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Banach algebra structure on simple extensions

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Abstract: We study the existence of Banach algebra structures on simple extensions of a unitary commutative Banach algebra. The link with the integrality of these extensions is studied. For any simple extension, a characterization of the existence of the Banach algebra norm making continuous the canonical injection is also given.

Key words: Banach algebra structure, ideal, principal ideal, unitary polynomial, simple extension, canonical injection, integral extension

1. Introduction

Let A be a unital and commutative algebra. A unital and commutative algebra B is called an extension of A if there exists an isomorphism of A into B that carries the identity of A into the identity of B. When convenient, we simply view A as a subalgebra of B that contains the identity of B. An extension of A of the form $A[\theta]$ is called a simple extension of A. Such an algebra is then a quotient of the algebra of polynomials $A[X]$.

If J is an ideal of $A[X]$ and s be the canonical surjection of $A[X]$ on $A[X]/J$, then $(A[X]/J, s|_A\})$ is an extension of A if, and only, if $J \cap A = \{0\}$, where $s|_A$ is the restriction from s to A.
Let B be an extension of A. An element x of B is said to be an integral on A if it is the root of a unitary polynomial of $A[X]$. We say that the extension B is integral over A if every element of B is integral over A. The concept of integrality is a notion of finiteness. Indeed, an element x of B is an integral on A if, and only, if the A-module $A[x]$ is of finite type.

Throughout the sequel, the algebras considered are complex, unitary and commutative.

Let A be an algebra and J be the principal ideal of $A[X]$ generated by $X^n + a_{n-1}X^{n-1} + ... + a_0$, where $a_1, ..., a_{n-1}$ are elements of A. Then, by the classical method of field extension, the equation:

$$X^n + a_{n-1}X^{n-1} + ... + a_0 = 0 \tag{1.1}$$

has at least one solution in $B = A[X]/J$. In [1], the authors are interested in the problem of obtaining a normed linear algebra on B, where A is a given normed linear algebra. They have shown that if A is normed (resp. Banach), then B is likewise. If the leading coefficient of the polynomial $\alpha(X) = a_0 + a_1X + ... + a_nX^n$

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has an inverse, then $\alpha A[X] = (a_n^{-1}\alpha)A[X]$ and the polynomial $a_n^{-1}\alpha$ is unitary. On the other hand, if the leading coefficient of the polynomial $\alpha(X)$ has no inverse, then the algebra $A[X]/\alpha A[X]$ is not necessarily an extension of A i.e. $\alpha A[X] \cap A \neq \{0\}$. Likewise, if $A[X]/\alpha A[X]$ is an extension of A (it can happen even if $\alpha(X)$ is not unitary), this quotient does not have good algebraic properties as it is the case where the polynomial is unitary. In the latter case, the extension is integral over A and therefore has very good properties.

In this work, we consider extensions B of A in the topological sense, i.e. a Banach algebra topology on B which makes continuous the canonical injection $A \longrightarrow B$. Our framework is the general case $A[X]/\alpha A[X]$, where $\alpha(X)$ is not necessarily unitary and more generally in the case $A[X]/J$, where J is an arbitrary ideal of $A[X]$, with $J \cap A = \{0\}$ i.e. any simple extension. Notice that in ([2]–[4], [6], [7]), the authors considered only the case of a quotient by a unitary polynomial. These quotients enjoy an extremely important property, namely the integrality which gives them good algebraic properties.

We show (Theorem 2.2) that there exists at most one Banach algebra structure on $A[\theta]$ making continuous the canonical injection $A \longrightarrow A[\theta]$. We then establish (Theorem 2.5) that if there is a Banach algebra structure on $A[\theta]$ making continuous the canonical injection $A \longrightarrow A[\theta]$, then the simple extension $A[\theta]$ is integral over A. For a nonzero ideal J of $A[X]$ such that $J \cap A = \{0\}$, we prove (Theorem 2.13) that there exists a Banach algebra norm on $A[X]/J$ making continuous the canonical injection $A \longrightarrow A[X]/J$, if, and only, if J contains a unitary polynomial $\alpha(X)$ and $J/\alpha A[X]$ is a closed ideal of the Banach algebra $(A[X]/\alpha A[X], \|.\|_\alpha)$, where $\|.\|_\alpha$ is the Banach algebra norm on $A[X]/\alpha A[X]$ extending that one of A.

2. Banach algebra structure on simple extensions

We are mainly interested in the simple extension $A[\theta]$ of a unitary commutative Banach algebra $(A, \|.\|)$. We will denote by $\mathcal{A}$ the Banach space of sequences $(a_n)_n \subset A$ such that:

$$\sum_{n\geq 0} \|a_n\| < +\infty \tag{2.1}$$

If $\alpha(X) \in A[X]$ is unitary, we denote $\|.\|_\alpha$ the Banach algebra norm on $A[X]/\alpha A[X]$ extending that one of A. The existence of such a norm is provided by [1]. To make the paper self contained, we reproduce the construction of this norm on $A\,[X]\,/\alpha A\,[X]$. The procedure is to define a norm in $A\,[X]$, and then use the canonical method (cf. [8], p. 14) for norming the quotient algebra $A\,[X]\,/\alpha A\,[X]$. This last method gives only "pseudo-norms" on $A\,[X]\,/\alpha A\,[X]$ since the ideal $\alpha A\,[X]$ is not assumed to be closed. Among the many ways of norming $A\,[X]$, the authors present a one parameter family of norms. Namely, for a positive t, and then for every element $\gamma(X)$ of $A\,[X]$ of the form:

$$\gamma(X) = c_0 + c_1 X + ... + c_p X^p, \tag{2.2}$$

they consider

$$\|\gamma(X)\| = \|c_0\| + \|c_1\|\,t + ... + \|c\|_p\, t^p. \tag{2.3}$$

No matter what t is, the subalgebra of "constant" polynomials is isometric and isomorphic to A. They are left with the task of choosing t in such way that the canonical homomorphism

$$A\,[X] \longrightarrow A\,[X]\,/\alpha A\,[X] \tag{2.4}$$

shall be isometric as applied to the constant polynomials. If the polynomial $\alpha(X)$ has the monic form:

$$\alpha(X) = a_0 + a_1 X + ... + a_{n-1} X^{n-1} + X^n \tag{2.5}$$

and t satisfies the condition

$$\|a_0\| + \|a_1\| t + ... + \|a_{n-1}\| t^{n-1} \leq t^n, \tag{2.6}$$

then they show ([1], Lemma. 3.5, p. 204) that the ideal $\alpha A[X]$ is closed in $A[X]$, and the natural image of A in $A[X]/\alpha A[X]$ is isometrically isomorphic to A. Moreover, if A is complete, so is the extension $A[X]/\alpha A[X]$. The following sums up the result:

Theorem 2.1 *([1], Theorem 3.6, p. 206): Let the leading coefficient a_n of $\alpha(X)$ have an inverse. Then a norm of type (2.3) can be found for $A[X]$ such that the algebra $A[X]/\alpha A[X]$ (in which $\alpha(X)$ has a root) is an isometric extension of A, and is complete when A is complete.*

Notice that if $\delta(X) \in A[X]$, then it is equivalent, modulo $\alpha(X)$, to a unique polynomial of the form

$$\beta(X) = b_0 + b_1 X + ... + b_{n-1} X^{n-1} \tag{2.7}$$

The norm of the coset $\overline{\delta}(X)$ of $\delta(X)$, in the quotient algebra $A[X]/\alpha A[X]$, will be

$$\left\|\overline{\delta}\right\|_\alpha = \inf_\gamma \|\beta(X) + \alpha(X)\gamma(X)\|. \tag{2.8}$$

More precisely, in the case where $t = 1$, the authors have shown that

$$\left\|\overline{\delta}\right\|_\alpha = \|\beta(X)\| = |b_0| + |b_1| + ... + |b_{n-1}|. \tag{2.9}$$

In the general case, there exists $K' > 0$, such that

$$\|b_0\| + \|b_1\| + ... + \|b_{n-1}\| \leq K' \left\|\overline{\delta}\right\|_\alpha. \tag{2.10}$$

The existence of K' results by considering the Banach algebra A^n endowed with the norm:

$$\|(b_0, b_1, ..., b_{n-1})\| = \|b_0\| + \|b_1\| + ... + \|b_{n-1}\| \tag{2.11}$$

and the bijective continuous morphism $A^n \longrightarrow A[X]/\alpha A[X]$ of Banach spaces given by:

$$(b_0, b_1, ..., b_{n-1}) \longmapsto \overline{b_0 + b_1 X + ... + b_{n-1} X^{n-1}} \tag{2.12}$$

to which we apply the open mapping theorem. For simple extension $A[\theta]$ of A, the Banach algebra structure making continuous the canonical injection $A \longrightarrow A[\theta]$ is unique as the following result shows:

Theorem 2.2 *There is at most one Banach algebra structure on $A[\theta]$ making continuous the canonical injection $A \longrightarrow A[\theta]$.*

Proof Let $\|.\|'$ be a Banach algebra norm on $A[\theta]$ making continuous the canonical injection $A \longrightarrow A[\theta]$. As $A[\theta] = A[\gamma\theta]$, for all $\gamma \in \mathbb{C}^*$, it is no loss of generality, to assume that $\|\theta\|' \leq 1$. Consider $K > 0$ such that $\|a\|' \leq K \|a\|$, for every $a \in A$. Let us note $\varphi : \mathcal{A} \longrightarrow A[\theta]$ be a linear mapping, which is well defined, given by:

$$\varphi((a_n)_n) = \sum_{n\geq 0} a_n \theta^n. \tag{2.13}$$

It is continuous, in fact, for $(a_n)_n \in \mathcal{A}$, one has:

$$\|\varphi((a_n)_n)\|' = \left\| \sum_{n\geq 0} a_n \theta^n \right\|' \leq \sum_{n\geq 0} \|a_n\|' \|\theta^n\|' \leq \sum_{n\geq 0} K \|a_n\|. \tag{2.14}$$

So

$$\|\varphi((a_n)_n)\|' \leq K \|(a_n)_n\| \tag{2.15}$$

Moreover φ is surjective. Then, by the open mapping theorem, the Banach space $\mathcal{A}/\ker\varphi$ is isomorphic (algebraically and topologically) to the Banach space $A[\theta]$. Whence the result. □

Remark 2.3 *In Theorem 2.2, uniqueness is not necessarily guaranteed without the continuity of the canonical injection* $A \longrightarrow A[\theta]$. *Indeed, consider* B *a unitary commutative algebra provided with two nonequivalent complete algebra norms (cf. Feldman example [3]). Considering* $B = B[X]/(X-e)B[X]$, *we see that the latter has, at least, two different Banach algebra structures.*

Remark 2.4 *If* A *is of finite dimension, then a simple extension of* A *is provided with a Banach algebra norm if, and only, if it is itself of finite dimension. Indeed if* $(e_1, ..., e_n)$ *is a basis of* A, *then* $\left\{e_i\theta^j : i = 1, 2, ..., n \text{ and } j \geq 0\right\}$ *is a generating countable family of* $A[\theta]$. *So, by Baire argument,* $A[\theta]$ *must be of finite dimension.*

Here is the second main result of this paper.

Theorem 2.5 *If there is a Banach algebra structure on* $A[\theta]$ *making continuous the canonical injection* $A \longrightarrow A[\theta]$, *then* $A[\theta]$ *is integral over* A.

For the proof we will need the following lemma:

Lemma 2.6 *Let* E, F *be two Banach spaces,* $\mathcal{L}(E, F)$ *be the Banach space of continuous linear maps of* $E \to F$ *and*

$$\mathcal{S} = \{f \in \mathcal{L}(E, F) : f \text{ is surjective}\} \tag{2.16}$$

Then $\mathcal{S}$ *is an open set of* $\mathcal{L}(E, F)$.

Proof The proof is a classic consequence of the open mapping theorem. To make the paper self contained, we reproduce it here. Let $S \in \mathcal{L}(E, F)$ be a surjective mapping. By the open mapping theorem, there exists $c > 0$ such that $B_F \subset cS(B_E)$, where B_E (resp. B_F) designates the closed unit ball of E (resp. F). Without loss of generality, suppose that $c = 1$. Let $0 < r < 1$ and $T \in \mathcal{L}(E, F)$ such that $\|S - T\| < r$. Let's show that T

is surjective. Let $y = y_1 \in B_F$. Then there exists $x_1 \in B_E$ such that $S(x_1) = y_1$. Let $y_2 = (S - T)(x_1)$. One has $\|y_2\| \le r$. Thus, there exists $x_2 \in E$ such that $S(x_2) = y_2$. By induction, it is possible to build a sequence $(x_n)_{n \ge 1}$ having the following properties:

$$S(x_n) = y_n,\ y_{n+1} = (S - T)(x_n),\ \ \|y_{n+1}\| \le r^n \text{ and } \|x_{n+1}\| \le r^n. \tag{2.17}$$

We then have, for each integer $n \ge 1$,

$$y = y_1 = T(x_1) + T(x_2) + ... + T(x_n) + y_{n+1} \tag{2.18}$$

It follows that the series $\sum\limits_{n \ge 1} x_n$ is convergent in E for it is absolutely convergent. Let $x = \sum\limits_{n \ge 1} x_n$ be its sum. One has clearly $T(x) = y$. This completes the proof. □

[**Proof of theorem 2.5**] We keep the notations of Theorem 2.2. We can assume that $\|\theta\|' < 1$. For all $p \ge 1$, consider the linear map $\varphi_p : \mathcal{A} \longrightarrow A[\theta]$, defined by:

$$\varphi_p((a_n)_n) = \sum_{n \le p} a_n \theta^n \tag{2.19}$$

As for φ, we show that φ_p is continuous. Let's prove that the sequence $(\varphi_p)_p$ converges to φ in the Banach space $\mathcal{L}(\mathcal{A}, A[\theta])$. Let $(a_n)_n \in \mathcal{A}$ such that $\|(a_n)_n\|' \le 1$, where $\|(a_n)_n\|' = \sum\limits_{n \ge 0} \|a_n\|$. One has:

$$\|\varphi((a_n)_n) - \varphi_p((a_n)_n)\|' = \left\| \sum_{n \ge p+1} a_n \theta^n \right\|' \tag{2.20}$$

$$\le \sum_{n \ge p+1} \|a_n\|' \|\theta\|'^n \le K \sum_{n \ge p+1} \|\theta\|'^n \underset{p \longrightarrow +\infty}{\longrightarrow} 0 \tag{2.21}$$

According to Lemma 2.6, there exists p such that φ_p is surjective. Thus, $A[\theta]$ is an A-module of finite type, and we conclude with ([5], theoreme I.3, p. 135 and Corollaire 4, p. 137).

As an illustration, we give the following example:

Example 2.7 *Let $\mathcal{A}(D)$ be the disc algebra. It is clear that it is an integral domain. Let u be in the field of fractions of $\mathcal{A}(D)$ and consider the simple extension $\mathcal{A}(D)[u]$ of $\mathcal{A}(D)$. This extension cannot be provided with any Banach algebra norm making continuous the canonical injection $\mathcal{A}(D) \longrightarrow \mathcal{A}(D)[u]$. Otherwise, by Theorem 2.4, u must be integral on $\mathcal{A}(D)$. Suppose then that there exists $v_0, ..., v_{n-1} \in \mathcal{A}(D)$ such that*

$$u^n + \sum_{0 \le i \le n-1} v_i u^i = 0 \qquad (*) \tag{2.22}$$

Let $u = a/b$, where $a, b \in \mathcal{A}(D)$. From equality $(*)$, we get:

$$a^n + \sum_{0 \le i \le n-1} v_i a^i b^{n-i} = 0 \tag{2.23}$$

It follows that all root of b is also a root of a. Taking, $a(z) = z$, $b(z) = z - 1$, we get a contradiction. We can also consider the extension $\mathcal{A}(D)[v]$, where $v(z) = \overline{z}$. Also this extension cannot be provided with any Banach algebra norm making continuous the canonical injection $\mathcal{A}(D) \longrightarrow \mathcal{A}(D)[v]$. In fact, v is a transcendental element over $\mathcal{A}(D)$. Indeed, if v is integral on $\mathcal{A}(D)$, let $f_0, ..., f_r \in \mathcal{A}(D)$, with minimal r such that:

$$f_0 + f_1 v + ... + f_r v^r = 0. \tag{2.24}$$

The derivative with respect to $\overline{z}$, gives a polynomial of $\mathcal{A}(D)[X]$, of degree strictly less than r, of which v is root, which contradicts the choice of r.

The converse of the Theorem 2.5 is not valid as shown in the following example:

Example 2.8 *Let $(B, \|.\|)$ be a unitary commutative Banach algebra and $a \in B$ such that the ideal aB is not closed. For example, $B = C([0,1])$ and $a(t) = t$, $t \in [0,1]$. It is clear that $\overline{aB} \subset \{f \in B : f(0) = 0\}$. By Weierstrass approximation theorem, the reader can prove the other inclusion. So $\overline{aB} = \{f \in B : f(0) = 0\}$. From the above, it follows that the function $g(t) = \sqrt{t}$, $t \in [0,1]$, belongs to $\overline{aB} \backslash aB$. Now, we consider the extension $B[\theta] = B[X]/\langle aX, X^2 \rangle$, where $\langle aX, X^2 \rangle$ is the ideal generated by aX and X^2. It is clear that $B[\theta]$ is integral over B. Let us even note that any element of $B[\theta]$ is root of a unitary polynomial of $B[X]$ of degree at most 2. We will show that there is no algebra norm (complete or not) on $B[\theta]$ making continuous the canonical injection $B \longrightarrow B[\theta]$. Suppose such an algebra norm $\|.\|'$ exists. Let us consider a sequence $(a_n)_n \subset B$ such that $aa_n \longrightarrow b$, with $b \notin aB$. We must have $b\theta = 0$. Then there exists $u(X)$, $v(X)$ in $B[X]$ such that $bX = aXu(X) + X^2 v(X)$, therefore $b = au(X) + Xv(X)$ and consequently $b = au(0) \in aB$, which is not the case.*

As a consequence, one has the following fundamental result:

Corollary 2.9 *Let $\alpha(X) \in A[X]$ be a nonzero polynomial. If there is a Banach algebra structure on $A[X]/\alpha A[X]$, making continuous the canonical injection $A \longrightarrow A[X]/\alpha A[X]$, then the leading coefficient of the polynomial $\alpha(X)$ has an inverse.*

Note that the converse of Corollary 2.9 is true as shown by Theorem 2.1.

Remark 2.10 *The Corollary 2.9 establishes then that the difficulty raised by the authors in [1], concerning the quotient by a polynomial not necessarily unitary, will not take place seeing that the polynomial in question is necessarily with the leading coefficient invertible.*

As an immediate consequence, we have:

Corollary 2.11 *Let $\alpha(X) \in A[X]$ be nonzero polynomial. If the extension $A[X]/\alpha A[X]$ of A is provided with a Banach algebra structure and for which A is closed, then the leading coefficient of the polynomial $\alpha(X)$ has an inverse.*

In the case of any simple extension, we have the following result:

Corollary 2.12 *Let J be a nonzero ideal of $A[X]$ such that $J \cap A = \{0\}$. If there exists a Banach algebra norm on $A[X]/J$ making continuous the canonical injection $A \longrightarrow A[X]/J$, then J contains a unitary polynomial of $A[X]$.*

Note that here, unlike Corollary 2.9, the converse is false; see example 2.8. However, we have:

Theorem 2.13 *Let J be a nonzero ideal of $A[X]$ such that $J \cap A = \{0\}$. The following assertions are equivalent:*

- ***i)*** *There is a Banach algebra norm on $A[X]/J$ making continuous the canonical injection $A \longrightarrow A[X]/J$.*
- ***ii)*** *J contains a unitary polynomial $\alpha(X)$ and, moreover, $J/\alpha A[X]$ is a closed ideal of Banach algebra $(A[X]/\alpha A[X], \|.\|_\alpha)$.*

Proof **ii)**$\Longrightarrow$**i)**: We have the algebraic isomorphism:

$$(A[X]/\alpha A[X])/(J/\alpha A[X]) \simeq A[X]/J \tag{2.25}$$

The Banach algebra structure on $A[X]/J$, deduced from this isomorphism, is suitable.
i)$\Longrightarrow$**ii)**: By the corollary 2.12, J contains a unitary polynomial $\alpha(X)$, which one supposes of degree n. Let φ be the morphism of natural algebras:

$$A[X]/\alpha A[X] \longrightarrow A[X]/J \tag{2.26}$$

This morphism is continuous. Indeed, if we note by $\overline{g}$ the coset of g in $A[X]/\alpha A[X]$ and $\widetilde{\varphi(g)}$ be the coset of $\varphi(g)$ in $A[X]/J$, then

$$\overline{g} = \sum_{0 \leq i \leq n-1} \overline{a_i}\overline{X}^i \in A[X]/\alpha A[X], \tag{2.27}$$

($\overline{g}$ is written just up to $n-1$, where $n = \deg(\alpha)$). Consider $K > 0$ such that $\|\widetilde{a_i}\| \leq K \|a_i\|$, for every $i = 0, ..., n-1$ and $K' > 0$ given by (2.10). Then one has:

$$\begin{aligned}
\|\varphi(\overline{g})\|_J &= \left\| \sum_{0 \leq i \leq n-1} \widetilde{a_i}\widetilde{X}^i \right\|_J \leq \sum_{0 \leq i \leq n-1} \|\widetilde{a_i}\|_J \left\|\widetilde{X}^i\right\|_J & (2.28)\\
&\leq \sum_{0 \leq i \leq n-1} K \|a_i\| \left\|\widetilde{X}^i\right\|_J & (2.29)\\
&\leq K \max_{0 \leq i \leq n-1} \left\|\widetilde{X}^i\right\|_J \sum_{0 \leq i \leq n-1} \|\overline{a_i}\| & (2.30)\\
&\leq KK' \max_{0 \leq i \leq n-1} \left\|\widetilde{X}^i\right\|_J \|\overline{g}\| & (2.31)
\end{aligned}$$

Thus, φ is continuous. We would conclude by noting that $J/(\alpha A[X]) = \ker \varphi$. □

Remark 2.14 *If the ideal J contains a unitary polynomial:*

$$\alpha(X) = \alpha_0 + \alpha_1 X + ... + \alpha_{n-1}X^{n-1} + X^n, \text{ with} \sum_{0\leq i\leq n-1} \|\alpha_i\| \leq 1 \tag{2.32}$$

*then, taking into account to the fact that in this case $\|.\|_\alpha$ is the quotient norm of $\|.\|$, the assertion **ii)** is equivalent to the following one:*

$$\textbf{\textit{iii)}}\ J \text{ is a closed ideal of } (A[X], \|.\|). \tag{2.33}$$

Remark 2.15 *The algebra of polynomials $A[X]$ is itself a simple extension of A. It cannot be provided with any Banach algebra norm. However, it admits noncomplete algebra norms.*

Finally, here is an example of a simple extension which admits no algebra norm (complete or not).

Example 2.16 *Let A be a unital and commutative Banach algebra admitting an element x, which is not an algebraic divisor of 0, whose spectrum is zero. Such an algebra exists (for example, take A the unitization of the convolution algebra $L^1([0,1])$ (which is radical) and consider x the element of $L^1([0,1])$ defined by $x(t) = 1$, for every $t \in [0,1]$). We consider the simple extension:*

$$B = A[X]/(xX - e)A[X]. \tag{2.34}$$

It is indeed an extension of A since

$$(xX - e)A[X] \cap A = \{0\}. \tag{2.35}$$

One has $Sp_B(x) = \varnothing$. Then it follows that B contains a nontrivial (other than $\mathbb{C}$) subalgebra which is a field. By Gelfand–Mazur's theorem, algebra B cannot be provided with any algebra norm.

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