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$q-$Hamiltonian systems

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Abstract: In this paper, we develop the basic theory of linear $q-$ Hamiltonian systems. In this context, we establish an existence and uniqueness result. Regular spectral problems are studied. Later, we introduce the corresponding maximal and minimal operators for this system. Finally, we give a spectral resolution.

Key words: Regular $q-$Hamiltonian system, maximal operator, minimal operator, boundary conditions, eigenfunction expansions

1. Introduction

Quantum calculus has attracted a lot of attention in the recent past. Because it has numerous applications in different mathematical areas, such as number theory, orthogonal polynomials, fractal geometry, combinatorics, the calculus of variations, mechanics, orthogonal polynomials, statistic physics, nuclear and high energy physics, conformal quantum mechanics, and theory of relativity (see [18]). New results in this area can be found in [1, 3–11, 13–15, 17–21, 25–27].

It is well-known that Hamiltonian systems can be described for the modeling and analysis of some physical systems with negligible dissipation. After the pioneering work [16] the theory of continuous Hamiltonian systems has intensively been investigated by several authors. Their results have been summarized in the book of Krall [22]. In [24], the author established the Weyl–Titchmarsh theory for a class of discrete linear Hamiltonian systems over a half line. Spectral problems of discrete linear Hamiltonian systems have been studied (cf. [2, 16, 23]). In [12], Anderson investigated nonself-adjoint Hamiltonian systems on Sturmian time scales. He unified discrete and continuous Hamiltonian theories to dynamic equations on Sturmian time scales.

In this paper, we develop the basic theory of linear $q-$Hamiltonian systems defined as

$$Jy^{[q]}(x) = [\lambda V(x) + T(x)]\, y(x), \ x \in (0,a), \ a > 0, \tag{1.1}$$

where

$$y^{[q]}(x) = \begin{pmatrix} D_q y_1(x) \\ \frac{1}{q} D_{q^{-1}} y_2(x) \end{pmatrix}, V(x) = \begin{pmatrix} V_1(x) & O_n \\ O_n & V_2(x) \end{pmatrix},$$

$$J = \begin{pmatrix} O_n & -I_n \\ I_n & O_n \end{pmatrix}, T(x) = \begin{pmatrix} T_1(x) & T_2^*(x) \\ T_2(x) & T_3(x) \end{pmatrix}$$

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2010 *AMS Mathematics Subject Classification:* 39A13, 34B27, 47B25, 34L10 and $T(x)$ are $2n \times 2n$ complex Hermitian matrix-valued functions defined on $[0,a]$ and continuous at zero.

In the analysis that follows, we will largely follow a development of the theory in [16, 22, 28].

If we take $n=1$ and $V(x)=1$ in the system (1.1), we get the $q-$analogue of the one dimensional Dirac problem. In [5]–[9], the authors studied the $q-$Dirac problem defined by

$$\begin{pmatrix} 0 & -\frac{1}{q}D_{q^{-1}} \\ D_q & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{1.2}$$

where λ is a complex parameter, $p(.)$ and $r(.)$ are real-valued functions defined on $[0,a]$ and continuous at zero.

Our paper is organized as follows: Section 2 introduces fundamental concepts and basic results of the quantum calculus. In Section 3, we present linear $q-$Hamiltonian systems. Section 4 is devoted to regular $q-$Hamiltonian systems. Finally, an eigenfunction expansion theorem for regular $q-$Hamiltonian equations is presented in Section 5.

2. Preliminaries

In this section, we recall some basic concepts and useful results about quantum calculus. We refer to [15, 18, 21] and some references cited therein. Let q be a positive number with $0<q<1$. A set $A \subset \mathbb{R}$ is called $q-$geometric if for every $x \in A$, $qx \in A$. Let y be a complex-valued function on A. Then, the *$q-$difference operator D_q is* defined by

$$D_q y(x) = [y(qx) - y(x)]\frac{1}{qx-x}, \text{ for all } x \in A.$$

The $q-$derivative at zero is defined by

$$D_q y(0) = \lim_{n\to\infty}[y(xq^n) - y(0)]\frac{1}{xq^n}, \ x \in A,$$

if the limit exists and does not depend on x. Associated with this operator there is a nonsymmetric formula for the $q-$differentiation of a product

$$D_q[f(x)g(x)] = [D_q f(x)]\, g(x) + [D_q g(x)]\, f(qx).$$

For $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, the *$q-$shifted factorial* is defined by

$$(a;q)_0 = 1, \ (a;q)_n = \prod_{k=0}^{n-1}\left(1-aq^k\right), \ (a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right)$$

(see [15]).

The *Jackson $q-$integration* is given by

$$\int_0^x f(t)\, d_q t = (1-q)\, x \sum_{n=0}^{\infty} f(q^n x)\, q^n \ (x \in A),$$

provided that the series converges, and

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t \ (a, b \in A).$$

A function f which is defined on $A,\ 0\in A,$ is said to be $q-$regular at zero if

$$\lim_{n\to\infty} f\left(xq^{n}\right)=f\left(0\right),$$

for every $x\in A$. Through the remainder of the paper, we deal only with $q-$regular functions at zero.

If f and g are $q-$regular at zero, then we have

$$\int_{0}^{a} f\left(qt\right)\left[D_{q}g\left(t\right)\right]d_{q}t+\int_{0}^{a} g\left(t\right)\left[D_{q}f\left(t\right)\right]d_{q}t=g\left(a\right)f\left(a\right)-g\left(0\right)f\left(0\right).$$

3. $q-$Hamiltonian systems

Let us consider the following linear $q-$Hamiltonian system

$$l\left(y\right):=Jy^{[q]}(x)=\left[\lambda V\left(x\right)+T\left(x\right)\right]y\left(x\right),\ x\in\left(0,a\right), \tag{3.1}$$

under the following hypotheses:

i) $V\left(x\right)=\begin{pmatrix} V_{1}\left(x\right) & O_{n}\\ O_{n} & V_{2}\left(x\right)\end{pmatrix}$ and $T\left(x\right)=\begin{pmatrix} T_{1}\left(x\right) & T_{2}^{*}\left(x\right)\\ T_{2}\left(x\right) & T_{3}\left(x\right)\end{pmatrix}$ are $2n\times 2n$ complex Hermitian matrix-valued functions defined on $[0,a]$ and continuous at zero.

ii) $I+\left(q-1\right)xT_{2}\left(x\right)$ is invertible and $V(x)$ is nonnegative definite.

iii) λ is a complex spectral parameter, $y\left(x\right)$ is $2n\times 1$ vector-valued function and

$$y^{[q]}(x)=\begin{pmatrix} D_{q}y_{1}\left(x\right)\\ \frac{1}{q}D_{q^{-1}}y_{2}\left(x\right)\end{pmatrix},$$

where $y_{1},y_{2}:(0,a)\to\mathbb{C}^{n}$.

iv)

$$J=\begin{pmatrix} O_{n} & -I_{n}\\ I_{n} & O_{n}\end{pmatrix},$$

where I_{n} is the $n\times n$ identity matrix. It is clear that $J^{*}=-J=J^{-1}$.

We denote by $\mathbf{H}=L_{q,V}^{2}((0,a);E)$ $(E:=\mathbb{C}^{2n})$ Hilbert space of $2n-$dimensional vector-valued functions $y,z,$ generated by the inner product

$$(y,z)=\int_{0}^{a} z^{*}Vyd_{q}x,$$

and norm $\|y\|=\sqrt{(y,y)}$.

Throughout this work, we assume that the following definiteness condition holds: for every nontrivial solution y of (3.1), we have

$$\int_{0}^{a} y^{*}Vyd_{q}x>0.$$

Now, we shall investigate the fundamental solutions of the linear $q-$Hamiltonian system (3.1). Let $C_{q}^{2}((0,a);E)$ be the space of all vector-valued functions y such that y and $D_{q}y$ are $q-$regular at zero. It is clear that $C_{q}^{2}((0,a);E)\subset\mathbf{H}$.

Theorem 3.1 *For $k_1, k_2 \in \mathbb{C}^n$, the linear $q-$Hamiltonian system (3.1) with initial condition*

$$y(0,\lambda)=\begin{pmatrix} y_1(0,\lambda) \\ y_2(0,\lambda) \end{pmatrix}=K=\begin{pmatrix} k_1 \\ k_2 \end{pmatrix},\ \lambda\in\mathbb{C} \tag{3.2}$$

has a unique solution in $C_q^2((0,a);E)$.

Proof If y is a solution of the system (3.1)–(3.2), then an integration yields

$$y(x,\lambda)=K-q\int_0^x J\left[\lambda V(qt,\lambda)+T(qt,\lambda)\right]y(qt,\lambda)\,d_qt, \tag{3.3}$$

where $x\in(0,a)$. Conversely, every solution of the Eq. (3.3) is also a solution of the system (3.1)–(3.2). Define

$$\widehat{y}(x)=\begin{pmatrix} y_1(x) \\ y_2\left(q^{-1}x\right) \end{pmatrix}$$

and $\widehat{y}_0(x,\lambda)=K$,

$$y_{n+1}(x,\lambda)=K-q\int_0^x J\left[\lambda V(qt,\lambda)+T(qt,\lambda)\right]y_n(qt,\lambda)\,d_qt,\ n=0,1,2,... \tag{3.4}$$

where $x\in(0,a)$.

We now prove that the sequence $\{y_n : n\in\mathbb{N}:=\{1,2,...\}\}$ converges to a function y uniformly on each compact subset of $(0,a)$. There exist positive numbers $\mu(\lambda)$ and $\nu(\lambda)$ such that

$$\left\|J\left[\lambda V(x,\lambda)+T(x,\lambda)\right]\right\|_E\le\mu(\lambda),\ \left\|y_1(x,\lambda)\right\|_E\le\nu(\lambda),\ x\in(0,a).$$

Using mathematical induction, we get

$$\left\|y_{n+1}(x,\lambda)-y_n(x,\lambda)\right\|_E\le\mu(\lambda)\,q^{\frac{n(n+1)}{2}}\frac{\left(\nu(\lambda)\,x\,(1-q)\right)^n}{(q;q)_n},\ n\in\mathbb{N}.$$

It follows from Weierstrass M-test that the sequence $\{y_n : n\in\mathbb{N}\}$ converges to a function y uniformly on each compact subset of $(0,a)$. One can prove that y and D_qy are continuous at zero. It is clear that the function y satisfies the condition (3.2). To show that the system (3.1)–(3.2) has a unique solution, assume z is another one. Then z is continuous at zero. Therefore, there exists a positive number M such that $\|y-z\|\le M$. Proceeding as above, we get

$$\left\|y(x,\lambda)-z(x,\lambda)\right\|_E\le M\mu(\lambda)\,q^{\frac{n(n+1)}{2}}\frac{\left(x\,(1-q)\right)^n}{(q;q)_n},\ n\in\mathbb{N}$$

Since

$$\lim_{n\to\infty}M\mu(\lambda)\,q^{\frac{n(n+1)}{2}}\frac{\left(x\,(1-q)\right)^n}{(q;q)_n}=0,$$

then $y=z$ on $[0,a]$. □

For any function $y \in \mathbf{H}$, $y(0)$ can be defined as

$$y(0) := \lim_{n\to\infty} y(q^n). \tag{3.5}$$

Since y is $q-$regular at zero, the limit in (3.5) exists and is finite.

Let us denote by $D_{\min}$ those elements in $\mathbf{H}$ satisfying the following conditions:

$$\begin{array}{lc} (i) & y \text{ and } D_q y \text{ are } q-\text{regular at zero.} \\ (ii) & l(y) = Jy^{[q]}(x) - T(x)\,y(x) = V(x)\,f(x) \text{ exists in } (0,a) \text{ and } f \in \mathbf{H}. \\ (iii) & \widehat{y}(0) = \widehat{y}(a) = 0, \text{ where } \widehat{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(q^{-1}x) \end{pmatrix}. \end{array} \tag{3.6}$$

Then we define the *minimal operator* $L_{\min}$ on $D_{\min}$ by the equality

$$L_{\min} y = l(y).$$

Similarly, we denote by $D_{\max}$ those elements in $\mathbf{H}$ satisfying the following conditions: y and $D_q y$ are $q-$regular at zero and

$$l(y) = Jy^{[q]}(x) - T(x)\,y(x) = V(x)\,f(x)$$

exists in $(0,a)$ and $f \in \mathbf{H}$.

We define the *maximal operator* $L_{\max}$ on $D_{\max}$ by the equality

$$L_{\max} y = l(y).$$

Let

$$y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},\; z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} \in \mathbf{H}.$$

Then, the *Wronskian* of $y(x)$ and $z(x)$ is defined by

$$W(y,z) = z_2^*\left(q^{-1}x\right) y_1(x) - z_1^*(x)\, y_2\left(q^{-1}x\right).$$

One can prove that the Wronskian of any solutions of Eq. (3.1) is independent of x. Now, we have the following theorem.

Theorem 3.2 (Green's formula) *Let y and z be in $D_{\max}$. Then we have*

$$(L_{\max} y, z) - (y, L_{\max} z) = [y,z](a) - [y,z](0), \tag{3.7}$$

where $[y,z](x) := \widehat{z}^*(x) J \widehat{y}(x)$ and $x \in (0,a)$.

Proof For $y, z \in D_{\max}$, there exist $f, g \in \mathbf{H}$ such that $L_{\max} y = f$ and $L_{\max} z = g$. Then, the left side of the formula (3.7) is

$$(L_{\max}y, z) - (y, L_{\max}z) = (f, z) - (y, g)$$

$$= \int_0^a z^*(x)\, V(x)\, f(x)\, d_q x$$

$$- \int_0^a g^*(x)\, y(x)\, V(x)\, d_q x - \int_0^a \{l(z)\}^* y(x)\, d_q x$$

$$= \int_0^a z^*(x) \left\{ J y^{[q]}(x) - [\lambda V(x) + T(x)]\, y(x) \right\} d_q x$$

$$- \int_0^a \left\{ J z^{[q]}(x) - [\lambda V(x) + T(x)]\, z(x) \right\}^* y(x)\, d_q x$$

$$= \int_0^a z^*(x)\, J y^{[q]}(x) d_q x - \int_0^a \left\{ J z^{[q]}(x) \right\}^* y(x)\, d_q x$$

$$= \int_0^a \left\{ -\frac{1}{q} D_{q^{-1}} z_1^*(x)\, y_2(x) + z_2^*(x)\, D_q y_1(x) \right\} d_q x$$

$$- \int_0^a \left[\left\{ -\frac{1}{q} D_{q^{-1}} z_2^*(x) \right\} y_1(x) + D_q z_1^*(x)\, y_2(x) \right] d_q x$$

$$= \int_0^a \left\{ z_1^*(x) [-\frac{1}{q} D_{q^{-1}} y_2(x)] - D_q z_1^*(x)\, y_2(x) \right\} d_q x$$

$$+ \int_0^a \left\{ z_2^*(x)\, D_q y_1(x) - \left\{ -\frac{1}{q} D_{q^{-1}} z_2^*(x) \right\} y_1(x) \right\}.$$

Since

$$D_q \left(z_1^*(x)\, y_2\left(q^{-1}x\right) \right) = \left(D_q \left(q^{-1}x\right) z_1^*(x)\, D_q y_2\left(q^{-1}x\right) \right) + D_q z_1^*(x)\, y_2(x)$$

$$= z_1^*(x) \frac{1}{q} \left(D_{q^{-1}} y_2(x) \right) + (D_q z_1(x))^* y_2(x)$$

and

$$D_q\left(z_2^*\left(q^{-1}x\right)y_1(x)\right)=\left(D_q z_2^*\left(q^{-1}x\right)\right)D_q\left(q^{-1}x\right)y_1(x)+z_2^*(x)\left(D_q y_1(x)\right)$$

$$=\frac{1}{q}\left(D_{q^{-1}}z_2^*(x)\right)y_1(x)+z_2^*(x)\left(D_q y_1(x)\right).$$

Hence we get

$$(L_{\max}y,z)-(y,L_{\max}z)=\int_0^a D_q\left\{-z_1^*(x)\,y_2\left(q^{-1}x\right)+z_2^*\left(q^{-1}x\right)y_1(x)\right\}d_qx$$

$$=\widehat{z}^*(a)J\widehat{y}(a)-\widehat{z}^*(0)J\widehat{y}(0)$$

$$=[y,z](a)-[y,z](0).$$

□

The following result directly follows from Theorem 3.2.

Theorem 3.3 *Let $y(x,\eta)$ and $z(x,\xi)$ be any solutions of the equation (3.1). Then, for all $\xi,\eta\in\mathbb{C}$,*

$$\left(\eta-\overline{\xi}\right)\int_0^x z^*(t,\xi)\,V(t,\eta)\,y(t,\xi)d_qt=\widehat{z}^*(x,\xi)\,J\widehat{y}(x,\eta)-\widehat{z}^*(0,\xi)\,J\widehat{y}(0,\eta),$$

holds.

Lemma 3.4 *The operator $L_{\min}$ is Hermitian.*

Proof For $y,z\in D_{\min}$, there exist $f,g\in\mathbf{H}$ such that $l(y)=Vf$ and $l(z)=Vg$. From (3.6) and Theorem 3.3, we get

$$(L_{\min}y,z)-(y,L_{\min}z)=(f,z)-(y,g)$$

$$=\int_0^a z^*(t)\,Vf(t)\,d_qt-\int_0^a g^*(t)\,Vy(t)\,d_qt$$

$$=\int_0^a\left[z^*(t)\,l(y)-(l(z))^*y(t)\right]d_qt$$

$$=\widehat{z}^*(a)\,J\widehat{y}(a)-\widehat{z}^*(0)\,J\widehat{y}(0)=0.$$

□

The following lemma has a similar proof of Lemma 3.4.

Lemma 3.5 *The relation*

$$(L_{\min}y, z) = (y, L_{\max}z)$$

holds for all $y \in D_{\min}$ *and for all* $z \in D_{\max}$.

Lemma 3.6 *Let the null space and the range of an operator* L *be denoted by* $\mathcal{N}(L)$ *and* $\mathcal{R}(L)$, *respectively. Then we have*

$$\mathcal{R}(L_{\min}) = \mathcal{N}(L_{\max})^{\perp}.$$

Proof Given any $f \in \mathcal{R}(L_{\min})$, there exists $y \in D_{\min}$ such that $L_{\min}y = f$. From Lemma 3.5, for each $z \in \mathcal{N}(L_{\max})$, we get

$$(f, z) = (L_{\min}y, z) = (y, L_{\max}z) = 0,$$

i.e. $f \in \mathcal{N}(L_{\max})^{\perp}$.

We now show that $\mathcal{N}(L_{\max})^{\perp} \subset \mathcal{R}(L_{\min})$. For any given $f \in \mathcal{N}(L_{\max})^{\perp}$, we have $(f, z) = 0$ for all $z \in \mathcal{N}(L_{\max})$. Consider the following problem:

$$Jy^{[q]}(x) - T(x)\, y(x) = V(x)\, f(x), \ x \in (0, a)$$
$$\widehat{y}(0) = 0.$$

It follows from Theorem 3.1 that the above problem has a unique solution on $(0, a)$. Let $\Psi(x) = (\psi_1, \psi_2, ..., \psi_{2n})$ be the fundamental solution of the system

$$Jy^{[q]}(x) - T(x)\, y(x) = 0, \ \widehat{\Psi}(a) = J, \ x \in (0, a).$$

It is clear that $\psi_i \in \mathcal{N}(L_{\max})$ for $1 \leq i \leq 2n$. By Theorem 3.2, for $1 \leq i \leq 2n$, we get

$$0 = (f, \psi_i) = \int_0^a \psi_i^*(t)\, V f(t)\, d_q t = \int_0^a \psi_i^*(t)\, l(y)(t)\, d_q t$$

$$= \int_0^a \psi_i^*(t)\, l(y)(t)\, d_q t - \int_0^a l(\psi_i)^*(t)\, y(t)\, d_q t$$

$$= \widehat{\psi}_i^*(a)\, J\widehat{y}(a) - \widehat{\psi}_i^*(0)\, J\widehat{y}(0) = \widehat{\psi}_i^*(a)\, J\widehat{y}(a).$$

Thus, we have $\widehat{\Psi}^*(a)\, J\widehat{y}(a) = \widehat{y}(a) = 0$, i.e. $f \in \mathcal{R}(L_{\min})$. □

Theorem 3.7 *The operator* $L_{\min}$ *is symmetric operator and the operator* $L_{\max}$ *is densely defined operator. Furthermore* $L_{\min}^* = L_{\max}$.

Proof We first show that $D_{\min}^{\perp} = \{0\}$. Suppose that $f \in D_{\min}^{\perp}$. Then, for all $z \in D_{\min}$, we have $(f, z) = 0$. Set $L_{\min}z(x) = h(x)$. Let $y(x)$ be any solution of the system

$$Jy^{[q]}(x) - T(x)\, y(x) = V(x)\, f(x), \ x \in (0, a).$$

It follows from Theorem 3.2 that

$$(y,h)-(f,z)=\int_0^a h^*(t)\,Vy(t)\,d_qt-\int_0^a z^*(t)\,Vf(t)\,d_qt$$

$$=\int_0^a l(z)^*(t)\,y(t)\,d_qt-\int_0^a z^*(t)\,l(y)(t)\,d_qt$$

$$=-\widehat{z}^*(a)\,J\widehat{y}(a)+\widehat{z}^*(0)\,J\widehat{y}(0)=0.$$

Thus we get $(y,h)=(f,z)=0$. By Lemma 3.6, we have $y\in\mathcal{R}(L_{\min})=\mathcal{N}(L_{\max})^{\perp}$ and consequently $f=0$.

Now, we prove that the domain of the operator $L^*_{\min}$, $D^*_{\min}$ is equal to $D_{\max}$, and $L^*_{\min}y=L_{\max}y$ for all $y\in D^*_{\min}$. By Lemma 3.5, for any given $y\in D_{\max}$, we have

$$(y,L_{\min}z)=(L_{\max}y,z)\,,\ \text{for all}\ z\in D_{\min}.$$

Therefore the functional $(y,L_{\min}(.))$ is continuous on $D_{\min}$ and $y\in D^*_{\min}$, i.e. $D_{\max}\subset D^*_{\min}$.

We prove the reverse conclusion. If $y\in D^*_{\min}$, then y and $h:=L^*_{\min}y$ are all in $\mathbf{H}$. Assume that u is a solution of the system

$$Ju^{[q]}(x)-T(x)\,u(x)=V(x)\,h(x)\,. \tag{3.8}$$

By Lemma 3.5, we deduce that

$$(h,z)=(L_{\max}u,z)=(u,L_{\min}z)\,.$$

Thus we have

$$(y-u,L_{\min}z)=(y,L_{\min}z)-(u,L_{\min}z)=(L^*_{\min}y,z)-(h,z)=0,$$

i.e. $y-u\in\mathcal{R}(L_{\min})^{\perp}$. It follows from Lemma 3.6 that $y-u\in\mathcal{N}(L_{\max})$.

Using (3.8), we get

$$Jy^{[q]}(x)-T(x)\,y(x)=Ju^{[q]}u(x)-T(x)\,u(x)=V(x)\,h(x)\,,\ x\in(0,a)\,.$$

Since $y,h\in\mathbf{H}$, we have that $y\in D_{\max}$ and $L_{\max}y=h=L^*_{\min}y$. This completes the proof. □

4. Regular $q-$Hamiltonian boundary value problems

In this section, we introduce regular $q-$Hamiltonian boundary value problems.

We denote by D those elements y in $\mathbf{H}$ satisfying:

i) y and D_qy are $q-$regular at zero.

ii) $l(y)=Jy^{[q]}(x)-T(x)\,y(x)=V(x)\,f(x)$ exists in $(0,a)$ and $f\in\mathbf{H}$.

iii) Let Θ and Φ be $m\times 2n$ matrices such that $rank\,(\Theta:\Phi)=m$. Then set

$$\Theta\widehat{y}(0)+\Phi\widehat{y}(a)=0.$$

We define the operator L by setting

$$Ly=f\Leftrightarrow Jy^{[q]}(x)-T(x)\,y(x)=V(x)\,f(x)\,,$$

for all y in D.

Let Υ and Γ be $(4n-m)\times 2n$ matrices, chosen so that $rank\,(\Upsilon:\Gamma)=4n-m$ and

$$\begin{pmatrix} \Theta & \Phi \\ \Upsilon & \Gamma \end{pmatrix}$$

is nonsingular. Let

$$\begin{pmatrix} \widetilde{\Theta} & \widetilde{\Phi} \\ \widetilde{\Upsilon} & \widetilde{\Gamma} \end{pmatrix}$$

be chosen so that

$$\begin{pmatrix} \widetilde{\Theta} & \widetilde{\Phi} \\ \widetilde{\Upsilon} & \widetilde{\Gamma} \end{pmatrix}^{*}\begin{pmatrix} \Theta & \Phi \\ \Upsilon & \Gamma \end{pmatrix}=\begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}. \tag{4.1}$$

We can now rewrite the formula (3.7).

Theorem 4.1 *Let y and z be in $D_{\max}$. Then*

$$\begin{aligned}(L_{\max}y,z)-(y,L_{\max}z)=&\left[\widetilde{\Theta}\widehat{z}(0)+\widetilde{\Phi}\widehat{z}(a)\right]^{*}\left[\Theta\widehat{y}(0)+\Phi\widehat{y}(a)\right]\\ &+\left[\widetilde{\Upsilon}\widehat{z}(0)+\widetilde{\Gamma}\widehat{z}(a)\right]^{*}\left[\Upsilon\widehat{y}\,(0)+\Gamma\widehat{y}(a)\right].\end{aligned}$$

Proof From (3.7) and (4.1), we get

$$\begin{aligned}&\widehat{z}^{*}(a)J\widehat{y}(a)-\widehat{z}^{*}(0)J\widehat{y}(0)\\ &=(\widehat{z}^{*}(0),\widehat{z}^{*}(a))\begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}\begin{pmatrix} \widehat{y}(0) \\ \widehat{y}(a) \end{pmatrix}\\ &=(\widehat{z}^{*}(0),\widehat{z}^{*}(a))\begin{pmatrix} \widetilde{\Theta} & \widetilde{\Phi} \\ \widetilde{\Upsilon} & \widetilde{\Gamma} \end{pmatrix}^{*}\begin{pmatrix} \Theta & \Phi \\ \Upsilon & \Gamma \end{pmatrix}\begin{pmatrix} \widehat{y}(0) \\ \widehat{y}(a) \end{pmatrix}\\ &=\left[\begin{pmatrix} \widetilde{\Theta} & \widetilde{\Phi} \\ \widetilde{\Upsilon} & \widetilde{\Gamma} \end{pmatrix}\begin{pmatrix} \widehat{z}(0) \\ \widehat{z}(a) \end{pmatrix}\right]^{*}\left[\begin{pmatrix} \Theta & \Phi \\ \Upsilon & \Gamma \end{pmatrix}\begin{pmatrix} \widehat{y}(0) \\ \widehat{y}(a) \end{pmatrix}\right]\\ &=\begin{pmatrix} \widetilde{\Theta}\widehat{z}(0)+\widetilde{\Phi}\widehat{z}(a) \\ \widetilde{\Upsilon}\widehat{z}(0)+\widetilde{\Gamma}\widehat{z}(a) \end{pmatrix}^{*}\begin{pmatrix} \Theta\widehat{y}(0)+\Xi\widehat{y}(a) \\ \Upsilon\widehat{y}\,(0)+\Gamma\widehat{y}(a) \end{pmatrix}.\end{aligned}$$

□

Now, we describe the operator L^{*}, i.e. the adjoint operator of L.

Theorem 4.2 *Let D^{*} be the domain of the operator L^{*}. Then, it consists of those elements z in* **H** *satisfying* *i)* y and $D_q y$ are $q-$regular at zero.

ii) $l(z) = Jz^{[q]}(x) - T(x)\,z(x) = V(x)\,g(x)$ exists in $(0,a)$ and $g \in \mathbf{H}$.

iii) $\widetilde{\Upsilon}\widehat{z}(0) + \widetilde{\Gamma}\widehat{z}(a) = 0$.

For $z \in D^*$, $L^* z = \widehat{g}$ if and only if $Jz^{[q]} = Vg + Tz$.

Proof Since $L_{\min} \subset L \subset L_{\max}$, we have $L_{\min} \subset L^* \subset L_{\max}$. Let $y \in D$ and $z \in D^*$. From Theorem 4.1, we have

$$(Ly, z) - (y, L^* z) = \left[\widetilde{\Theta}\widehat{z}(0) + \widetilde{\Phi}\widehat{z}(a)\right]^* \left[\Theta\widehat{y}(0) + \Phi\widehat{y}(a)\right] + \left[\widetilde{\Upsilon}\widehat{z}(0) + \widetilde{\Gamma}\widehat{z}(a)\right]^* \left[\Upsilon\widehat{y}(0) + \Gamma\widehat{y}(a)\right].$$

Then we get

$$0 = \left[\widetilde{\Upsilon}\widehat{z}(0) + \widetilde{\Gamma}\widehat{z}(a)\right]^* \left[\Upsilon\widehat{y}(0) + \Gamma\widehat{y}(a)\right].$$

Since $\Upsilon\widehat{y}(0) + \Gamma\widehat{y}(a)$ is arbitrary, this forces z to satisfy $\widetilde{\Upsilon}\widehat{z}(0) + \widetilde{\Gamma}\widehat{z}(a) = 0$.

Conversely, if z satisfies the criteria listed above then $z \in D^*$. □

Now, we find parametric boundary conditions for D and D^*. Recall that

$$\Theta\widehat{y}(0) + \Phi\widehat{y}(a) = 0, \Upsilon\widehat{y}(0) + \Gamma\widehat{y}(a) = F, \tag{4.2}$$

where F is arbitrary. Hence, we have

$$\begin{pmatrix} \Theta & \Phi \\ \Upsilon & \Gamma \end{pmatrix} \begin{pmatrix} \widehat{y}(0) \\ \widehat{y}(a) \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix}. \tag{4.3}$$

Multiplying both sides of (4.3) by

$$\begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} \widetilde{\Theta} & \widetilde{\Phi} \\ \widetilde{\Upsilon} & \widetilde{\Gamma} \end{pmatrix}^*$$

we obtain

$$\begin{pmatrix} \widehat{y}(0) \\ \widehat{y}(a) \end{pmatrix} = \begin{pmatrix} J\widetilde{\Upsilon}^* F \\ -J\widetilde{\Gamma}^* F \end{pmatrix}. \tag{4.4}$$

Similarly, we find parametric boundary conditions for D^*. Since

$$\widetilde{\Theta}\widehat{z}(0) + \widetilde{\Phi}\widehat{z}(a) = G,\ \widetilde{\Upsilon}\widehat{z}(0) + \widetilde{\Gamma}\widehat{z}(a) = 0,$$

where G is arbitrary, we get

$$\begin{pmatrix} \widehat{z}^*(0) & \widehat{z}^*(a) \end{pmatrix} \begin{pmatrix} \widetilde{\Theta} & \widetilde{\Phi} \\ \widetilde{\Upsilon} & \widetilde{\Gamma} \end{pmatrix}^* = \begin{pmatrix} G^* & 0 \end{pmatrix}. \tag{4.5}$$

Multiplying both sides of (4.5) by

$$\begin{pmatrix} \Theta & \Phi \\ \Upsilon & \Gamma \end{pmatrix} \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}$$

we obtain

$$\widehat{z}(0) = -J\Theta^{*}G, \ \widehat{z}(a) = J\Phi^{*}G. \tag{4.6}$$

Using these parametric boundary conditions, we develop a criterion under which L is self-adjoint. Then we have the following theorem.

Theorem 4.3 *The operator L is self-adjoint if and only if rank $(\Theta : \Phi) = m = 2n$ and $\Theta J\Theta^{*} = \Phi J\Phi^{*}$.*

Proof Let L be self-adjoint operator. Then z satisfies the boundary conditions for D, i.e.

$$\Theta\widehat{z}\,(0) + \Phi\widehat{z}\,(a) = 0.$$

Using (4.6), we get

$$\Theta\,(-J\Theta^{*}G) + \Phi\,(J\Phi^{*}G) = 0$$
$$[\Theta J\Theta^{*} - \Phi J\Phi^{*}]\,G = 0.$$

Since G is arbitrary, we obtain

$$\Theta J\Theta^{*} = \Phi J\Phi^{*}.$$

Conversely, if $\Theta J\Theta^{*} = \Phi J\Phi^{*}$, then we have

$$\begin{pmatrix} -\Theta J & \Phi J \end{pmatrix}\begin{pmatrix} \Theta^{*} \\ \Phi^{*} \end{pmatrix} = 0,$$

i.e. the columns of $\begin{pmatrix} \Theta^{*} \\ \Phi^{*} \end{pmatrix}$ for n independent solutions to the equation $\begin{pmatrix} -\Theta J & \Phi J \end{pmatrix} X = 0$.

By virtue of (4.2) and (4.4), we have

$$\begin{pmatrix} -\Theta J & \Phi J \end{pmatrix}\begin{pmatrix} \widetilde{\Upsilon}^{*} \\ \widetilde{\Gamma}^{*} \end{pmatrix} = 0.$$

Hence, there must be a constant, nonsingular matrix K such that

$$\begin{pmatrix} \widetilde{\Upsilon}^{*} \\ \widetilde{\Gamma}^{*} \end{pmatrix} K = \begin{pmatrix} \Theta^{*} \\ \Phi^{*} \end{pmatrix}.$$

That is, the boundary conditions $\Theta\widehat{y}\,(0) + \Phi\widehat{y}\,(a) = 0$ and $\Upsilon\widehat{y}\,(0) + \Gamma\widehat{y}(a) = 0$ are equivalent. Since the forms of L and L^{*} are the same, $L = L^{*}$. □

5. Eigenfunction expansions

In this section, we shall give an eigenfunction expansion using by the spectral theory of self-adjoint operators in a Hilbert space.

Let us consider the operator L of the previous section. We set in $\mathbf{H}$, $Ly = f$ if and only if $Jy^{[q]}(x) - T\,(x)\,y\,(x) = V\,(x)\,f\,(x)$, where y is constrained in part by the self-adjointness criterion $\Theta J\Theta^{*} = \Phi J\Phi^{*}$. Suppose that the matrix $Y\,(x,\lambda)$ is a fundamental matrix for $Jy^{[q]}(x) = [\lambda V\,(x) + T\,(x)]\,y\,(x)$ satisfying $\widehat{Y}\,(0,\lambda) = I$.

Now, we solve the equation $(L-\lambda I)\, y = f$ for y. y satisfies the nonhomogeneous equation $Jy^{[q]}(x) = [\lambda V(x) + T(x)]\, y(x) + V(x)\, f(x)$. The method of variation of constants suggests to search the general solution under the form $y(x,\lambda) = Y(x,\lambda)\, C(x,\lambda)$, where $C(x,\lambda)$ is $2n \times 1$ vector function. We get

$$Jy^{[q]} = JY^{[q]}C + J\widetilde{Y}(x)\, C^{[q]},$$
$$[\lambda V(x) + T(x)]\, y(x) = [\lambda V(x) + T(x)]\, YC,$$

where

$$\widetilde{Y}(x) = \begin{pmatrix} Y_1(qx) \\ Y_2\left(q^{-1}x\right) \end{pmatrix}.$$

Hence

$$\begin{aligned} V(x)\, f(x) &= Jy^{[q]}(x) - [\lambda V(x) + T(x)]\, y(x) \\ &= JY^{[q]}C + J\widetilde{Y}C^{[q]} - [\lambda V(x) + T(x)]\, YC \\ &= \left\{ JY^{[q]} - [\lambda V(x) + T(x)]\, Y \right\} C + J\widetilde{Y}C^{[q]} = J\widetilde{Y}C^{[q]}, \end{aligned}$$

i.e.

$$C^{[q]} = \left[J\widetilde{Y}(x)\right]^{-1} V(x)\, f(x).$$

Then, the general solution is given by

$$y(x,\lambda) = Y(x,\lambda) \int_0^x \left[J\widetilde{Y}(t)\right]^{-1} V(t)\, f(t)\, d_q t + Y(x,\lambda)\, M.$$

If we impose the boundary condition $\Theta\widehat{y}(0) + \Phi\widehat{y}(a) = 0$, then we obtain

$$\begin{aligned} &\widehat{y}(0) = M, \\ &\widehat{y}(a) = \widehat{Y}(a,\lambda) \int_0^a \left[J\widetilde{Y}(t)\right]^{-1} V(t)\, f(t)\, d_q t + \widehat{Y}(a,\lambda)\, M. \end{aligned}$$

These yield

$$y(x,\lambda) = Y(x,\lambda) \left[\Theta + \Phi\widehat{Y}(a)\right]^{-1} \Theta \int_0^x \left[J\widetilde{Y}(t)\right]^{-1} V(t)\, f(t)\, d_q t$$

$$- Y(x,\lambda) \left[\Theta + \Phi\widehat{Y}(a)\right]^{-1} \Phi\widehat{Y}(a) \int_x^a \left[J\widetilde{Y}(t)\right]^{-1} V(t)\, f(t)\, d_q t.$$

In the next results, we use the notation

$$y(x,\lambda) = \int_0^a G(x,t,\lambda)\, V(t)\, f(t)\, d_q t,$$

where

$$G(x,t,\lambda) = \begin{cases} Y(x,\lambda) \left[\Theta + \Phi\widehat{Y}(a)\right]^{-1} \Theta \left[J\widetilde{Y}(t)\right]^{-1}, & 0 \le t \le x \le a \\ -Y(x,\lambda) \left[\Theta + \Phi\widehat{Y}(a)\right]^{-1} \Phi\widehat{Y}(a) \left[J\widetilde{Y}(t)\right]^{-1}, & 0 \le x \le t \le a. \end{cases}$$

Theorem 5.1 *For all nonreal* λ, *the operator* $R(\lambda) = (L - \lambda I)^{-1}$ *exists and is a bounded operator. It exists also for all real* λ *for which* $\det[\Theta + \Phi\widehat{y}(a)] \neq 0$ *as a bounded operator. The spectrum of the operator* L *consists entirely of isolated eigenvalues, zeros of* $\det[\Theta + \Phi\widehat{y}(a)] = 0$.

Proof It is clear that the operator $R(\lambda) = (L - \lambda I)^{-1}$ exists for all real λ except the zeros of $\det[\Theta + \Phi\widehat{y}(a)] = 0$. Since the operator L is self-adjoint operator, it follows that the operator $R(\lambda) = (L - \lambda I)^{-1}$ exists for all nonreal λ. The spectrum of the operator L consists entirely of isolated eigenvalues, zeros of $\det[\Theta + \Phi\widehat{y}(a)] = 0$ because $\det[\Theta + \Phi\widehat{y}(a)]$ is analytic in λ and is not identically zero. These zeros can accumulate only at $\pm\infty$.

To prove that the operator $R(\lambda) = (L - \lambda I)^{-1}$ is a bounded operator, write $F(\eta) = V^{1/2}(\eta) f(\eta)$ and $W(x, \eta, \lambda) = V^{1/2}(\eta) G(x, t, \lambda) V^{1/2}(x)$, where $V^{1/2}$ is a square root of the matrix V. Then, we have

$$\left\|(L - \lambda I)^{-1} f\right\|^2 = \|y\|^2 = \int_0^a y^*(x) V(x) y(x) d_q x$$

$$= \int_0^a \left[\int_0^a G(x, \eta, \lambda) V(\eta) \widehat{f}(\eta) d_q\eta\right]^* V(x) \left[\int_0^a G(x, t, \lambda) V(t) \widehat{f}(t) d_q t\right] d_q x$$

$$= \int_0^a \left[\int_0^a F^*(\eta) W^*(x, \eta, \lambda) d_q\eta\right] \left[\int_0^a W(x, t, \lambda) F(t) d_q t\right] d_q x.$$

If we apply Schwarz's inequality to both terms, then we obtain

$$\left\|(L - \lambda I)^{-1} f\right\|^2 \leq \|W\|^2 \|f\|^2,$$

where

$$\|W\|^2 = \int_0^a \int_0^a \sum_{i=1}^n \sum_{j=1}^n |W_{ij}(x, \eta, \lambda)|^2 d_q\eta d_q x.$$

□

Theorem 5.2 *Eigenfunctions associated with different eigenvalues are mutually orthogonal. For each eigenvalue* μ_j, *its eigenfunctions can be made mutually orthogonal.*

Proof Let y_1 and y_2 be eigenfunctions associated with μ_1 and μ_2, respectively. It follows from Green's formula that

$$(\mu_1 - \mu_2)(y_1, y_2) = 0.$$

Since $\mu_1 \neq \mu_2$, $(y_1, y_2) = 0$.

Let $y_1, y_2, ..., y_m$ be eigenfunctions associated with μ. Let us define N_k by

$$N_k = y_k - \sum_{j=1}^{k-1} u_j (y_k, u_j),$$

$$u_1 = \frac{y_1}{\|y_1\|},$$

$$u_k = \frac{N_k}{\|N_k\|}, \ k = 2, 3, ..., m.$$

It is clear that u_k is orthogonal to $y_1, y_2, ..., y_{k-1}$. □

Without loss of generality, we can assume that 0 is not an eigenvalue. Then, the solution to

$$Jy^{[q]}(x) - T(x) y(x) = V(x) f(x),$$

$$\Theta \widehat{y}(0) + \Phi \widehat{y}(a) = C,$$

is given by

$$y(x) = \int_0^a G(x,t) V(t) f(t) d_q t,$$

where $G(x,t) = G(x,t,0)$. In the next results, we use the following notation

$$y = \Upsilon f = L^{-1} f.$$

Theorem 5.3 Υ *is bounded and* $\|\Upsilon\| = \sup \left\{ \left| \frac{1}{\lambda_n} \right| : \lambda_n \in \sigma(L) \right\}$.

Proof It is clear that if $L\chi_n = \lambda_n \chi_n \ (n \in \mathbb{N})$, then $\Upsilon \chi_n = \frac{1}{\lambda_n} \chi_n$. □

Now, we shall order the eigenvalues of Υ, $\tau_n = \frac{1}{\lambda_n}$, such that $|\tau_1| \geq |\tau_2| \geq ... \geq |\tau_n| \geq ...$, where

$$\lim_{n \to \infty} |\tau_n| = 0. \tag{5.1}$$

Let us define $\{\Upsilon_n\}_{n=1}^{\infty}$ by

$$\Upsilon_n f = \Upsilon f - \sum_{i=1}^{n-1} \tau_i \chi_i (f, \chi_i).$$

Then we have the following theorem.

Theorem 5.4 *The following results hold.*

$$\|\Upsilon_n\| = |\tau_n|, \ n \in \mathbb{N},$$

and

$$\lim_{n \to \infty} \Upsilon_n = 0. \tag{5.2}$$

Proof We have

$$\Upsilon_n \chi_j = \begin{cases} 0, & \text{if } 1 \le j \le n-1 \\ \tau_j & \text{if } n \le j < \infty. \end{cases}$$

Further Υ_n is bounded and self-adjoint. Hence, we get

$$\|\Upsilon_n\| = \sup_{\chi \in \mathbf{H}, \|\chi\|=1} |(\Upsilon_n \chi, \chi)| = \sup_{\substack{\chi \in \mathbf{H}, \|\chi\|=1 \\ \chi \neq \chi_1, \ldots, \chi_n}} |(\Upsilon_n \chi, \chi)| = |\tau_n| \, .$$

For a finite n, we can stop this process such that $\Upsilon_n = 0$. Hence, for all $f \in \mathbf{H}$, we get

$$\Upsilon f = \sum_{i=1}^{n-1} \tau_i \chi_i (f, \chi_i) \, . \tag{5.3}$$

If we apply L to the equality (5.3), then we obtain

$$f = \sum_{i=1}^{n-1} \chi_i (f, \chi_i) \, ,$$

which says that f is differentiable. Since there are f 's which are not, the process cannot stop. From (5.1), we get $\lim_{n\to\infty} \Upsilon_n = 0$. □

Theorem 5.5 *For all* $f \in \mathbf{H}$, *we have*

$$f = \sum_{i=1}^{\infty} \chi_i (f, \chi_i) \, , \; \Upsilon f = \sum_{i=1}^{\infty} \tau_i \chi_i (f, \chi_i) \, .$$

For all $y \in D$, *we get*

$$Ly = \sum_{i=1}^{\infty} \lambda_i \chi_i (y, \chi_i) \, .$$

Proof It follows from (5.2) that

$$\Upsilon f = \sum_{i=1}^{\infty} \tau_i \chi_i (f, \chi_i) \, . \tag{5.4}$$

Applying L to the equality (5.4), we get

$$f = \sum_{i=1}^{\infty} \chi_i (f, \chi_i) \, .$$

Further,

$$(f, \chi_i) = (Ly, \chi_i) = (y, L\chi_i) = \lambda_i (y, \chi_i) \, .$$

Thus, we get

$$Ly = \sum_{i=1}^{\infty} \lambda_i \chi_i (y, \chi_i) \, .$$

□

Theorem 5.6 *There exists a collection of projection operators* $\{E(\lambda)\}$ *satisfying*
(a) $\lim_{\lambda\to\infty} E(\lambda) = I,\ \lim_{\lambda\to-\infty} E(\lambda) = 0,$
(b) $E(\lambda_1) \leq E(\lambda_2)$ *when* $\lambda_1 \leq \lambda_2,$
(c) $E(\lambda)$ *is continuous from above,*
(d) For all $f \in \mathbf{H}$ *and* $y \in D,$ *we have*

$$f = \int_{-\infty}^{\infty} dE(\lambda) f,\ \Upsilon f = \int_{-\infty}^{\infty} \frac{1}{\lambda} dE(\lambda) f,\ Ly = \int_{-\infty}^{\infty} \lambda dE(\lambda) y.$$

Proof Let us define

$$P_i f = \chi_i (f, \chi_i),$$

where P_i is a projection operator. If we define

$$E(\lambda) f = \sum_{\lambda_i \leq \lambda} P_i f,$$

then $E(\lambda)$ generates a Stieltjes measure. The integrals in (d) are obtained from this series. □

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