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Isotropic Riemannian submersions

Feyza Esra ERDOĞAN*, **Bayram ŞAHİN**
Department of Mathematics, Faculty of Science, Ege University, İzmir, Turkey
Dedicated to the 60th birthday of Professor Rıfat Güneş

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Abstract: In this paper, we present the notion of isotropic submersions between Riemannian manifolds. We first give examples to illustrate this new notion. Then we express a characterization in terms of O'Neill's tensor field T and examine certain relations between sectional curvatures of the total manifold and the base manifold. We also study λ-isotropic submersions with pointwise planar horizontal sections.

Key words: Riemannian submersion, planar geodesic immersion, isotropic submersion, isotropic immersion

1. Introduction

A submanifold (M,g) of a Riemannian manifold (N,g) is called isotropic if and only if for any point p and any tangent vector X at a point p, we have that

$$g(h(X,X),h(X,X)) = \lambda(p)g(X,X)^2 \tag{1.1}$$

where h denotes the second fundamental form of the immersion and λ is a function on the submanifold. Isotropic submanifolds were first introduced by O'Neill in [24]. It is known that every umbilical submanifold is an isotropic submanifold, but the converse is not true. Several properties of isotropic immersions were explored in [5, 21, 22].

Riemannian submersions were independently defined and studied by O'Neill [25] and Gray [15] as counterpart of isometric immersions in semi-Riemannian, Lorentzian, almost Hermitian, contact and quaternionic geometry. The most studied notion in terms of smooth maps in differential geometry is isometric immersion and many textbooks and monographs about this concept have been published [10, 11, 13]. Although not as much as isometric immersions, there are a few books on Riemannian submersions [14, 17, 18, 29]. Today, many research articles on Riemannian submersions have been published and new Riemannian submersions under various names, such as semi-Riemannian submersion, Lorentzian submersion [14], almost Hermitian submersion [33], contact-complex submersion [17], quaternionic submersion [18] etc., have been introduced into the literature. Riemannian submersions have applications in other research areas. Indeed, Riemannian submersions have been used in Kaluza-Klein model and superstring theories of mathematical physics, [6, 14] and in robotic theory as forward kinematics [3].

Recently, Şahin [27] introduced the notion of antiinvariant Riemannian submersions which are Riemannian submersions from almost Hermitian manifolds such that the vertical distribution is antiinvariant under the

*Correspondence: feyza.esra.erdogan@ege.edu.tr
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almost complex structure of the total manifold. Later this notion has been extended for several cases, see: [1, 2, 4, 8, 9, 16, 19, 20, 23, 26, 29, 30, 32].

We note that the concept of submanifold with planar normal sections was introduced by Chen in [12]. Such submanifolds were studied widely by many authors and many nice geometric properties of such submanifolds have been found, see [11]. On the other hand, the notion of planar horizontal sections for Riemannian submersions has been introduced in [7].

In this paper, we introduce isotropic Riemannian submersions, give examples, obtain a characterization and investigate relations between the total manifold and the base manifold of isotropic Riemannian submersions. We also obtain a relation between isotropic submersions and Riemannian submersions having planar horizontal sections. The paper is organized as follows. In Section 2, we recall some concepts which are necessary for this paper. In Section 3, we define isotropic Riemannian submersions and we give two examples. Then we give necessary and sufficient conditions for such submersions to be totally geodesic, totally umbilical and minimal. We examined the correlation between curvatures of the total manifold and the base manifold. Moreover we relate the new concept with the one of Riemannian submersion with pointwise planar horizontal sections. In fact, we show that if a Riemannian submersion has geodesic 2-planar horizontal sections, such submersions are constant isotropic.

2. Preliminaries

Let M be an m-dimensional Riemannian manifold and N an n-dimensional Riemannian manifold. A Riemannian submersion $G : M \longrightarrow N$ is a map of M onto N satisfying the following axioms:

(S1) G has maximal rank.

(S2) The differential G_* preserves the lengths of horizontal vectors.

For any point $x \in N$, the leaf $G_x = G^{-1}(x)$ is a submanifold of M with $r = (m-n)-$dimensional. The integrable distribution of G is defined by $\vartheta_p = KerG_{*p}$ and ϑ is called the vertical distribution of submersion G. For any point $p \in M$, putting $G(p) = x$, the tangent space at p of G_x and ϑ_p coincide. The distribution $\varkappa$ orthogonal to ϑ is called the horizontal distribution. Thus for every $p \in M$, M has the follow decomposition:

$$T_pM = \vartheta_p \oplus \varkappa_p = \vartheta_p \oplus (\vartheta_p)^{\perp}.$$

The geometry of Riemannian submersion is characterized by O'Neill's tensors T and A defined for vector fields E, F on M by

$$A_E F = \varkappa \nabla_{\varkappa E} \vartheta F + \vartheta \nabla_{\varkappa E} \varkappa F, \;\; T_E F = \varkappa \nabla_{\vartheta E} \vartheta F + \vartheta \nabla_{\vartheta E} \varkappa F \tag{2.1}$$

where ∇ is the Levi-Civita connection of M. One can easily see that a Riemannian submersion $G : (M^m, g) \to (\bar{M}^n, g')$ has totally geodesic fibres if and only if T vanishes identically. It is also easy to see that T is vertical, $T_E = T_{\vartheta E}$, and A is horizontal, $A_E = A_{\varkappa E}$. We note that the tensor field T satisfy

$$T_U W = T_W U \qquad \forall U, W \in \Gamma(\ker G_*),$$

from here we have

$$\nabla_V W = T_V W + \hat{\nabla}_V W \qquad \forall V, W \in \Gamma(\ker G_*),$$

where $\hat{\nabla}_V W = \vartheta \nabla_V W$.

We now recall some theorems and Lemmas which will be used in this paper.

Theorem 2.1 *[25] Let (M^m,g), $(\bar{M}^n,g')$ be Riemannian manifolds and $G:(M^m,g)\to(\bar{M}^n,g')$ a Riemannian submersion. If $\alpha:I\to M$ is regular curve and $E(t)$, $W(t)$ denote the horizontal and vertical parts of its tangent vector field, then α is a geodesic on M if and only if*

$$(\bar{\nabla}_{\dot{\alpha}}W+T_WE)(t)=0$$

and

$$(\bar{\nabla}_EE+2A_EW+T_WW)(t)=0,$$

where $\bar{\nabla}$ is Schouten connection.

Lemma 2.2 *[25] Let (M^m,g), $(\bar{M}^n,g')$ be Riemannian manifolds and $G:(M^m,g)\to(\bar{M}^n,g')$ a Riemannian submersion. We have*

$$\begin{aligned}(\nabla_VA)_W &= -A_{T_vW}, \qquad (\nabla_XT)_Y=-T_{A_XY},\\ (\nabla_XA)_W &= -A_{A_XW,} \qquad (\nabla_VT)_Y=-T_{T_vY},\end{aligned}$$

where $X,Y\in\chi^{\vartheta}(M)$, $W,V\in\chi^h(M)$ and $E\in\chi(M)$. where $X,Y\in\chi^{\vartheta}(M)$, $W,V\in\chi^h(M)$ and $E\in\chi(M)$. Also $\hat{R}$, stand for the Riemannian curvature of any fibres $(G^{-1}(x),\hat{g}_x)$. Then the corresponding Gauss and Codazzi equations lead to:

$$\begin{aligned}R(U,V,F,W) &= \hat{R}(U,V,F,W)+g(T_UW,T_VF)-g(T_VW,T_UF),\\ R(U,V,W,X) &= g((\nabla_VT)(U,W),X)-g((\nabla_UT)(V,W),X),\end{aligned}\tag{2.2}$$

for any $U,V,W,F\in\chi^{\vartheta}(M)$, $X\in\chi^h(M)$. On the other hand for a vector field $E\in\chi(M)$ and $V\in\chi^{\vartheta}(M)$, we have

$$(\nabla_VT)_VE=\nabla_VT_VE-T_{\nabla_VV}E-T_V\nabla_VE.\tag{2.3}$$

Theorem 2.3 *[25] Let (M^m,g), $(\bar{M}^n,g')$ be Riemannian manifolds and $G:(M^m,g)\to(\bar{M}^n,g')$ a Riemannian submersion. Therefore, we have for the sectional curvature K, K', $\hat{K}$ of the total space, the base space and the fibres, one has respectively*

$$\begin{aligned}K(U.V) &= \hat{K}(U,V)+\|T_UV\|^2-g(T_UU,T_VV),\\ K(X,Y) &= K'(X',Y')\circ\pi-3\|A_XY\|^2,\\ K(X,V) &= g((\nabla_XT)_VV,X)+\|T_VX\|^2-\|A_XV\|^2,\end{aligned}$$

for any $X,Y\in\varkappa$, $\quad U,V\in\vartheta$.

We now recall that a Riemannian submersion G is called a Riemannian submersion with totally umbilical fibers if

$$T_VW=g(V,W)H,$$

for $V,W\in\Gamma(\ker G_*)$, where H is the mean curvature vector field of the fibers.

Finally in this section, we will recall the notion of Riemannian submersion with planar horizontal sections.

Definition 2.4 *Let (M,g) be a m-dimensional Riemannian manifold and $(E^n,\langle,\rangle)$ be a n-dimensional Euclidean space. Consider a Riemannian submersion $G:E^n\to M$ and denote its vertical distribution and horizontal distribution by ϑ and $\varkappa$, respectively. It is known that the vertical distribution ϑ is always integrable. We denote the integral manifold of ϑ by $\tilde{M}$. For $p\in\tilde{M}$ and a nonzero vector $X\in\vartheta_p$, we define the $(m+1)-$dimensional affine subspace $E(p,X)$ of E^n by*

$$E(p,X)=p+Span\{X,\varkappa_p\}.$$

In a neighbourhood of p, the intersection $\tilde{M}\cap E(p,X)$ is a regular curve $\alpha:(-\varepsilon,\varepsilon)\to\tilde{M}$. We suppose that the parameter $t\in(-\varepsilon,\varepsilon)$ is a multiple of the arc-length such that $\alpha(0)=p$ and $\alpha'(0)=X$. Each choice of $X\in\vartheta_p$ yields a different curve. We will call α the horizontal section curve of $\tilde{M}$ at p in the direction of X. The Riemannian submersion G is said to have " pointwise k-planar horizontal sections $(Pk-PHS)$ " if for each horizontal section α, the higher order derivatives

$$\{\alpha'(0),\alpha''(0),...,\alpha^{k+1}(0)\}$$

are linearly dependent.

Thus a horizontal section can be written as below:

$$\alpha(t)=p+\lambda(t)X+U(t)$$

where $U\in\chi^{\varkappa}(E^n)$, $\lambda(t)\in\mathbb{R}$ [7].

Theorem 2.5 *Let $G:(E^n,\langle,\rangle)\to(M,g_M)$ be a Riemannian submersion. Then G has $P2-PHS$ if and only if $(\nabla_XT)_XX$ and T_XX satisfy*

$$(\nabla_XT)_XX\wedge T_XX=0,$$

for any $X\in\chi^{\vartheta}(E^n)$ [7].

3. Isotropic submersions

In this section, we are going to introduce isotropic submersions, give examples, obtain a characterization and investigate the effect of this notion on the geometry of the total manifold and the base manifold of the Riemannian submersion.

Definition 3.1 *Let $G:M\longrightarrow N$ be a Riemannian submersion. For $u\in\Gamma(\ker G_*)$, if the following condition is satisfied*

$$g(T_uu,T_uu)=\lambda g(u,u)^2,$$

for all $p\in M$, then G is called λ-isotropic. If λ is constant for any $p\in M$, then G is called $\lambda-$ constant isotropic.

We first give the following theorem which will be very useful for investigating isotropic Riemannian submersions.

Theorem 3.2 *Let $G:(M^m,g)\to(\tilde{M}^n,g')$ be a Riemannian submersion. If G is λ-isotropic, then we have*

$$g(T_XX,T_XY)=0$$

for all orthogonal $X,Y\in\chi^{\vartheta}(M)$.

Proof For all $U \in \chi^{\vartheta}(M)$ we have

$$T_U U = \varkappa \nabla_U U.$$

Hence, if T is isotropic, we find

$$g(\varkappa \nabla_U U, \varkappa \nabla_U U) = \lambda g(U,U)^2.$$

Now, let $\triangle$ be the quadrilinear function on M,

$$\triangle : \chi^{\vartheta}(M) \times \chi^{\vartheta}(M) \times \chi^{\vartheta}(M) \times \chi^{\vartheta}(M) \to C^{\infty}(M, \mathbb{R})$$

as

$$\triangle(X,Y,U,V) = g(T_X Y, T_U V) - \lambda g(X,Y)g(U,V).$$

Because T is symmetric, $\triangle(X,Y,U,V)$ is symmetric in X and Y, and also in U and V. Also $\triangle$ is symmetric by pairs: $\triangle(X,Y,U,V) = \triangle(U,V,X,Y)$. If G is λ isotropic, then we have

$$\psi(U) = \triangle(U,U,U,U) = 0,$$

for all $U \in \chi^{\vartheta}(M)$. From here,

$$\psi(X+Y) + \psi(X-Y) = 0, \tag{3.1}$$

$$\psi(X+Y) - \psi(X-Y) = 0 \tag{3.2}$$

for all $X, Y \in \chi^{\vartheta}(M)$. Now, $\psi(X+Y)$, $\psi(X-Y)$ can be found as

$$\begin{aligned} \psi(X+Y) = \; & 4[g(T_X X, T_X Y) - \lambda g(X,X)g(X,Y)] \\ & +2[g(T_X X, T_Y Y) - \lambda g(X,X)g(Y,Y)] \\ & +4[g(T_X Y, T_X Y) - \lambda g(X,Y)^2] \\ & +4[g(T_X Y, T_Y Y) - \lambda g(X,Y)g(Y,Y)], \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \psi(X-Y) = \; & -4[g(T_X X, T_X Y) - \lambda g(X,X)g(X,Y)] \\ & +2[g(T_X X, T_Y Y) - \lambda g(X,X)g(Y,Y)] \\ & +4[g(T_X Y, T_X Y) - \lambda g(X,Y)^2] \\ & -4[g(T_X Y, T_Y Y) - \lambda g(X,Y)g(Y,Y)], \end{aligned} \tag{3.4}$$

Using (3.3),(3.4),(3.1) and (3.2) we have

$$\begin{aligned} \psi(X+Y) = \; & 4\triangle(X,X,X,Y) + 2\triangle(X,X,Y,Y) \\ & +4\triangle(X,Y,X,Y) + 4\triangle(X,Y,Y,Y), \end{aligned} \tag{3.5}$$

$$\begin{aligned} \psi(X-Y) = \; & -4\triangle(X,X,X,Y) + 2\triangle(X,X,Y,Y) \\ & +4\triangle(X,Y,X,Y) - 4\triangle(X,Y,Y,Y), \end{aligned} \tag{3.6}$$

and hence we find

$$\frac{1}{4}(\psi(X+Y)+\psi(X-Y)) = \triangle(X,X,Y,Y)+2\triangle(X,Y,X,Y) = 0. \tag{3.7}$$

Also, we have

$$\frac{1}{8}(\psi(X+Y)-\psi(X-Y)) = \triangle(X,X,X,Y)+\triangle(Y,Y,Y,X) = 0. \tag{3.8}$$

If we replace Y by $X+Y$ in (3.7), we obtain

$$6\triangle(X,X,X,Y)+\triangle(X,X,Y,Y)+2\triangle(X,Y,X,Y) = 0. \tag{3.9}$$

If we consider (3.7) in (3.9), we get

$$\triangle(X,X,X,Y) = 0.$$

Thus we arrive at

$$g(T_X X, T_X Y) = \lambda g(X,X)g(X,Y).$$

So, we get the assertion. □

We now give examples of isotropic submersions.

Example 3.3 *Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds, $f : M_1 \to (0,\infty)$ and $\pi : M_1 \times M_2 \to M_1$, $\sigma : M_1 \times M_2 \to M_2$ the projection maps given by $\pi(x,y) = x$ and $\sigma(x,y) = y$ for every $(x,y) \in M_1 \times M_2$. Denote the warped product manifold $M = (M_1 \times_f M_2, g)$, where*

$$g(X, Y) = g_1(\pi_* X, \pi_* Y) + f(\pi(x,y))\, g_2(\sigma_* X, \sigma_* Y)$$

for every X and Y of M and $*$ is symbol for the tangent map. The manifolds M_1 and M_2 are called the base and the fiber of M. It is easy to prove that the first projection $\pi : M_1 \times_f M_2 \longrightarrow M_1$ is a Riemannian submersion whose vertical and horizontal spaces at any point $p = (p_1, p_2)$ are respectively identified with $T_{p_2}M_2, T_{p_1}M_1$. For the invariant $\mathcal{T}$, for any $U, V \in \mathfrak{X}^{\vartheta}(M)$, one obtains:

$$\mathcal{T}_U V = -\frac{1}{2f} g(U,V) gradf. \tag{3.10}$$

(3.10) shows that π is an isotropic Riemannian submersion[14].

The following example includes the above isotropic Riemannian submersions.

Example 3.4 *Every Riemannian submersion with totally umbilical fibers is an isotropic Riemannian submersion.*

We now give an another example of an isotropic Riemannian submersion.

Example 3.5 *Let* $(\mathbb{R}^4, g)$ *and* $(\mathbb{R}^2, \bar{g})$ *be Riemannian manifolds with the Riemannian metrics* $g = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$ *and* $\bar{g} = \frac{1-\lambda}{\lambda}(dx_1^2 + dx_2^2)$, $0 < \lambda < 1$. *Let's consider the following map*

$$\begin{array}{rcl} G : \mathbb{R}^4 & \to & \mathbb{R}^2 \\ (x_1, x_2, x_3, x_4) & \to & (\sqrt{1 - x_1^2 - x_2^2}, \sqrt{1 - x_3^2 - x_4^2}), \end{array}$$

such that $x_2^2 + x_4^2 = \lambda < 1, \qquad x_1^2 + x_3^2 = \lambda < 1$. *It follows that* G *is a submersion. By direct computation, If we consider above condition, we have*

$$\ker G_* = \vartheta = span\left\{ v = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}, \qquad u = x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4} \right\}$$

and

$$\ker G_*^{\perp} = \varkappa = span\left\{ X = -x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}, \qquad Y = -x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4} \right\}.$$

Since we have

$$g(G_* X, G_* X) = \bar{g}(X, X) \qquad g(G_* Y, G_* Y) = \bar{g}(Y, Y).$$

G *is a Riemannian submersion. Also, we obtain*

$$T_v v = X, \;\; T_u u = Y$$

and

$$g(T_u u, T_u u) = g(T_v v, T_v v) = \lambda.$$

Hence, we show that G *is an isotropic Riemannian submersion.*

Lemma 3.6 *Let* $G : (M^m, g) \to (\bar{M}^n, g')$ *be a* $\lambda-$ *isotropic Riemannian submersion. Then we have*

1. $$g(T_{u_1} u_1, T_{u_2} u_2) + 2g(T_{u_1} u_2, T_{u_1} u_2) = \lambda, \;\; for \; \|u_1\| = \|u_2\| = 1,$$

2. $$g(T_{u_1} u_1, T_{u_3} u_4) + 2g(T_{u_1} u_3, T_{u_1} u_4) = 0,$$

3. $$g(T_{u_1} u_2, T_{u_3} u_4) + g(T_{u_1} u_3, T_{u_2} u_4) + g(T_{u_1} u_4, T_{u_2} u_3) = 0,$$

for all orthogonal $u_1, u_2, u_3, u_4 \in \chi^{\vartheta}(M)$.

Proof For any $u_1, u_2, u_3, u_4 \in \chi^{\vartheta}(M)$, we have (3.8) and (3.7), namely

$$\triangle(u_1, u_1, u_1, u_2) + \triangle(u_2, u_2, u_2, u_1) = 0, \tag{3.11}$$

and

$$\triangle(u_1, u_1, u_2, u_2) + 2\triangle(u_1, u_2, u_1, u_2) = 0. \tag{3.12}$$

If we replace u_2 by $u_1 + u_2$ in (3.12), we find

$$\triangle(u_1, u_1, u_1, u_2) = 0. \tag{3.13}$$

If we replace u_2 by $u_3 + u_4$ in equation (3.12), we obtain

$$\triangle(u_1, u_1, u_3, u_4) + 2\triangle(u_1, u_3, u_1, u_4) = 0. \tag{3.14}$$

If we replace u_1 by $u_1 + u_2$ in equation (3.14), we have

$$\triangle(u_1, u_2, u_3, u_4) + \triangle(u_1, u_3, u_2, u_4) + \triangle(u_2, u_3, u_1, u_4) = 0. \tag{3.15}$$

Now suppose that u_1, u_2, u_3, u_4 are orthogonal and use (3.12), we can find

$$g(T_{u_1}u_1, T_{u_2}u_2) + 2g(T_{u_1}u_2, T_{u_1}u_2) = \lambda. \tag{3.16}$$

Using (3.14), we obtain

$$g(T_{u_1}u_1, T_{u_3}u_4) + 2g(T_{u_1}u_3, T_{u_1}u_4) = 0,$$

and equation (3.15), implies

$$g(T_{u_1}u_2, T_{u_3}u_4) + g(T_{u_1}u_3, T_{u_2}u_4) + g(T_{u_1}u_4, T_{u_2}u_3) = 0.$$

Thus the proof is completed. □

Let $G : (M^m, g) \to (\bar{M}^n, g')$ be a λ isotropic Riemannian submersion. Considering Theorem 2.3 and Lemma 3.6, from (3.16), we have

$$g(T_u u, T_v v) + 2\left\|T_u v\right\|^2 = \lambda. \tag{3.17}$$

By direct computations we find

$$K(u, v) = \hat{K}(u, v) + 3\left\|T_u v\right\|^2 - \lambda,$$

and

$$2K(u, v) = 2\hat{K}(u, v) - 3g(T_u u, T_v v) + \lambda.$$

Therefore, we have the following result.

Lemma 3.7 *Let $G : (M^m, g) \to (\bar{M}^n, g')$ be a λ isotropic Riemannian submersion. For orthonormal $u, v \in \chi^{\vartheta}(M)$ we have*

$$K(u, v) = \hat{K}(u, v) + 3\left\|T_u v\right\|^2 - \lambda, \tag{3.18}$$

$$2K(u, v) = 2\hat{K}(u, v) - 3g(T_u u, T_v v) + \lambda. \tag{3.19}$$

By using the above Lemma 3.7 we have the following result.

Proposition 3.8 *Let $G : (M^m, g) \to (\bar{M}^n, g')$ be a λ isotropic Riemannian submersion. For ortonormal $u, v \in \chi^{\vartheta}(M)$ the following expressions are equivalent to each other;*

1. $K(u,v) = \hat{K}(u,v) - \lambda$,

2. $T_u v = 0$.

Proof Let us assume that $K(u,v) = \hat{K}(u,v) - \lambda$. From (3.18) we have $\|T_u v\| = 0$, namely $T_u v = 0$. Conversely, if $T_u v = 0$, from Lemma 3.6 (1), we find $g(T_u u, T_v v) = \lambda$ and from (3.19) we obtain $K(u,v) = \hat{K}(u,v) - \lambda$. □

Using Proposition 3.8, we have the following result.

Theorem 3.9 *Let $G : (M^m, g) \to (\bar{M}^n, g')$ be a λ isotropic Riemannian submersion.For any vertical plane P spanned by vectors u, v one has $K(P) = \hat{K}(P) - \lambda$ if and only if G is a Riemannian submersion with minimal fibres.*

Proof From Proposition 3.8, we have $T_u v = 0$ if and only if $K(u,v) = \hat{K}(u,v) - \lambda$. Namely,

$$T_u v = g(u,v)H,$$

where H is the mean curvature vector field. If we take x and $x+y$ instead of u and v in $T_u v = 0$, respectively, we have

$$T_u v = 0 \Longrightarrow T_x x = 0.$$

If we take $x+y$ and y instead of u and v in $T_u v = 0$, respectively, we have

$$T_u v = 0 \Longrightarrow T_y y = 0.$$

Therefore, we find $H = 0$. □

Using Proposition 3.8 we have the following results.

Corollary 3.10 *Let $G : (M^m, g) \to (\bar{M}^n, g')$ be a $\lambda-$ isotropic Riemannian submersion. $\{u_j\}_{1 \leq j \leq r}$ is local orthonormal frame of vertical distribution of M^m. For horizontal vector field $N = \sum_{j=1}^{r} T_{u_j} u_j$ on (M^m, g), we have*

$$\|N\|^2 = r^2 \lambda.$$

Corollary 3.11 *Let $G : (M^m, g) \to (\bar{M}^n, g')$ be a $\lambda-$ isotropic Riemannian submersion. For a geodesic curve $\alpha : I \to M^m, \gamma = G \circ \alpha$ is geodesic if and only if*

$$\|A_E W\|^2 = \frac{\lambda}{4} \|W\|^2,$$

where $E(t), W(t)$ are horizontal, vertical components of the vector field tangent to α, respectively.

Let $G : (M^m, g) \to (\bar{M}^n, g')$ be a constant λ isotropic Riemannian submersion. For any (local) unitary vector field u, we have

$$g(T_u u, T_u u) = \lambda.$$

Hence we obtain

$$g(\nabla_u T_u u, T_u u) = 0 \Rightarrow g((\nabla_u T)_u u, T_u u) + 2g(T_u \nabla_u u, T_u u) = 0.$$

If $(\nabla_u T)_u u = 0$, then we have $g(T_u \nabla_u u, T_u u) = 0$. If $\nabla_u u \in \chi^h(M^m)$, then we can conclude that $\nabla_u u \perp u$. If $\nabla_u u \in \chi^\vartheta(M^m)$, then we find $T_u u = 0$. Hence we have, G is a Riemannian submersion totally geodesic fibres from $T = 0$. The converse is obvious. Thus we find the following theorem.

Theorem 3.12 *Let $G : (M^m, g) \to (\bar{M}^n, g')$ be a constant λ isotropic Riemannian submersion. Then if $(\nabla_u T)_u u = 0$ for any $u \in \vartheta_p$ one of the following assertions are valid*

1. *$\nabla_u u \perp u$ in $\chi^\vartheta(M^m)$.*
2. *G is Riemannian submersion with totally geodesic fibres.*

We now investigate a Riemannian submersion G from Euclidean space $\mathbb{E}^n$ to a Riemannian manifold M having geodesic $P2 - PHS$.

Theorem 3.13 *Let $G : \mathbb{E}^n \to M$ be a Riemannian submersion from Euclidean space $\mathbb{E}^n$ to a Riemannian manifold M. If G has geodesic 2-planar horizontal sections, then G is a constant isotropic Riemannian submersion at $p \in \tilde{M}$. Here, $\tilde{M}$ is the integral manifold of vertical distribution.*

Proof We suppose that G has geodesic $2-$planar horizontal sections. Let α be a horizontal section curve at $\alpha(0) = p \in M$ and direction of $\alpha'(0) = v \in \vartheta$. Since G has geodesic $2-$planar horizontal sections, we have

$$\begin{aligned} \alpha'(0) &= v, \\ \alpha''(0) &= \nabla_v v = T_v v + \hat{\nabla}_v v = T_v v \end{aligned}$$

and

$$\alpha'''(0) = \nabla_v T_v v = T_v T_v v + h(\nabla_v T_v v).$$

Here α''', is linear combination of v and $\varkappa_p$, thus we find

$$\begin{aligned} T_v T_v v &= av, \\ h(\nabla_v T_v v) &= bT_v v, \end{aligned}$$

$a, b \in \mathbb{R}$. In this case, from $T_v T_v v = av$, we have

$$T_v T_v v \wedge v = 0.$$

Hence, we find that

$$< T_v T_v v, z >= 0,$$

for any orthogonal vertical vectors v, z at $p \in \tilde{M}$. On the other hand, from the property of tensor field T, we have

$$< T_v T_v v, z >= 0 \Leftrightarrow < T_v v, T_v z >= 0. \tag{3.20}$$

Since G has geodesic $P2 - PHS$, we have $cT_v v = (\nabla_v T)_v v,\ c \in \mathbb{R}$ and $\hat{\nabla}_v v = 0$, thus, we obtain

$$\begin{aligned} < \quad & T_v v, T_v z >= 0 \Leftrightarrow < (\nabla_v T)_v v, T_v z >= 0, \\ \Leftrightarrow \quad & < \nabla_v T_v v - T_{\nabla_v v} v - T_v \nabla_v v, T_v z >= 0, \\ \Leftrightarrow \quad & < \nabla_v T_v v - T_{T_v v + \hat{\nabla}_v v} v - T_v(T_v v + \hat{\nabla}_v v), T_v z >= 0, \\ \Leftrightarrow \quad & < \nabla_v T_v v - T_{T_v v} v - T_v T_v v, T_v z >= 0, \end{aligned}$$

because of $T_v v \in \varkappa_p$, we get $T_{T_v v} v = 0$. Therefore, we obtain

$$< \nabla_v T_v v, T_v z > - < T_v T_v v, T_v z >= 0.$$

Since $T_v T_v v \in \vartheta_p$ and $T_v z \in \varkappa_p$, we find

$$< \nabla_v T_v v, T_v z >= 0,$$

from (3.20), we get

$$< T_v v, \nabla_v T_v z >= 0.$$

From the covariant derivative of tensor field T, we can write

$$< T_v v, (\nabla_v T)_v z + T_{\nabla_v v} z + T_v \nabla_v z >= 0.$$

Since $T_{\nabla_v v} z = T_{T_v v} z + T_{\hat{\nabla}_v v} z = 0$, if we expand z to parallel vector field Z through horizontal section curve α, we can get $\nabla_v Z = 0$, thus we have

$$< T_v v, (\nabla_v T)_v Z >= 0.$$

From (2.2), we can write

$$< T_v v, (\nabla_Z T)_v v >= 0.$$

In this equation, if we take the covariant derivative of the tensor field T, we can write

$$\begin{aligned} < \quad & T_v v, \nabla_Z T_v v - T_{\nabla_Z v} v - T_v \nabla_Z v >= 0, \\ < \quad & T_v v, \nabla_Z T_v v - T_{T_Z v} v - T_{\hat{\nabla}_Z v} v - T_v T_Z v - T_v \hat{\nabla}_Z v >= 0. \end{aligned} \tag{3.21}$$

Since $T_v T_Z v \in \vartheta_p$ and $T_v v \in \varkappa_p$, we have,

$$\begin{aligned} < \quad & T_v v, \nabla_Z T_v v - 2 T_v \hat{\nabla}_Z v >= 0, \\ < \quad & T_v v, \nabla_Z T_v v >= 2 < T_v v, T_v \hat{\nabla}_Z v > . \end{aligned} \tag{3.22}$$

From the property of tensor field T and $T_v T_v v = av$, we find

$$< T_v v, T_v \hat{\nabla}_Z v >=< T_v T_v v, \hat{\nabla}_Z v >,$$

$$< T_v T_v v, \hat{\nabla}_Z v >= a < v, \hat{\nabla}_Z v > .$$

Also, for $v \in \vartheta_p$ unit vertical vector, we have $< v, \nabla_Z v >= 0 \Leftrightarrow < v, T_v Z + \hat{\nabla}_Z v >= 0 \Leftrightarrow < v, \hat{\nabla}_Z v >= 0$. Thus, from (3.21) we obtain

$$< T_v v, \nabla_Z T_v v - T_{T_Z v} v - T_{\hat{\nabla}_Z v} v >= 0.$$

Since, for $\hat{\nabla}_Z v \in \vartheta_p$, $T_{\hat{\nabla}_Z v} v = 0$ and $< T_v v, T_{T_Z v} v >=< T_v v, T_v T_Z v >= 0$. That is, from (3.22), we get

$$\begin{aligned} < \quad & T_v v, \nabla_Z T_v v >= 0, \\ < \quad & T_v v, \nabla_Z T_v v >= 0 \Leftrightarrow \frac{1}{2} Z < T_v v, T_v v >= 0. \end{aligned}$$

Therefore, we can say that, G is a constant isotropic Riemannian submersion.

□

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