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Symmetric polynomials in Leibniz algebras and their inner automorphisms

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Abstract: Let L_n be the free metabelian Leibniz algebra generated by the set $X_n = \{x_1, \dots, x_n\}$ over a field K of characteristic zero. This is the free algebra of rank n in the variety of solvable of class 2 Leibniz algebras. We call an element $s(X_n) \in L_n$ *symmetric* if $s(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = s(x_1, \dots, x_n)$ for each permutation σ of $\{1, \dots, n\}$. The set $L_n^{S_n}$ of symmetric polynomials of L_n is the algebra of invariants of the symmetric group S_n. Let $K[X_n]$ be the usual polynomial algebra with indeterminates from X_n. The description of the algebra $K[X_n]^{S_n}$ is well known, and the algebra $(L_n')^{S_n}$ in the commutator ideal L_n' is a right $K[X_n]^{S_n}$-module. We give explicit forms of elements of the $K[X_n]^{S_n}$-module $(L_n')^{S_n}$. Additionally, we determine the description of the group $\mathrm{Inn}(L_n^{S_n})$ of inner automorphisms of the algebra $L_n^{S_n}$. The findings can be considered as a generalization of the recent results obtained for the free metabelian Lie algebra of rank n.

Key words: Leibniz algebras, metabelian identity, automorphisms, symmetric polynomials

1. Introduction

Hilbert's fourteen problem is one of those famous twenty three problems suggested by German mathematician David Hilbert [10] in 1900 at the Paris conference of the International Congress of Mathematicians, and it is related with the finite generation of the algebra $K[X_n]^G$ of invariants of $G < GL_n(K)$, where $K[X_n] = K[x_1, \dots, x_n]$ is the usual polynomial algebra over a field K, and $GL_n(K)$ is the general linear group. Nagata [15] showed that the problem is not true in general in 1959. Earlier in 1916, Noether [14] solved the problem in affirmative for finite groups. In particular let $G = S_n$ be the symmetric group acting on the algebra $K[X_n]$ by permuting the variables: $\pi \cdot p(x_1, \dots, x_n) = p(x_{\pi(1)}, \dots, x_{\pi(n)})$, $p \in K[X_n]$, $\pi \in S_n$. The algebra $K[X_n]^{S_n}$ is generated by the set $\{\sum_{i=1}^n x_i^k \mid k = 1, \dots, n\}$, by the fundamental theorem of symmetric polynomials. Elementary symmetric polynomials $e_j = \sum x_{i_1} \cdots x_{i_j}$, $i_1 < \cdots < i_j$, $j = 1, \dots, n$, form another generating set.

A noncommutative analogue of the problem is the algebra $K\langle X_n\rangle^{S_n}$ of symmetric polynomials in the free associative algebra $K\langle X_n\rangle$. One may see the works [2, 9, 18] on the algebra $K\langle X_n\rangle^{S_n}$. Another analogue is working in relatively free Lie algebras, which are not associative and commutative. The algebras $F_n^{S_n}$, and $M_n^{S_n}$ are not finitely generated via [3] and [4], where F_n and M_n are the free Lie algebra and the free metabelian Lie algebra of rank n, respectively. One may see the papers [5, 7] for the explicit elements of the algebra $M_n^{S_n}$. See also [8] for the inner automorphisms of $M_n^{S_n}$.

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We consider the Leibniz algebras which can be thought as a generalization of the Lie algebras. Leibniz algebras are defined by the identity $[x,[y,z]] = [[x,y],z] - [[x,z],y]$, where the bracket is bilinear; however, nonnecessarily skew-symmetric. In the case of skew-symmetry the identity turns into the Jacobi identity, and we obtain a Lie algebra. Leibniz algebras are related with many branches of mathematics. See the papers [1, 11–13, 16] for more details.

In the present study, we consider the free metabelian Leibniz algebra L_n and we determine the algebra $L_n^{S_n}$ of symmetric polynomials. Additionally, we describe the group $\mathrm{Inn}(L_n^{S_n})$ of inner automorphisms of $L_n^{S_n}$.

2. Preliminaries

Let K be a field of characteristic zero. A Leibniz algebra L over K is a vector space furnished with bilinear commutator $[.,.]$ satisfying the Leibniz identity

$$[[x,y],z] = [[x,z],y] + [x,[y,z]],$$

or

$$[x,y]r_z = [xr_z,y] + [x,yr_z],$$

$x,y,z \in L$. Here r_z stands for the adjoint operator $\mathrm{ad}z$ acting from right side by commutator multiplication. The Leibniz algebra L is nonassociative and noncommutative.

Now consider the free algebra L_n of rank n generated by $X_n = \{x_1,\ldots,x_n\}$ in the variety of metabelian Leibniz algebras over the base field K. The algebra L_n satisfies the metabelian identity $[[x,y],[z,t]] = 0$, and is a solvable of class 2 Leibniz algebra. Hence every element in the commutator ideal $L_n' = [L_n, L_n]$ of the free metabelian Leibniz algebra L_n can be expressed as a linear combination of left-normed monomials of the form

$$\begin{aligned}[][[\cdots[[x_{i_1},x_{i_2}],x_{i_3}],\ldots],x_{i_k}] &= [x_{i_1},x_{i_2},x_{i_3},\ldots,x_{i_k}]\\ &= [x_{i_1},x_{i_2}]r_{i_3}\cdots r_{i_k} = [x_{i_1},x_{i_2}]r_{i_{\pi(3)}}\cdots r_{i_{\pi(k)}}\end{aligned}$$

where π is a permutation of the set $\{3,\ldots,k\}$. In this way the commutator ideal L_n' can be considered as a right $K[R_n] = K[r_1,\ldots,r_n]$-module, where $r_i = r_{x_i} = \mathrm{ad}x_i$, $i = 1,\ldots,n$. It is well known, see Proposition 3.1. of the paper [6], that the elements

$$x_{i_1}, [x_{i_1},x_{i_2}], [x_{i_1},x_{i_2}]r_{i_3}\cdots r_{i_k}, \quad 1 \le i_1,i_2 \le n, \quad 1 \le i_3 \le \cdots \le i_k,$$

form a basis for L_n. The next result is a direct consequence of this basis.

Corollary 2.1 *The commutator ideal L_n' of the free metabelian Leibniz algebra L_n is a free right $K[R_n]$-module with generators $[x_i,x_j]$, $1 \le i,j \le n$.*

A polynomial $s = s(x_1,\ldots,x_n)$ in the free metabelian Leibniz algebra L_n is said to be *symmetric* if

$$\sigma s = s(x_{\sigma(1)},\ldots,x_{\sigma(n)}) = s(x_1,\ldots,x_n), \quad \sigma \in S_n.$$

The set $L_n^{S_n}$ of symmetric polynomials forms a Leibniz subalgebra, which is the algebra of invariants of the symmetric group S_n. The $K[R_n]$-module structure of the commutator ideal L_n' implies that the algebra $(L_n')^{S_n}$ is a right $K[R_n]^{S_n}$-module. One of the set of generators of the algebra $K[R_n]^{S_n}$ of symmetric polynomials is well known: $\{r_1^k + \cdots + r_n^k \mid 1 \le k \le n\}$, see [17].

3. Main results

3.1. Symmetric polynomials

In this section we determine the algebra $L_n^{S_n}$ of symmetric polynomials in the free metabelian Leibniz algebra L_n. Clearly the linear symmetric polynomials are included in the K-vector space spanned on a single element $x_1 + \cdots + x_n$. Hence it is sufficient to work in the commutator ideal L_n' of L_n, and describe the algebra $(L_n')^{S_n}$. Let us fix the notations $a_i = [x_i, x_i]$, $1 \leq i \leq n$, and $b_{jk} = [x_j, x_k]$, $1 \leq j \neq k \leq n$, which are the free generators of $K[R_n]$-module L_n'. We provide explicit elements of $K[R_n]^{S_n}$-module $(L_n')^{S_n}$. For this purpose, we study in the $K[R_n]$-submodules

$$A_n = \left\{ \sum_{i=1}^{n} a_i p_i \mid p_i \in K[R_n] \right\} \quad \text{and} \quad B_n = \left\{ \sum_{1 \leq j \neq k \leq n} b_{jk} q_{jk} \mid q_{jk} \in K[R_n] \right\}$$

of the $K[R_n]$-module $L_n' = A_n \oplus B_n$, generated by a_i, $1 \leq i \leq n$, and b_{jk}, respectively, $1 \leq j \neq k \leq n$, due to the fact that they are invariant under the action of S_n; i.e., $A_n^{S_n} \subset A_n$, and $B_n^{S_n} \subset B_n$.

Let us denote the subgroups $\Pi_i = \{\pi \in S_n \mid \pi(i) = i\}$, $1 \leq i \leq n$, and $\Pi_{jk} = \{\pi \in S_n \mid \pi(j) = j,\ \pi(k) = k\}$, $1 \leq j \neq k \leq n$, of S_n. In the next theorems, we determine symmetric polynomials in the $K[R_n]$-modules A_n, and B_n, respectively.

Theorem 3.1 *Let $p = \sum_{i=1}^{n} a_i p_i$ be a polynomial in A_n, for some $p_i \in K[R_n]$, $1 \leq i \leq n$. Then p is symmetric if and only if*

$$p_1(r_1, r_2, \ldots, r_n) = \pi p_1(r_1, r_2, \ldots, r_n) = p_1(r_1, r_{\pi(2)}, \ldots, r_{\pi(n)}), \quad \pi \in \Pi_1,$$

$\sigma p_i = p_i$, $\sigma \in \Pi_i$, and $p_i = (1i)p_1 = p_1(r_i, r_2, \ldots, r_{i-1}, r_1, r_{i+1}, \ldots, r_n)$, for transpositions $(1i) \in S_n$, $i = 2, \ldots, n$.

Proof Let $p \in A_n$ be an element of the form

$$p = \sum_{i=1}^{n} a_i p_i(r_1, \ldots, r_n), \quad p_i \in K[R_n].$$

If p is a symmetric polynomial, then $p = \pi p$; i.e.

$$\sum_{i=1}^{n} a_i p_i(r_1, \ldots, r_n) = \sum_{i=1}^{n} a_{\pi(i)} p_i(r_{\pi(1)}, \ldots, r_{\pi(n)})$$

for each $\pi \in S_n$ by definition, and by Corollary 2.1 we may compare the coefficients of a_i, $i = 1, \ldots, n$, from $K[R_n]$, in the last equality. In particular, $p = \pi p$ for each $\pi \in \Pi_i$, and comparing the coefficients of a_i, $i = 1, \ldots, n$, we obtain that

$$p_i(r_1, \ldots, r_n) = p_i(r_{\pi(1)}, \ldots, r_{\pi(i-1)}, r_i, r_{\pi(i+1)}, \ldots, r_{\pi(n)}).$$

Now consider $(1i)p = p$ for every transposition $(1i) \in S_n$, $i = 2, \ldots, n$. Then these equalities give that $p_i = (1i)p_1$, and thus

$$\begin{aligned} p_i(r_1, \ldots, r_n) &= (1i)p_1(r_1, \ldots, r_n) \\ &= p_1(r_i, r_2, \ldots, r_{i-1}, r_1, r_{i+1}, \ldots, r_n) \end{aligned}$$

Conversely consider the element $p = \sum_{i=1}^{n} a_i p_i$ satisfying the conditions in the theorem. It is sufficient to show that $(1k)p = p$, $k = 2, \dots, n$, since these transpositions generate the symmetric group S_n. Note that if $i \neq 1, k$, then $(1k)p_i = p_i$, since $(1k) \in \Pi_i$. The following computations complete the proof:

$$\begin{aligned}(1k)p =& (1k)\left(a_1p_1 + a_kp_k + \sum_{i\neq 1,k} a_ip_i\right)\\ &= a_k((1k)p_1) + a_1((1k)p_k) + \sum_{i\neq 1,k} a_i((1k)p_i)\\ &= a_kp_k + a_1p_1 + \sum_{i\neq 1,k} a_ip_i = p.\end{aligned}$$

□

Theorem 3.2 *Let* $q = \sum b_{ij}q_{ij}$ *be a polynomial in* B_n, *for some* $q_{ij} \in K[R_n]$, $1 \leq i \neq j \leq n$. *Then* q *is symmetric if and only if* $q_{ij} = \sigma q_{kl}$ *for every* $\sigma : i \to k, j \to l$, *in particular,*

$$\begin{aligned} q_{1i} =& (2i)q_{12}, \quad q_{i2} = (1i)q_{12}, \quad q_{2i} = (1i)q_{21}, \quad q_{i1} = (2i)q_{21},\\ q_{21} =& (12)q_{12}, \quad q_{ij} = (1i)(2j)q_{12}, \quad 3 \leq i \neq j \leq n,\end{aligned}$$

and $q_{ij} = \pi q_{ij}$, *for all* $\pi \in \Pi_{ij}$.

Proof Assume that a polynomial $q = \sum b_{ij}q_{ij} \in B_n$, $1 \leq i \neq j \leq n$, is symmetric. Then $\pi q = q$ for each $\pi \in \Pi_{12}$ gives that

$$q_{12}(r_1, r_2, \dots, r_n) = q_{12}(r_1, r_2, r_{\pi(3)}, \dots, r_{\pi(n)}).$$

Relations on q_{ij}'s in the theorem are straightforward, by making use of Corollary 2.1, and comparing the coefficients of b_{12}, b_{21}, and b_{ij} from the equalities $q = (12)q = (1i)q = (2i)q = (1i)(2j)q = (ij)q$, where $3 \leq i \neq j \leq n$.

Now let the polynomial $q \in B_n$ satisfy the conditions of the theorem, and $(1k) \in S_n$ be a transposition for a fixed $k \in \{3, \dots, n\}$. We have to show that $(1k)q = q$. Let express q in the following form

$$q = b_{1k}q_{1k} + b_{k1}q_{k1} + \sum_{i,j\neq 1,k} b_{ij}q_{ij} + \sum_{i\neq 1,k} (b_{1i}q_{1i} + b_{ki}q_{ki} + b_{ik}q_{ik} + b_{i1}q_{i1}).$$

Note that $(1k) \in \Pi_{ij}$, and hence $(1k)q_{ij} = q_{ij}$, for $i, j \neq 1, k$. Then we have that

$$\begin{aligned}(1k)q =& b_{k1}((1k)q_{1k}) + b_{1k}((1k)q_{k1}) + \sum_{i,j\neq 1,k} b_{ij}((1k)q_{ij})\\ &+ \sum_{i\neq 1,k} (b_{ki}((1k)q_{1i}) + b_{1i}((1k)q_{ki}) + b_{i1}((1k)q_{ik}) + b_{ik}((1k)q_{i1}))\\ =& b_{k1}q_{k1} + b_{1k}q_{1k} + \sum_{i,j\neq 1,k} b_{ij}q_{ij} + \sum_{i\neq 1,k} (b_{ki}q_{ki} + b_{1i}q_{1i} + b_{i1}q_{i1} + b_{ik}q_{ik}) = q.\end{aligned}$$

□

We obtain the next corollary by combining Theorems 3.1 and 3.2.

Corollary 3.3 *If s is a symmetric polynomial in the free metabelian Leibniz algebra L_n, then it is of the form*

$$
\begin{aligned}
s = &\sum_{1\leq i\leq n} \alpha x_i + \sum_{1\leq i\leq n} [x_i,x_i]((1i)f) + \sum_{3\leq i\neq j\leq n} [x_i,x_j]((1i)(2j)g) \\
&+ [x_1,x_2]g + \sum_{3\leq i\leq n} ([x_1,x_i]((2i)g) + [x_i,x_2]((1i)g) \\
&+ [x_2,x_1]h + \sum_{3\leq i\leq n} ([x_i,x_1]((2i)h) + [x_2,x_i]((1i)h)),
\end{aligned}
$$

where $\alpha \in K$, $f,g,h \in K[R_n]$, such that $\pi f = f$ for $\pi \in \Pi_1$, $\sigma g = g$ for $\sigma \in \Pi_{12}$, and $h = (12)g$.

Example 3.4 *Let $n = 2$ and the free metabelian Leibniz algebra L_2 be generated by x_1, x_2. Then each symmetric polynomial $s \in L_2^{S_2}$ is of the form*

$$
\begin{aligned}
s =&\alpha(x_1+x_2) + [x_1,x_1]f(r_1,r_2) + [x_2,x_2]f(r_2,r_1) \\
&+ [x_1,x_2]g(r_1,r_2) + [x_2,x_1]g(r_2,r_1),
\end{aligned}
$$

where $\alpha \in K$, $f,g \in K[R_2]$. Note that the Lie correspondence of this result (modulo the annihilator) is that if $s(x_1,x_2)$ is a symmetric polynomial in the free metabelian Lie algebra generated by x_1, x_2, then

$$s = \alpha(x_1+x_2) + [x_1,x_2]t(r_1,r_2),$$

such that $t(r_1,r_2) = -t(r_2,r_1)$, which is compatible with the recent result given in [7].

3.2. Inner automorphisms

Let u be an element in the commutator ideal L_n' of the free metabelian Leibniz algebra L_n. The adjoint operator

$$\mathrm{ad}u : L_n \to L_n, \quad \mathrm{ad}u(v) = [v,u], \quad v \in L_n$$

is nilpotent since $\mathrm{ad}^2 u = 0$, and that $\psi_u = \exp(\mathrm{ad}u) = 1 + \mathrm{ad}u$ is called an inner automorphism of L_n with inverse ψ_{-u}. Clearly the group $\mathrm{Inn}(L_n)$ consisting of all inner automorphisms is abelian due to the fact that $\psi_{u_1}\psi_{u_2} = \psi_{u_1+u_2}$.

Let $\mathrm{Ann}_R(L_n) = \{u \in L_n \mid [x,u] = 0,\ x \in L_n\}$ be the right annihilator of the free metabelian Leibniz algebra L_n. In the next theorem we determine the group $\mathrm{Inn}(L_n^{S_n})$ of inner automorphisms preserving symmetric polynomials.

Theorem 3.5 $\mathrm{Inn}(L_n^{S_n}) = \{\psi_{u_1+u_2} \mid u_1 \in \mathrm{Ann}_R(L_n),\ u_2 \in (L_n')^{S_n}\}$.

Proof Let $v \in L_n^{S_n}$, $u = u_1 + u_2$, for some $u_1 \in \mathrm{Ann}_R(L_n)$, and $u_2 \in (L_n')^{S_n}$. Then clearly

$$\psi_u(v) = v + [v, u_1 + u_2] = v + [v, u_2] \in L_n^{S_n}.$$

Conversely, let $\psi_u(v) \in L_n^{S_n}$ for $v \in L_n^{S_n}$, and $u \in L_n'$. The action of ψ_u is identical when $v \in L_n'$. Hence we assume that the linear counterpart $v_l = \alpha(x_1 + \cdots + x_n)$, $\alpha \in K$, of v is nonzero. We may express $u = u_1 + u_2$,

$u_1 \in \mathrm{Ann}_R(L_n)$, $u_2 \in L'_n$, where $u_2 \notin \mathrm{Ann}_R(L_n)$. Hence we have $\psi_u(v) \in L_n^{S_n}$, which implies that $[v_l, u_2]$ is a symmetric polynomial. Let $\pi \in S_n$ be an arbitrary permutation. Then

$$[v_l, u_2] = \pi[v_l, u_2] = [\pi v_l, \pi u_2] = [v_l, \pi u_2]$$

or $[x_1 + \cdots + x_n, u_2 - \pi u_2] = 0$, and thus $u_2 - \pi u_2 = 0$. Therefore $u_2 \in (L'_n)^{S_n}$. □

We complete the paper by releasing the following problem.

Problem 3.6 *Determine the group* $\mathrm{Aut}(L_n^{S_n})$ *of all automorphisms preserving the symmetric polynomials.*

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