \sum_{k=3}^{\infty} \binom{k+r-2}{r+1} q^{k-3} (5\beta+1) r(r+1)$$
$$\left. + q \sum_{k=2}^{\infty} \binom{k+r-2}{r} q^{k-2} (4\beta+3-\alpha) r + (1-\alpha) \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1} \right]$$

$$= (1-q)^r \left[q^3 \sum_{k=0}^{\infty} \binom{k+r+2}{r+2} q^{k} \beta r(r+1)(r+2) \right.$$
$$+ q^2 \sum_{k=0}^{\infty} \binom{k+r+1}{r+1} q^{k} (5\beta+1) r(r+1)$$
$$\left. + q \sum_{k=0}^{\infty} \binom{k+r}{r} q^{k} (4\beta+3-\alpha) r + (1-\alpha) \sum_{k=1}^{\infty} \binom{k+r-1}{r-1} q^{k} \right]$$

$$= \frac{q^3 \beta r(r+1)(r+2)}{(1-q)^3} + \frac{q^2 r(r+1)(5\beta+1)}{(1-q)^2} + \frac{qr(4\beta+3-\alpha)}{1-q} + (1-\alpha)\left[1-(1-q)^r\right].$$

The last expression is bounded above by $(1-\alpha)$ if the inequality (2.3) is satisfied. This completes the proof. □

For $\beta = 0$, we obtain the following corollary for the class of convex functions of order α:

Corollary 2. $\mathbf{\Phi}_q^r(z)$ given by (1.2) is in the class $\mathcal{C}(\alpha)$ if

$$\frac{q^2 r(r+1)}{(1-q)^{r+2}} + \frac{qr(3-\alpha)}{(1-q)^{r+1}} \leq (1-\alpha).$$

Theorem 2.3 *(i) If the condition*

$$\frac{q^3 \beta r(r+1)(r+2)}{(1-q)^3} + \frac{q^2 r(r+1)(5\beta+1)}{(1-q)^2} + \frac{qr(4\beta+3-\alpha)}{1-q} \leq (1-\alpha)(1-q)^r. \tag{2.5}$$

holds, then the operator $\mathbf{\Theta}_q^r(z)$ *defined by (1.3) maps the class* $\mathcal{S}^*$ *to the class* $\mathcal{G}(\beta,\alpha)$, *that is* $\mathbf{\Theta}_q^r(\mathcal{S}^*) \subset \mathcal{G}(\beta,\alpha)$

(ii) If the condition (2.1) is satisfied, then the operator $\mathbf{\Theta}_q^r$ *maps the class* $\mathcal{C}$ *to the class* $\mathcal{G}(\beta,\alpha)$, *that is* $\mathbf{\Theta}_q^r(\mathcal{C}) \subset \mathcal{G}(\beta,\alpha)$

Proof According to Lemma 1, to prove that $\mathbf{\Theta}_q^r(z) \in \mathcal{G}(\beta,\alpha)$ it is sufficient to show that

$$\sum_{k=2}^{\infty}[k+\beta k(k-1)-\alpha]\binom{k+r-2}{r-1}|q|^{k-1}|1-q|^r|a_k| \leq (1-\alpha). \tag{2.6}$$

(i) If f has the form (1.1) is in the class $\mathcal{S}^*$, then $|a_k| \leq k$ holds for all $k \geq 2$ (see [4], p. 44). Now using (2.5), we obtain

$$\sum_{k=2}^{\infty}[k+\beta k(k-1)-\alpha]\binom{k+r-2}{r-1}|q|^{k-1}|1-q|^r|a_k|$$

$$\leq \sum_{k=2}^{\infty}k\,[k+\beta k(k-1)-\alpha]\binom{k+r-2}{r-1}q^{k-1}(1-q)^r$$

Then, in order to avoid repetition, we can skip a few steps of the calculations and write directly the following inequality:

$$\frac{q^3\beta r(r+1)(r+2)}{(1-q)^3}+\frac{q^2r(r+1)(5\beta+1)}{(1-q)^2}+\frac{qr(4\beta+3-\alpha)}{1-q}+(1-\alpha)\left[1-(1-q)^r\right] \leq (1-\alpha).$$

That is (2.6) holds, hence $\mathbf{\Theta}_q^r(f) \in \mathcal{G}(\beta,\alpha)$.

(ii) If f has the form (1.1) is in the class $\mathcal{C}$, then $|a_k| \leq 1$ holds for all $k \geq 2$ (see [4], pp.45). Now, using (2.1), we obtain

$$\sum_{k=2}^{\infty}[k+\beta k(k-1)-\alpha]\binom{k+r-2}{r-1}|q|^{k-1}|1-q|^r|a_k|$$

$$\leq \sum_{k=2}^{\infty}[k+\beta k(k-1)-\alpha]\binom{k+r-2}{r-1}q^{k-1}(1-q)^r$$

If we use similar techniques as in the proof of Theorem 2.1, we obtain the following inequality:

$$\frac{q^2r(r+1)\beta}{(1-q)^2}+\frac{qr(1+2\beta)}{1-q}+(1-\alpha)\left[1-(1-q)^r\right] \leq (1-\alpha).$$

Hence (2.1) holds, and therefore $\mathbf{\Theta}_q^r(f) \in \mathcal{G}(\beta,\alpha)$.

□

3. Integral operators

In this section, we give conditions for the integral operator defined as follows:

$$H_q^r(z) = \int_0^z \frac{\mathbf{\Phi}_q^r(t)}{t} dt \tag{3.1}$$

where $\mathbf{\Phi}_q^r(t)$ is given by (1.2).

Theorem 3.1 *A sufficient condition for the function H_q^r to be in the class $\mathcal{G}(\beta, \alpha)$ is*

$$1 + \frac{\beta q r}{(1-q)} - \frac{\alpha(1-q)}{q(r-1)} \left[1 - (1-q)^{r-1}\right] - (1-\alpha)(1-q)^r \leq 1 - \alpha. \tag{3.2}$$

Proof From (3.1), we can write

$$H_q^r(z) = \int_0^z \frac{\mathbf{\Phi}_q^r(t)}{t} dt = z + \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1}(1-q)^r \frac{z^k}{k}. \tag{3.3}$$

According to Lemma 1, it is enough to show that

$$\sum_{k=2}^{\infty} \frac{[k + \beta k(k-1) - \alpha]}{k} \binom{k+r-2}{r-1} |q|^{k-1} |1-q|^r \leq 1 - \alpha. \tag{3.4}$$

Using the assumption (3.2), a simple computation shows that

$$\sum_{k=2}^{\infty} \frac{[k + \beta k(k-1) - \alpha]}{k} \binom{k+r-2}{r-1} q^{k-1}(1-q)^r$$

$$= (1-q)^r \left[\sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1} \beta(k-1) + \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1} \right.$$
$$\left. - \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1} \frac{\alpha}{k} \right]$$

$$= (1-q)^r \left[q \sum_{k=2}^{\infty} \binom{k+r-2}{r} q^{k-2} \beta r + \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1} \right.$$
$$\left. - \frac{\alpha}{q(r-1)} \sum_{k=2}^{\infty} \binom{k+r-2}{r-2} q^k \right]$$

$$
\begin{aligned}
=&(1-q)^r\left\{\beta qr\sum_{k=0}^{\infty}\binom{k+r}{r}q^k+\left(\sum_{k=0}^{\infty}\binom{k+r-1}{r-1}q^k-1\right)-\frac{\alpha}{q(r-1)}\left[\sum_{k=0}^{\infty}\binom{k+r-2}{r-2}q^k-1-q(r-1)\right]\right\}\\
=&\frac{\beta qr}{(1-q)}+1-(1-q)^r-\frac{\alpha}{q(r-1)}\left[(1-q)-(1-q)^r-q(r-1)(1-q)^r\right]\\
=&\frac{\beta qr}{(1-q)}-\frac{\alpha(1-q)}{q(r-1)}\left[1-(1-q)^{r-1}\right]-(1-\alpha)(1-q)^r+1.
\end{aligned}
$$

The last expression is bounded above by $1-\alpha$ by the given condition. This completes the proof. □

Theorem 3.2 *A sufficient condition for the function H_q^r to be in the class $\mathcal{K}(\beta,\alpha)$ is*

$$
\frac{q^2r(r+1)\beta}{(1-q)^2}+\frac{qr(1+2\beta)}{1-q}\leq(1-\alpha)(1-q)^r. \tag{3.5}
$$

Proof The proof of this theorem is same as to proof of Theorem 2.1. Therefore we omit the details involved. □

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