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Games relating to weak covering properties in bitopological spaces

Ali Emre EYSEN[1], **Selma ÖZÇAĞ**[2,*]
[1]Department of Mathematics, Faculty of Science, Trakya University, Edirne, Turkey
[2]Department of Mathematics, Faculty of Science, Hacettepe University, Ankara, Turkey

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Abstract: We study topological games related to weak forms of the Menger property in bitopological spaces. In particular we investigate almost Menger game and its connections to games which are associated with the covering properties consisting of covers containing G_δ subsets.

Key words: Almost Menger, almost Alster, bitopological space, selection principle, topological game

1. Introduction

It was shown in [10] that almost σ-compact bitopological spaces are almost Alster. In this paper, after observing that the almost σ-compact property implies the property that TWO has a winning strategy in almost Menger game, it is of interest to know whether the almost Alster property and TWO has a winning strategy in the almost Menger game may be related in bitopological spaces.

We start by recalling the Menger covering property.

Definition 1.1 *[11] A topological space X is Menger if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a cover of X.*

After Hurewicz proved the theorem explaining the relationship between the Menger property and games (see Theorem 1.3), many mathematicians have worked on infinitely long, two-person games which are naturally associated to selection principles (see [1, 3, 4, 11, 20]).

In recent years many papers on weak covering properties in topological spaces have appeared in the literature. On the other hand, there are quite a few studies on the weak versions of covering properties in bitopological spaces. They are mainly related to weak versions of the Menger property such as almost Menger, weakly Menger and their connections with the Menger property in bitopological spaces (see [9, 10, 18]). However there is no systematic study of weak covering properties and their relations with game theory in bitopological context. In this paper we begin such a study and investigate game-theoretic properties of selection principles related to weaker forms of the Menger properties in bitopological spaces.

The layout of the paper is as follows: The remainder of this introduction is given over to some background material.

*Correspondence: sozcag@hacettepe.edu.tr
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Section 2 gives the definition of almost Menger game in bitopological spaces and a characterization of almost Menger bitopological spaces in terms of the almost Menger game.

Section 3 begins by the question whether we can find a game dual to the almost Alster game in bitopological spaces. Based on this question we turned our direction to almost Alster bitopological spaces. Our topological terminology and notations are as in the book [7] and standard reference for bitopological spaces is [6]. For a detailed review on weak forms of classical covering properties see [15].

Now we recall some basic definitions and results which will enable the reader to follow the paper. Some other notions will be defined throughout the sections as will be needed.

Selection principles

Many topological properties are defined in terms of the following two classical selection principles [19].

Let $\mathcal{A}$ and $\mathcal{B}$ be families of sets. Then: $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ is the selection hypothesis: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that for each n, $B_n \in A_n$ and $\{B_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subset A_n$ and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

In [11] Hurewicz proved that the Menger basis property which was introduced by Menger in [16] is equivalent to the property $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$, where $\mathcal{O}$ denotes the family of open covers of the space and nowadays this selection principle is known as the Menger property.

Topological games

The concept of topological games was introduced by Telgársky [20]. There are games $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ and $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$ which are associated to the selection principles $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ and $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$, respectively.

The symbol $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ denotes the infinitely long game for two players, ONE and TWO, who play an inning for each $n \in \mathbb{N}$. In the n-th inning ONE chooses a set $A_n \in \mathcal{A}$, while TWO responds by choosing an element $B_n \in A_n$. A play $(A_0, B_0, A_1, B_1, \ldots, A_n, B_n, \ldots)$ is won by TWO if $\{B_n : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

The symbol $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$ denotes the infinitely long game for two players, ONE and TWO, who play an inning for each for each $n \in \mathbb{N}$. In the n-th inning ONE chooses a set $A_n \in \mathcal{A}$, while TWO responds by choosing a finite subset $B_n \subset A_n$. A play $(A_0, B_0, A_1, B_1, \ldots, A_n, B_n, \ldots)$ is won by TWO if $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$; otherwise, ONE wins.

It is evident that if ONE does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ (resp. $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$), then the selection hypothesis $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ (resp. $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$) holds. The converse implication need not be always true.

If ONE is a player of a game G, we denote by ONE $\uparrow \mathsf{G}$ the fact that ONE has a winning strategy in G, and by ONE $\not\uparrow \mathsf{G}$ the fact that ONE does not have a winning strategy in G.

Definition 1.2 [11, 22] The Menger game on a topological space X is played as follows: In each inning $n \in \mathbb{N}$, ONE chooses an open cover $\mathcal{U}_n$ of X, and then TWO chooses a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$. TWO wins the play if $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a cover of X; otherwise, ONE is the winner.

In [11] Menger spaces were characterized by the Menger game and the following theorem was obtained by Hurewicz. For the proof of the theorem see [19].

Theorem 1.3 *[11, 19] A topological space X is Menger if and only if* ONE *does not have a winning strategy in the Menger game on X.*

The relation between Menger game and the other topological games were considered in [3, 20, 22]. In [5] Babinkostova, Pansera and Scheepers investigated game-theoretic properties of selection principles related to weaker forms of the Menger properties and characterized the almost Menger space by ONE does not have a winning strategy in the game $\mathsf{G}_{fin}(\mathcal{O},\overline{\mathcal{O}})$ where $\overline{\mathcal{O}}$ denotes the collection of families $\mathcal{U}$ of open sets in X with $\bigcup\{\mathrm{Cl}(U) : U \in \mathcal{U}\} = X$. More recently in [3] Aurichi and Dias studied the connections between Menger games and the games involving covers by G_δ subsets and obtained the following result:

A cover $\mathcal{U}$ of (X,τ) by G_δ subsets is said to be an Alster cover if every compact subset of X is included in some element of $\mathcal{U}$. A topological space X is an Alster space if every Alster cover $\mathcal{U}$ of X has a countable subcover. The reader is referred to [1–4, 17] for more background material on Alster spaces.

Theorem 1.4 *[3] If* TWO *has a winning strategy in the Menger game on a regular space X, then X is an Alster space.*

We conclude this section with a few words on bitopological spaces.

A set X equipped with two arbitrary topologies τ_1 and τ_2 is called a bitopological space and is denoted by (X,τ_1,τ_2) which was first introduced by Kelly in [12]. For a subset A of X, $\mathrm{Cl}_{\tau_i}(A)$ will denote the closure of A in (X,τ_i), $(i = 1,2)$. τ_i-open set means the open set with respect to topology τ_i on X. By τ_i-open cover we mean the cover of X by τ_i-open sets. Let $\mathcal{P}$ be some topological property. Then (i,j)-$\mathcal{P}$ denotes an analogue of this property for τ_i with respect to τ_j where $i,j \in \{1,2\}$ and $i \neq j$, and p-$\mathcal{P}$ denotes the conjunction $(1,2)$-$\mathcal{P}$ $\wedge$ $(2,1)$-$\mathcal{P}$ where "p" is the abbreviation for "pairwise". We note that (X,τ_i) has a property $\mathcal{P}$ if and only if the bitopological space (X,τ_1,τ_2) has a property i-$\mathcal{P}$ and d-$\mathcal{P}$ $\iff$ 1-$\mathcal{P}$ $\wedge$ 2-$\mathcal{P}$, where "d" is the abbreviation for "double".

This paper is largely based on previously unpublished work from the PhD thesis of the first author [8].

2. Games and the almost Menger property

In [14] Kočinac introduced the notion of almost Menger property in topological spaces and in [9, 10, 18] this notion was investigated in the bitopological context. Now in this section we consider the almost Menger game on bitopological spaces and relate this notion to (i,j)-almost Menger property.

Definition 2.1 [18] A bitopological space (X,τ_1,τ_2) is said to be (i,j)-*almost Menger* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite families such that for each n, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}} \bigcup_{V\in\mathcal{V}_n} \mathrm{Cl}_{\tau_j}(V)$.

The following game will be naturally associated to the above property.

Definition 2.2 The (i,j)-almost Menger game on a bitopological space (X,τ_1,τ_2) is played as follows: In each inning $n \in \mathbb{N}$, ONE chooses a τ_i-open cover $\mathcal{U}_n$ of X and then TWO chooses a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$. The play is won by TWO if

$$\bigcup_{n\in\mathbb{N}}\bigcup_{V\in\mathcal{V}_n} Cl_{\tau_j}(V)=X$$

otherwise, ONE is the winner.

We will denote the (i,j)-almost Menger game by $\mathsf{G}_{fin}(\mathcal{O}_i,\overline{\mathcal{O}}_j)$.

Recall that [10] a bitopological space (X,τ_1,τ_2) is (i,j)-almost σ-compact if it is the union of τ_j-closures of countably many τ_i-compact subsets. Then we have the following theorem:

Theorem 2.3 *If the bitopological space $\left(X,\tau_1,\tau_2\right)$ is (i,j)-almost σ-compact then* TWO *has a winning strategy in the (i,j)-almost Menger game.*

Proof Since $\left(X,\tau_1,\tau_2\right)$ is (i,j)-almost σ-compact there exists a sequence $(K_n:n\in\mathbb{N})$ of τ_i-compact subsets of X such that

$$\bigcup_{n\in\mathbb{N}} Cl_{\tau_j}(K_n)=X.$$

If $\mathcal{U}_n$ is n-th move of ONE in (i,j)-almost Menger game, then there exists a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$ such that $K_n\subseteq\cup\mathcal{V}_n$. If $\sigma(\mathcal{U}_0,\mathcal{U}_1,\ldots,\mathcal{U}_n)=\mathcal{V}_n$ for each $n\in\mathbb{N}$ then σ is a winning strategy for TWO. □

Theorem 2.4 *Let $\left(X,\tau_1,\tau_2\right)$ be a bitopological space and $\left(X,\tau_i\right)$ be a Lindelöf space. The following are equivalent:*

(1) *$\left(X,\tau_1,\tau_2\right)$ has the (i,j)-almost Menger property.*

(2) ONE *does not have a winning strategy in the game (i,j)-almost Menger.*

Proof The implication $(2)\Rightarrow(1)$ is clear.

$(1)\Rightarrow(2)$: We must prove that if $\left(X,\tau_1,\tau_2\right)$ is (i,j)-almost Menger bitopological space, then ONE has no winning strategy in (i,j)-almost Menger game.

The argument used here is due to Hurewicz [11] for Menger spaces and in Theorem 27 and Theorem 28 [5] Babinkostova, Pansera and Scheepers proved the similar argument for weakly Menger and almost Menger spaces.

The proof proceeds like for Theorem 27 in [5]. We will not repeat the whole proof but we will go through some important steps as follows. Let σ be a strategy for ONE. We may assume that each move of ONE according to the strategy σ, is an ascending ω-chain of τ_i-open sets covering X. Set $\sigma(\emptyset)=\{U_{(n)}:n\in\mathbb{N}\}$, listed in $\subset$-increasing order. For each n, list $\sigma(U_{(n)})=\{U_{(n,m)}:m\in\mathbb{N}\}$ in $\subset$-increasing order. Supposing that U_τ has been defined for each finite sequences τ of length at most k of positive integers, we now define $\sigma(U_{(n_1,n_2,\ldots,n_k)})=\{U_{(n_1,n_2,\ldots,n_k,m)}:m\in\mathbb{N}\}$ for each $(n_1,n_2,\ldots,n_k)$.

Now we define for each n and k:

$$U_k^n=\begin{cases}U_{(k)} & ,n=0\\ U_k^{n-1}\cap\Big(\bigcap_{\tau\in\mathbb{N}^n}U_{\tau\frown(k)}\Big) & ,n\neq 0\end{cases}$$

Then for each n the set $\mathcal{U}_n = \{U_k^n : k \in \mathbb{N}\}$ is an increasing chain of τ_i-open sets with each $\mathcal{U}_n$ is an τ_i-open cover of X. Now we apply the fact that (X, τ_1, τ_2) is (i,j)-almost Menger to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$. Then there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $X = \bigcup_{n \in \mathbb{N}} Cl_{\tau_j}(U_{f(n)}^n)$ so the sequence of moves $(U_{(f(0))}, U_{(f(0),f(1))}, \ldots, U_{(f(0),f(1),\ldots,f(n))}, \ldots)$ by TWO defeats ONE's strategy σ. □

Now we note that the following implications are always true for a bitopological space (X, τ_1, τ_2).

$$\begin{array}{c} X \text{ is } (i,j)\text{-almost } \sigma\text{-compact} \\ \Downarrow \\ \mathsf{TWO} \uparrow \mathsf{G}_{fin}(\mathcal{O}_i, \overline{\mathcal{O}}_j) \\ \Downarrow \\ \mathsf{ONE} \uparrow \mathsf{G}_{fin}(\mathcal{O}_i, \overline{\mathcal{O}}_j) \\ \Downarrow \\ X \text{ is } (i,j)\text{-almost Menger} \end{array}$$

3. Almost Alster property and related games

In this section our motivation is to define a game associated with the (i,j)-almost Alster property and to find a game which is dual to (i,j)-almost Alster game, so that the relation between almost Alster property and TWO has a winning strategy in almost Menger game in bitopological spaces may be more understood.

We recall the following definition.

Definition 3.1 [10] A bitopological space (X, τ_1, τ_2) is said to be (i,j)-*almost Alster*, if for every τ_i-Alster cover $\mathcal{U}$ of X there is a countable subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $\bigcup_{V \in \mathcal{V}} \mathrm{Cl}_{\tau_j}(V) = X$.

Let (X, τ_1, τ_2) be a bitopological space. We use the following notations which are similar to those used in the papers [4, 13].

$\mathcal{G}^{\tau_i}$: The family of all covers $\mathcal{U}$ of X for which each element of $\mathcal{U}$ is a τ_i-G_δ set.

$\mathcal{G}_{\mathcal{A}}^{\tau_i}$: The family of all τ_i-Alster covers of X.

$\mathcal{G}_{\Omega}^{\tau_i}$: The family of all covers $\mathcal{U} \in \mathcal{G}^{\tau_i}$ such that every finite subset of X is contained by an element of $\mathcal{U}$.

$\mathrm{Cl}_{\tau_j}(\mathcal{G}^{\tau_i})$: The family, consisting of sets $\mathcal{U}$ with each element of $\mathcal{U}$ is a τ_i-G_δ subset of X and $\{\mathrm{Cl}_{\tau_j}(U) : U \in \mathcal{U}\}$ covers X.

$\mathrm{Cl}_{\tau_j}(\mathcal{G}_{\Omega}^{\tau_i})$: The family of all sets $\mathcal{U} \in \mathrm{Cl}_{\tau_j}(\mathcal{G}^{\tau_i})$ such that for each finite subset $F \subseteq X$ there is a $U_F \in \mathcal{U}$ such that $F \subseteq \mathrm{Cl}_{\tau_j}(U_F)$.

In [10] the authors characterized (i,j)-almost Alster property in terms of selection principle S_1.

Theorem 3.2 *[10] Let (X, τ_1, τ_2) be a bitopological space. The following are equivalent.*

1. *X is (i,j)-almost Alster;*

2. *X satisfies the selection principle* $S_1\big(\mathcal{G}^{\tau_i}_{\mathcal{A}}, \mathrm{Cl}_{\tau_j}(\mathcal{G}^{\tau_i})\big)$*;*

3. *X satisfies the selection principle* $S_1\big(\mathcal{G}^{\tau_i}_{\mathcal{A}}, \mathrm{Cl}_{\tau_j}(\mathcal{G}^{\tau_i}_{\Omega})\big)$*.*

Clearly, (i,j)-almost Alster bitopological spaces are (i,j)-almost Menger.

The following game is naturally associated with the (i,j)-almost Alster property.

Definition 3.3 The (i,j)-almost Alster game on a bitopological space (X,τ_1,τ_2) is defined as follows: In each inning $n \in \mathbb{N}$, ONE chooses a τ_i-Alster cover $\mathcal{U}_n$ of X, and then TWO chooses $U_n \in \mathcal{U}_n$. The play $\mathcal{U}_0, U_0, \mathcal{U}_1, U_1, \ldots, \mathcal{U}_n, U_n, \ldots$ is won by TWO if

$$\bigcup_{n\in\mathbb{N}} Cl_{\tau_j}(U_n) = X$$

otherwise, ONE is the winner.

Now we introduce another game.

Definition 3.4 (i,j)-almost compact-G_δ game on a bitopological space (X,τ_1,τ_2) is defined as follows: In each inning $n \in \mathbb{N}$, ONE chooses a τ_i-compact subset K_n of X, and TWO chooses a τ_i-G_δ subset U_n of X such that $K_n \subseteq U_n$; the play

$$K_0, U_0, K_1, U_1, \ldots, K_n, U_n, \ldots$$

is won by ONE if the τ_j-closures of chosen sets by TWO covers X, that is

$$\bigcup_{n\in\mathbb{N}} Cl_{\tau_j}(U_n) = X$$

otherwise, TWO is the winner.

Now we show that the (i,j)-almost compact-G_δ game is essentially the same as the (i,j)-almost Alster game on a bitopological space in the following sense. In order to do that we need the following concept and lemma.

Definition 3.5 Two games G and G' are dual if

- ONE has a winning strategy in G if and only if TWO has a winning strategy in G'; and
- TWO has a winning strategy in G if and only if ONE has a winning strategy in G'.

The following result and its proof follow from the proof of Lemma 6.1 in [21].

Lemma 3.6 *Let σ be a strategy for* TWO *in (i,j)-almost Alster game and $\big(\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n\big)$ be a sequence of τ_i-Alster covers of (X,τ_1,τ_2). Then there is a τ_i-compact subset K of X so that for each τ_i-G_δ subset U of X containing K, there is a τ_i-Alster cover $\mathcal{U}$ of X so that $U = \sigma(\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n, \mathcal{U})$.*

Proof Let $\mathcal{W}$ be a family of τ_i-G_δ subsets of X that are not of the form $\sigma(\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n, \mathcal{U})$ for some τ_i-Alster cover $\mathcal{U}$ of X. That is let:

$$\mathcal{W} = \{G \subset X : G \text{ is a } \tau_i\text{-}G_\delta \text{ set}, G \neq \sigma(\mathcal{U}_0,\mathcal{U}_1,\ldots,\mathcal{U}_n,\mathcal{U}), \mathcal{U} \text{ is a } \tau_i\text{-Alster cover}\}$$

Then it follows that $\mathcal{W}$ is not a τ_i-Alster cover of X. Otherwise $\sigma(\mathcal{U}_0,\mathcal{U}_1,\ldots,\mathcal{U}_n,\mathcal{W}) \in \mathcal{W}$ contradicts the definition of $\mathcal{W}$.

If the claim of the lemma is not true we will get a contradiction. Suppose for each τ_i-compact subset K of X there is a τ_i-G_δ subset U of X containing K such that $U \neq \sigma(\mathcal{U}_0,\mathcal{U}_1,\ldots,\mathcal{U}_n,\mathcal{U})$ for each τ_i-Alster cover $\mathcal{U}$ of X. Then $\mathcal{W}$ is a τ_i-Alster cover of X and this is a contradiction. □

Theorem 3.7 *The (i,j)-almost Alster game and (i,j)-almost compact-G_δ game are dual on a bitopological space (X,τ_1,τ_2).*

Proof Let σ be a winning strategy of TWO in (i,j)-almost Alster game. We define a winning strategy σ' for ONE in (i,j)-almost compact-G_δ game.

We set $\sigma'(\emptyset) = K_0$ where K_0 is a τ_i-compact subset of X with the property: each τ_i-G_δ nbhd of K_0 has the form $\sigma(\mathcal{U})$ for some τ_i-Alster cover $\mathcal{U}$ of X. The existence of K_0 is assured by Lemma 3.6. Now let U_0 be any τ_i-G_δ subset of X containing K_0. Then there is a τ_i-Alster cover $\mathcal{U}_0$ of X with $U_0 = \sigma(\mathcal{U}_0)$. We note that the first move of TWO in the (i,j)-almost compact-G_δ game is U_0, and the first moves of ONE and TWO in the (i,j)-almost Alster game are $\mathcal{U}_0$ and $\sigma(\mathcal{U}_0)$ respectively. Again by Lemma 3.6 there is a τ_i-compact subset K_1 of X with the property: each τ_i-G_δ nbhd of K_1 has the form $\sigma(\mathcal{U}_0,\mathcal{U})$ for some τ_i-Alster cover $\mathcal{U}$ of X. Now we set that $\sigma'(U_0) = K_1$. Let U_1 be any τ_i-G_δ subset of X containing K_1. Then there is a τ_i-Alster cover $\mathcal{U}_1$ of X with $U_1 = \sigma(\mathcal{U}_0,\mathcal{U}_1)$. By continuing in this way we define a strategy σ' for ONE in (i,j)-almost compact-G_δ game. Since the strategy σ is a winning strategy for TWO in (i,j)-almost Alster game, we obtain

$$\bigcup_{n\in\mathbb{N}} Cl_{\tau_j}(U_n) = X.$$

Thus σ' is a winning strategy for ONE in (i,j)-almost compact-G_δ game.

Conversely let σ be a winning strategy for ONE in (i,j)-almost compact-G_δ game and we define a winning strategy σ' for TWO in (i,j)-almost Alster game.

Let $\mathcal{U}_0$ be a τ_i-Alster cover of X. Then there is a set $U_0 \in \mathcal{U}_0$ with $\sigma(\emptyset) \subset U_0$. Now set $\sigma'(\mathcal{U}_0) = U_0$. Let $\mathcal{U}_1$ be a τ_i-Alster cover of X then we set $\sigma'(\mathcal{U}_0,\mathcal{U}_1) = U_1$ where U_1 is a set in $\mathcal{U}_1$ containing $\sigma(U_0)$. This procedure define a strategy σ' for TWO in (i,j)-almost Alster game. Since σ is a winning strategy for ONE in (i,j)-almost compact-G_δ game, we obtain

$$\bigcup_{n\in\mathbb{N}} Cl_{\tau_j}(U_n) = X.$$

Thus σ' is a winning strategy for TWO in (i,j)-almost Alster game.

Now, let σ be a winning strategy for ONE in (i,j)-almost Alster game. We define a winning strategy σ' for TWO in almost compact-G_δ game. Let K_0 be a τ_i-compact subset of X. Then the τ_i-Alster cover $\sigma(\emptyset)$ contains an element U_0 containing K_0. Set $\sigma'(K_0) = U_0$. Similarly let K_1 be a τ_i-compact subset of X. Then the τ_i-Alster cover $\sigma(U_0)$ contains an element U_1 containing K_1. Set $\sigma'(K_0, K_1) = U_1$ and so on. Since

$$\bigcup_{n\in\mathbb{N}} Cl_{\tau_j}(U_n) \neq X$$

it follows that σ' is the winning strategy for TWO in (i,j)-almost compact-G_δ game.

Finally, let σ be a winning strategy for TWO in (i,j)-almost compact-G_δ game. We define a winning strategy σ' for ONE in (i,j)-almost Alster game.

We set $\mathcal{U}_0 = \{\sigma(K) : K \subset X, \tau_i\text{-compact}\}$ and $\sigma'(\emptyset) = \mathcal{U}_0$. If $U_0 \in \mathcal{U}_0$ then there exists a τ_i-compact subset K_0 with $U_0 = \sigma(K_0)$. Let $\mathcal{U}_1 = \{\sigma(K_0, K) : K \subset X, \tau_i\text{-compact}\}$ and $\sigma'(U_0) = \mathcal{U}_1$. If $U_1 \in \mathcal{U}_1$ then there exists a τ_i-compact subset K_1 with $U_1 = \sigma(K_0, K_1)$, and so on. Since σ is a winning strategy of TWO in (i,j)-almost compact-G_δ game, we have

$$\bigcup_{n\in\mathbb{N}} Cl_{\tau_j}(U_n) \neq X.$$

Thus σ' is a winning strategy for ONE. □

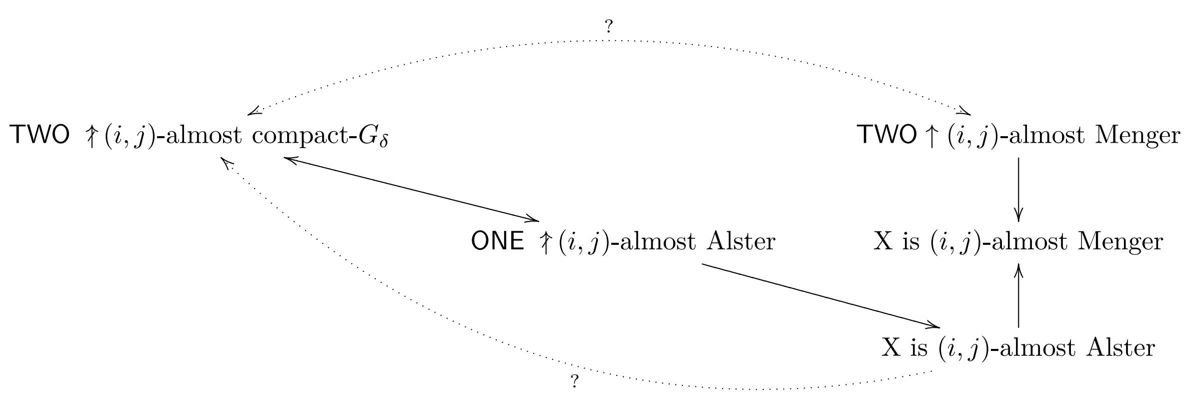
Proposition 3.8 *If* ONE *does not have a winning strategy in* (i,j)*-almost Alster game on a bitopological space* (X,τ_1,τ_2)*, then* X *is* (i,j)*-almost Alster bitopological space.*

Proof Straightforward by Theorem 3.2. □

Since (i,j)-almost compact-G_δ and (i,j)-almost Alster games are dual, we have the following.

Corollary 3.9 *If* TWO *does not have a winning strategy in* (i,j)*-almost compact-*G_δ *game, the bitopological space* (X,τ_1,τ_2) *is* (i,j)*-almost Alster.*

Relations among the covering properties we consider in this paper are given in the following diagram.



4. Conclusion

In conclusion we have managed to relate the almost Alster property to TWO not having a winning strategy in almost compact-G_δ games in bitopological spaces. However the picture is still incomplete.

Returning to the question posed at the beginning of this study, it is still not possible to state the connection between the almost Alster property in bitopological spaces and TWO has a winning strategy in the almost Menger game. Further works needs to be done in line with the problems suggested below.

Problem 4.1 *Is the* (i,j)*-almost Alster property equivalent to* TWO *not having a winning strategy in the* (i,j)*-almost compact-*G_δ *game?*

Problem 4.2 *Is there any relation between the* (i,j) *-almost Alster property and* TWO *having a winning strategy in the almost Menger game?*

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