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Neutral multivalued integro-differential evolution equations with infinite state-dependent delay

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Abstract: Our problem through this work is to give the existence of mild solutions for the first order class of neutral functional multivalued integro-differential evolution equations with infinite state-dependent delay using the nonlinear alternative of Frigon for multivalued contraction maps in Fréchet spaces combined with the semi-group theory.

Key words: Neutral integro-differential evolution inclusions, mild solution, state-dependent delay, fixed-point, semi-group theory, Fréchet spaces

1. Introduction

In this paper, the existence of mild solutions in a real separable Banach space $(\mathbb{E}, |\cdot|)$ is proved for neutral multivalued integro-differential evolution equations with delay. The solutions are defined on the positive real interval $\mathcal{J} = \mathbb{R}^+$, the delay is depending on the solution and is infinite.

Consider firstly in Section 3, the problem

$$\frac{d}{dx}[u(x) - g(x, u_{\rho(x,u_x)})] \quad \in \quad \mathcal{A}(x)u(x) + \int_0^x \mathcal{I}(x,y)\, \mathfrak{F}(y, u_{\rho(y,u_y)})dy, \qquad \text{a.e. } x \in \mathcal{J}, \tag{1.1}$$

$$u_0 = \psi \in \mathcal{B}, \tag{1.2}$$

where $\mathcal{B}$ is a given specified phase space, $\Lambda = \{(x,y) \in \mathcal{J}^2 / y \le x\}$, $g : \mathcal{J} \times \mathcal{B} \to \mathbb{E}$, $\rho : \mathcal{J} \times \mathcal{B} \to \mathbb{R}$, $\mathcal{I} : \Lambda \to \mathbb{R}$ and $\psi \in \mathcal{B}$ are given functions, $\mathfrak{F} : J \times \mathcal{B} \to \mathcal{P}(\mathbb{E})$ is a multivalued map with nonempty compact values, $\mathcal{P}(\mathbb{E})$ is the set of all subsets of $\mathbb{E}$ and $\{\mathcal{A}(x)\}_{x\ge 0}$ is a family of linear closed operators from $\mathbb{E}$ into $\mathbb{E}$ which yields to an evolution system of operators $\{\mathcal{U}(x,y)\}_{(x,y)\in\Lambda}$.

Let us consider u_x the element of $\mathcal{B}$ which is the history of the state from time $x \in \overline{\mathcal{J}} = \mathbb{R}^-$ up to the present time x defined for any continuous function u and any $x \in \mathcal{J}$ by $u_x(\theta) = u(x+\theta)$ for any $\theta \in \overline{\mathcal{J}}$.

One example is given in the last section to illustrate the abstract theory.

Differential equations with delay are employed in modeling scientific phenomena during many years and so on, often when the delay is a fixed constant, it's called distributed delay as in the books of Hale and Lunel in [26], Kolmanovskii and Myshkis in [31] and Wu in [35] and the papers of Corduneanu and Lakshmikantham [18] and Hale and Kato [25].

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For evolution equations, Ahmed and Freidmann developed an extensive theory in [5, 6, 21]. Uniqueness, existence and controllability results was established for various evolution problems with constant delay by Baghli et al. in [2], [9]–[14].

However, complicated situations modeled differential equations when the delay depends on the unknown function which is the researched solution, called equations with state-dependent delay. Existence and other results were established recently for functional differential equations when the solution is depending on the delay into a bounded interval for impulsive problems by Abada et al. [1], Benchohra et al. [16], Anguraj et al. [7], Hernandez et al. [27] and Li et al. [32]. Perturbed and nonperturbed evolution equations with state-dependent delay was given by Aoued and Baghli-Bendimerad [3, 4], Baghli-Bendimerad [8], and Baghli et al. in [15].

Our purpose in this paper is to give an extension of the above results for neutral multivalued integro-differential evolution inclusions with infinite state-dependent delay $(1.1)-(1.2)$. The existence results of mild solutions are given using the nonlinear alternative of Frigon given in [22, 23] in Fréchet spaces for contractive multivalued maps, combined with semigroup theory [6, 33].

2. Preliminaries

Notations, definitions and theorems which are used throughout this paper are introduced in this section.

Let $C(\mathcal{J};\mathbb{E})$ be the Banach space of all functions from $\mathcal{J}$ into $\mathbb{E}$ which are continuous and let $B(\mathbb{E})$ be the space of all linear operators from $\mathbb{E}$ into $\mathbb{E}$ which are bounded, with the next norm

$$\|\aleph\|_{B(\mathbb{E})} = \sup_{|u|=1} |\aleph(u)|.$$

Each measurable function $u : \mathcal{J} \to \mathbb{E}$ which is measurable, u is said Bochner-integrable if $|u|$ is Lebesgue-integrable (see the monograph of Yosida [36]).

Let $L^1(\mathcal{J}, \mathbb{E})$ be the Banach space of all Bochner-integrable measurable functions $u : \mathcal{J} \to \mathbb{E}$ with the norm defined by

$$\|u\|_{L^1} = \int_0^{+\infty} |u(x)|\ dx.$$

Then, let us consider the family $\{\mathcal{A}(x)\}_{x\geq 0}$ of closed, unbounded, linear and densely defined operators on the Banach space $\mathbb{E}$ when the domain $D(\mathcal{A}(x))$ is independent of x.

Definition 2.1 *For every $(x,y) \in \Lambda$, a family $\{\mathcal{U}(x,y)\}_{(x,y)\in\Lambda}$ is called an evolution system when the next properties are verified :*

(P1) $\mathcal{U}(x,x) = I$ *where I is the identity operator,*

(P2) $\mathcal{U}(x,y)\,\mathcal{U}(y,z) = \mathcal{U}(x,z)$ *for $z \leq y \leq x$,*

(P3) *For every $\mathcal{U} \in B(\mathbb{E})$ and for each $u \in \mathbb{E}$, the mapping $(x,y) \to \mathcal{U}(x,y)\ u$ is continuous.*

We refer to the books of Ahmed [5], Engel and Nagel [20] and Pazy [33] for more details on evolution systems.

Let $\mathbb{X}$ be a Fréchet space with a family of seminorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. Assume that the family of seminorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ verifies for every $u \in \mathbb{X}$

$$\|u\|_1 \leq \|u\|_2 \leq \|u\|_3 \leq \dots$$

Let $\mathbb{Y} \subset \mathbb{X}$, say that $\mathbb{Y}$ is bounded if for every $n \in \mathbb{N}$, there exists $w_n > 0$ such that for all $v \in \mathbb{Y}$ such that :

$$\|v\|_n \leq w_n.$$

For every $n \in \mathbb{N}$, consider the equivalence relation $\sim_n$ defined for $u, v \in \mathbb{X}$ by

$$u \sim_n v \text{ if and only if } \|u - v\|_n = 0.$$

Associate to $\mathbb{X}$ a family $\{\mathbb{X}^n\}_n$ in Banach spaces $(\mathbb{X}^n, \|\cdot\|_n)$. Denote the completion of $\mathbb{X}^n$ with respect to $\|\cdot\|_n$ by the quotient space by $\mathbb{X}^n = (\mathbb{X}|_{\sim_n}, \|\cdot\|_n)$,. Associate to every $\mathbb{Y} \subset \mathbb{X}$ a family $\{\mathbb{Y}^n\}_n$ of subsets $\mathbb{Y}^n \subset \mathbb{X}^n$ as follows: For every $u \in \mathbb{X}$, denote $[u]_n$. the equivalence class of u of subset $\mathbb{X}^n$ and defined $\mathbb{Y}^n = \{[u]_n : u \in \mathbb{Y}\}$. Denote respectively the notion of closure, the interior and the boundary of $\mathbb{Y}^n$ by $\overline{\mathbb{Y}^n}$, $int_n(\mathbb{Y}^n)$ and $\partial_n \mathbb{Y}^n$ with respect to $\|\cdot\|_n$ in $\mathbb{X}^n$.

In this paper, the phase space $\mathcal{B}$ considered is introduced by Hale and Kato in [25] and Hino et al. in [29]. Then, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions from $\overline{\mathcal{J}}$ into $\mathbb{E}$ verifying

(A_1) For a fixed $b > 0$, if $u : (-\infty, b) \to \mathbb{E}$ is continuous on $[0, b]$ and $u_0 \in \mathcal{B}$, then for every $x \in [0, b)$, the next conditions hold

- **(i)** $u_x \in \mathcal{B}$;
- **(ii)** There exists a nonnegative constant ϵ such that

$$|u(x)| \leq \epsilon \|u_x\|_{\mathcal{B}};$$

- **(iii)** There exist two functions $\kappa(\cdot), \Pi(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ nondependent of u with κ continuous and Π locally bounded such that

$$\|u_x\|_{\mathcal{B}} \leq \kappa(x) \sup_{y \leq x} |u(y)| + \Pi(x)\|u_0\|_{\mathcal{B}}.$$

(A_2) For each function u defined from (A_1), u_x is a $\mathcal{B}-$valued continuous function on $[0, b]$.

(A_3) The space $\mathcal{B}$ is complete.

Set $\kappa_b = \sup\limits_{x\in[0,b]} \kappa(x)$ and $\Pi_b = \sup\limits_{x\in[0,b]} \Pi(x)$.

Remark 2.2 *Note that*

1. *(ii) is equivalent to $|\psi(0)| \leq \epsilon\|\psi\|_{\mathcal{B}}$ for every $\psi \in \mathcal{B}$.*
2. *Two functions ψ and $\overline{\psi}$ of $\mathcal{B}$ can verify $\|\psi - \overline{\psi}\|_{\mathcal{B}} = 0$ without necessarily verify $\psi(\theta) = \overline{\psi}(\theta)$ for all $\theta \in \overline{\mathcal{J}}$ because $\|\cdot\|_{\mathcal{B}}$ is a seminorm.*
3. *For all ψ and $\overline{\psi}$ of $\mathcal{B}$, $\|\psi - \overline{\psi}\|_{\mathcal{B}} = 0$ implies necessarily that $\psi(0) = \overline{\psi}(0)$ from Remark 1.*

Here are cited samples of the specified phase spaces from the book by Hino et al. [29].

Example 2.3 *Let*

BC *denote the space of all bounded continuous functions defined from* $\overline{\mathcal{J}}$ *to* $\mathbb{E}$ *;*

BUC *denote the space of all* BC *functions which are uniformly continuous;*

$C^{\infty} \;= \left\{\psi \in BC : \lim_{t\to-\infty} \psi(t) \text{ exist in } \mathbb{E}\right\};$

$C^{0} \;= \left\{\psi \in BC : \lim_{t\to-\infty} \psi(t) = 0\right\}$, *endowed with the uniform norm* $\|\psi\| = \sup_{t\le 0} |\psi(t)|$.

Then, $(A_1)-(A_3)$ *are verified for the cited spaces* BUC, C^{∞} *and* C^0. *But,* (A_1) *and* (A_3) *are verified for* BC *and* (A_2) *is not.*

Example 2.4 $C_g = \left\{\psi \in C(\overline{\mathcal{J}}, \mathbb{E}) : \frac{\psi(t)}{g(t)} \text{ is bounded on } \overline{\mathcal{J}}\right\}$; $C_g^0 = \left\{\psi \in C_g : \lim_{t\to-\infty} \frac{\psi(t)}{g(t)} = 0\right\}$, *endowed with the uniform norm* $\|\psi\| = \sup_{t\in\overline{\mathcal{J}}} \frac{|\psi(t)|}{g(t)}$. *Then* (A_3) *is verified for the spaces* C_g *and* C_g^0.

Consider the next condition on the function g.

(g_1) *For all* $b>0$, $\sup_{0\le x\le b} \sup_{t\le -x} \frac{g(x+t)}{g(t)} < +\infty.$

The conditions (A_1) *and* (A_2) *are verified if* (g_1).

Example 2.5 *For any real constant* γ, *define the functional space* C_γ *by*
$C_\gamma = \left\{\psi \in C(\overline{\mathcal{J}}, \mathbb{E}) : \lim_{t\to-\infty} e^{\gamma t}\psi(t) \text{ exists in } \mathbb{E}\right\}$ *endowed with the next norm* $\|\psi\| = \sup_{t\in\overline{\mathcal{J}}}\{e^{\gamma t}|\psi(t)|\}$. *Then, the conditions* $(A_1)-(A_3)$ *are verified in the space* C_γ.

Consider the next space

$$B_{+\infty} = \{u : \mathbb{R} \to \mathbb{E} : u|_J \in C(\mathcal{J}, \mathbb{E}), \quad u_0 \in \mathcal{B}\}$$

where $u|_{\mathcal{J}}$ is the restriction of u to $\mathcal{J}$.

Set $\mathcal{R}(\rho^-) = \{\rho(s,\psi) \text{ is continuous for } (s,\psi) \in \mathcal{J}\times\mathcal{B} \text{ with } \rho(s,\psi) \in \mathbb{R}^-\}$. Consider the next hypothesis

(H_ψ) The function ψ_t from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ is continuous and there exists a function $\mathcal{L}^\psi : \mathcal{R}(\rho^-) \to (0,\infty)$ which is bounded and continuous such that for every $t \in \mathcal{R}(\rho^-)$

$$\|\psi_t\|_{\mathcal{B}} \le \mathcal{L}^\psi(t)\|\psi\|_{\mathcal{B}}.$$

Remark 2.6 (H_ψ) *is verified usually by bounded and continuous functions [29].*

Lemma 2.7 *[27] If $u:(-\infty,b]\to\mathbb{E}$ is a function such that $u_0=\psi$, then for every $s\in\mathcal{R}(\rho^-)\cup\mathcal{J}$*

$$\|u_s\|_{\mathcal{B}}\le\kappa_b\sup_{t\in[0,\breve{s}]}|u(t)|+(\Pi_b+\mathcal{L}^{\psi})\|\psi\|_{\mathcal{B}}$$

where $\mathcal{L}^{\psi}=\sup_{t\in\mathcal{R}(\rho^-)}\mathcal{L}^{\psi}(t)$ and $\breve{s}=max\{0,s\}$.

Proposition 2.8 *By (H_ψ), (A_1) and Lemma 2.7, we have for each $x\in[0,n]$ and $n\in\mathbb{N}$*

$$\|u_{\rho(x,u_x)}\|_{\mathcal{B}}\le\kappa_n|u(x)|+(\Pi_n+\mathcal{L}^{\varphi})\,\|u_0\|_{\mathcal{B}}.$$

Let $(\mathbb{E},d)$ be a metric space. The next notations are used

$$\mathcal{P}_{cl}(\mathbb{E})=\{\mathbb{F}\in\mathcal{P}(\mathbb{E}):\mathbb{F}\ closed\},\quad \mathcal{P}_{b}(\mathbb{E})=\{\mathbb{F}\in\mathcal{P}(\mathbb{E}):\mathbb{F}\ bounded\},$$

$$\mathcal{P}_{cv}(\mathbb{E})=\{\mathbb{F}\in\mathcal{P}(\mathbb{E}):\mathbb{F}\ convexe\},\quad \mathcal{P}_{cp}(\mathbb{E})=\{\mathbb{F}\in\mathcal{P}(\mathbb{E}):\mathbb{F}\ compact\}.$$

Consider $H_d:\mathcal{P}(\mathbb{E})\times\mathcal{P}(\mathbb{E})\longrightarrow\mathbb{R}_+\cup\{\infty\}$, given by

$$H_d(\mathcal{A},\mathcal{B})=\max\left\{\sup_{a\in\mathcal{A}}d(a,\mathcal{B})\ ,\ \sup_{b\in\mathcal{B}}d(\mathcal{A},b)\right\}$$

where $d(\mathcal{A},b)=\inf_{a\in\mathcal{A}}d(a,b)$, $d(a,\mathcal{B})=\inf_{b\in\mathcal{B}}d(a,b)$. Then $(\mathcal{P}_{b,cl}(\mathbb{E}),H_d)$ is a metric space and $(\mathcal{P}_{cl}(\mathbb{E}),H_d)$ is a complete (generalized) metric space (see [30]).

Definition 2.9 *A multivalued map $\mathcal{G}:\mathcal{J}\to\mathcal{P}_{cl}(\mathbb{E})$ is called* measurable *if for each $x\in E$, the function $\mathfrak{F}:\mathcal{J}\to\mathbb{E}$ defined by*

$$\mathfrak{F}(t)=d(x,\mathcal{G}(t))=\inf_{z\in\mathcal{G}(t)}|x-z|$$

is measurable where d is the metric endowed by the norm of $\mathbb{E}$.

Definition 2.10 *A multivalued map $\mathfrak{F}:\mathcal{J}\times\mathcal{B}\longrightarrow\mathcal{P}(\mathbb{E})$ is called L^1_{loc}-Carathéodory multivalued map if it verifies:*

(i) *for all $x\in\mathcal{J}$, $u\mapsto\mathfrak{F}(x,u)$ is continuous with respect to the metric H_d;*

(ii) *for almost each $u\in\mathcal{B}$, $x\mapsto\mathfrak{F}(x,u)$ is measurable;*

(iii) *for every nonnegative constant k, there exists $h_k\in L^1_{loc}(\mathcal{J};\mathbb{R}^+)$ such that for almost all $x\in\mathcal{J}$ and for all $\|u\|_{\mathcal{B}}\le k$*

$$\|\mathfrak{F}(x,u)\|\le h_k(x).$$

Let $(\mathbb{E},\|\cdot\|)$ be a Banach space. A multivalued map $\mathcal{G}:\mathbb{E}\to\mathcal{P}(\mathbb{E})$ has *closed (convex) values* if $\mathcal{G}(x)$ is closed (convex) for all $x\in\mathbb{E}$. Say that $\mathcal{G}$ is *bounded* on bounded sets if $\mathcal{G}(B)$ is bounded in $\mathbb{E}$ for each bounded set B of $\mathbb{E}$, i.e.

$$\sup_{x\in B}\sup_{u\in\mathcal{G}(x)}\|u\|<\infty.$$

Finally, $\mathcal{G}$ has a fixed-point if there exists $x \in \mathbb{E}$ such that $x \in \mathcal{G}(x)$.

For each $u \in B_{+\infty}$ let the set $S_{\mathfrak{F},u}$ known as *the set of selectors* from $\mathfrak{F}$ defined by

$$S_{\mathfrak{F},u} = \{\mathfrak{f} \in L^1(\mathcal{J};\mathbb{E}) : \mathfrak{f}(x) \in \mathfrak{F}(x, u_x)\ ,\ a.e.\ x \in \mathcal{J}\}.$$

Refer to the books of Deimling [19], Górniewicz [24], Hu and Papageorgiou [28] and Tolstonogov [34] for more details on multivalued maps.

Let us recall here the definition of admissible multivalued contraction.

Definition 2.11 *A multivalued map $\mathfrak{F} : \mathbb{E} \to \mathcal{P}(\mathbb{E})$ is called an* admissible contraction *with constant $\{\kappa_n\}_{n\in\mathbb{N}}$ if for each $n \in \mathbb{N}$ there exists $k_n \in (0,1)$ such that*

i) *for all $x, y \in \mathbb{E}$,*

$$H_d(\mathfrak{F}(x), \mathfrak{F}(y)) \leq \kappa_n\ \|x-y\|_n.$$

ii) *for every $x \in \mathbb{E}$ and every $\epsilon \in (0,\infty)^n$, there exists $y \in \mathfrak{F}(x)$ such that*

$$\|x-y\|_n \leq \|x - \mathfrak{F}(x)\|_n + \epsilon_n \text{ for every } n \in \mathbb{N}.$$

We are going to use the nonlinear alternative of Frigon [22, 23] for contractive multivalued maps in Fréchet spaces as follows:

Theorem 2.12 *Let $\mathbb{E}$ be a Fréchet space and U be an open neighborhood of the origin in $\mathbb{E}$. Let $\aleph : \overline{U} \longrightarrow \mathbb{E}$ be an admissible multi-valued contraction. Assume that $\aleph$ is bounded. Then only one of the next statements holds :*

$(Fr1)$ *The operator $\aleph$ has a fixed-point;*

$(Fr2)$ *There exists $\lambda \in [0,1)$ and $x \in \partial U$ such that $x \in \lambda \aleph (x)$.*

3. Neutral integro-differential evolution inclusions

The definition of a mild solution for the neutral multivalued integro-differential evolution problem (1.1) – (1.2) is given hereafter.

Definition 3.1 *We say that the function $u : \mathbb{R} \to \mathbb{E}$ is a mild solution of the evolution system (1.1) – (1.2) if $u(x) = \psi(x)$ for all $x \in \overline{\mathcal{J}}$ and the restriction of u to the interval J is continuous and there exists $\mathfrak{f} \in L^1(\mathcal{J};\mathbb{E})$ such that $\mathfrak{f}(x) \in \mathfrak{F}(x, u_{\rho(x,u_x)})$ almost every $x \in \mathcal{J}$ when u verifies the next integral equation*

$$\begin{aligned} u(x) &= \mathcal{U}(x,0)[\psi(0) - g(0,\psi)] + g(x, u_{\rho(x,u_x)}) + \int_0^x \mathcal{U}(x,y)\mathcal{A}(y)\ g(y, u_{\rho(y,u_y)})\ dy \\ &+ \int_0^x \mathcal{U}(x,y) \int_0^y \mathcal{I}(y,r)\mathfrak{f}(r)\ drdy. \end{aligned} \tag{3.1}$$

Here is given our main result after assuming the next needed hypotheses

$(H1)$ $\mathcal{U}(x,y)$ is compact for $x>y$ and there exists a constant $\widehat{\mathcal{M}}\geq 1$ such that for every $(x,y)\in\Lambda$

$$\|\mathcal{U}(x,y)\|_{B(\mathbb{E})}\leq \widehat{\mathcal{M}}.$$

$(H2)$ For all $x,y\in J$, $\mathcal{I}(x,y)$ is measurable on $[0,x]$ and

$$\mathcal{I}(x)=\sup_{0\leq y\leq x}|\mathcal{I}(x,y)|$$

is bounded on $[0,n]$; let

$$S_n=\sup_{x\in[0,n]}\mathcal{I}(x).$$

$(H3)$ The multifunction $\mathfrak{F}:\mathcal{J}\times\mathcal{B}\longrightarrow\mathcal{P}(\mathbb{E})$ is L^1_{loc}-Carathéodory with compact and convex values for each $u\in\mathcal{B}$ and there exist a function $p\in L^1_{loc}(\mathcal{J};\mathbb{R}_+)$ and a continuous increasing function $\Psi:\mathcal{J}\to\mathbb{R}^0_+$ and such that for almost every $x\in\mathcal{J}$ and each $u\in\mathcal{B}$

$$\|\mathfrak{F}(x,u)\|_{\mathcal{P}(\mathbb{E})}\leq p(x)\Psi(\|u\|_{\mathcal{B}}).$$

$(H4)$ For all $R>0$, there exists $l_R\in L^1_{loc}(\mathcal{J};\mathbb{R}_+)$ such that for each $x\in\mathcal{J}$ and for all $u,v\in\mathcal{B}$

$$H_d(\mathfrak{F}(x,u),\mathfrak{F}(x,v))\leq l_R(x)\ \|u-v\|_{\mathcal{B}}$$

with $\|u\|_{\mathcal{B}}\leq R$ and $\|v\|_{\mathcal{B}}\leq R$ and

$$d(0,\mathfrak{F}(x,0))\leq l_R(x)\quad \text{a.e.}\quad x\in\mathcal{J}.$$

$(H5)$ There exists a constant $\overline{M}_0>0$ such that for all $x\in\mathcal{J}$

$$\|\mathcal{A}^{-1}(x)\|_{B(\mathbb{E})}\leq\overline{M}_0.$$

$(H6)$ There exists a constant $0<L<\dfrac{1}{\overline{M}_0\kappa_n}$, such that for all $x\in\mathcal{J}$ and $\psi\in\mathcal{B}$

$$|\mathcal{A}(x)\ g(x,\psi)|\leq L\ (\|\psi\|_{\mathcal{B}}+1).$$

$(H7)$ There exists a constant $L_*>0$ such that

$$|\mathcal{A}(y)\ g(y,\psi)-\mathcal{A}(\overline{y})\ g(\overline{y},\overline{\psi})|\leq L_*\ (|y-\overline{y}|+\|\psi-\overline{\psi}\|_{\mathcal{B}})$$

for all $y,\overline{y}\in\mathcal{J}$ and $\psi,\overline{\psi}\in\mathcal{B}$.

Consider the following space

$$B_{+\infty}=\left\{y:\mathbb{R}\to E: y|_{[0,T]}\text{ continuous for }T>0\text{ and }y_0\in\mathcal{B}\right\}$$

where $y|_{[0,T]}$ is the restriction of y to the real compact interval $[0,T]$.

Let us fix $\tau > (1-\overline{M}_0 L_* \kappa_n)^{-1}$ for every $n \in \mathbb{N}$ and define in $B_{+\infty}$ the seminorms by

$$\|u\|_n := \sup_{x\in[0,n]} e^{-\tau\ L_n^*(x)}\ |u(x)|,$$

where $L_n^*(x) = \int_0^x \bar{l}_n(y)\ dy$ with $\bar{l}_n(x) = \widehat{\mathcal{M}} n S_n \kappa_n l_n(x)$ and l_n is the function from $(H4)$. Then $B_{+\infty}$ is a Fréchet space with those family of seminorms $\|\cdot\|_{n\in\mathbb{N}}$.

Theorem 3.2 *Under the hypotheses* (H_ψ), $(H1)-(H7)$ *and moreover for all* $n \in \mathbb{N}$, *if*

$$\int_{\delta_n}^{+\infty} \frac{dy}{y+\Psi(y)} > \frac{\kappa_n \widehat{\mathcal{M}}}{(1-\overline{M}_0 L\kappa_n)} \int_0^n \max(L; nS_n p(y))\ dy \tag{3.2}$$

with

$$\delta_n = \left(\Pi_n + \mathcal{L}^\psi + \kappa_n \widehat{\mathcal{M}}\epsilon + \frac{\kappa_n \overline{M}_0 L\left[\Pi_n + \mathcal{L}^\psi + (\kappa_n\epsilon+1)\widehat{\mathcal{M}}\right]}{(1-\overline{M}_0 L\kappa_n)}\right)\|\psi\|_{\mathcal{B}} + \frac{\kappa_n[\overline{M}_0 L(\widehat{\mathcal{M}}+1)+\widehat{\mathcal{M}}Ln]}{(1-\overline{M}_0 L\kappa_n)},$$

then problem (1.1) $-$ (1.2) *has at least one mild solution on* $\mathbb{R}$.

Proof. We will transform the problem (1.1) $-$ (1.2) into a problem of fixed-point. Consider the multivalued operator $\aleph : B_{+\infty} \to B_{+\infty}$ defined by

$$\aleph(u) = \left\{ \mathfrak{h} \in B_{+\infty} : \mathfrak{h}(x) = \begin{cases} \psi(x), & \text{if } x \in \overline{\mathcal{J}}; \\ \mathcal{U}(t,0)[\psi(0)-g(0,\psi)] + g(x,u_{\rho(x,u_x)}) + \int_0^x \mathcal{U}(x,y)\mathcal{A}(y)g(y,u_{\rho(y,u_y)})dy & \\ +\int_0^x \mathcal{U}(x,y)\int_0^y \mathcal{I}(y,z)\mathfrak{f}(z)\ dz\ dy, & \text{if } x \in \mathcal{J} \end{cases} \right\}$$

where $\mathfrak{f} \in S_{\mathfrak{F},u} = \{v \in L^1(\mathcal{J},\mathbb{E}) : v(z) \in \mathfrak{F}(z, u_{\rho(z,u_z)})$ for almost every $z \in \mathcal{J}\}$. Clearly, $\aleph$ fixed-points are mild solutions of (1.1) $-$ (1.2)'s problem. We see also that, for each $y \in B_{+\infty}$, the set $S_{\mathfrak{F},u}$ is nonempty since by $(H3)$, $\mathfrak{F}$ has a measurable selection as in Theorem III.6 of [17]. For $\psi \in \mathcal{B}$, let define the function $\omega(.) : \mathbb{R} \to \mathbb{E}$ by

$$\omega(x) = \begin{cases} \psi(x), & \text{if } x \in \overline{\mathcal{J}}; \\ \mathcal{U}(x,0)\ \psi(0), & \text{if } x \in \mathcal{J}. \end{cases}$$

Then $\omega_0 = \psi$. For each function $\varpi \in B_{+\infty}$, set that

$$u(x) = \varpi(x) + \omega(x).$$

Obviously u verifies (3.1) if and only if ϖ verifies $\varpi_0 = 0$ and

$$\begin{aligned}\varpi(x) &= g(x,\Theta(x,\rho,\varpi,\omega)) - \mathcal{U}(t,0)g(0,\psi) + \int_0^x \mathcal{U}(x,y)\mathcal{A}(y)g(y,\Theta(x,\rho,\varpi,\omega))dy \\ &+ \int_0^x \mathcal{U}(x,y)\int_0^y \mathcal{I}(y,z)\mathfrak{f}(z)\ dz\ dy,\end{aligned}$$

where $\Theta(x,\rho,\varpi,\omega) = \varpi_{\rho(x,\varpi_x+\omega_x)} + \omega_{\rho(x,\varpi_x+\omega_x)}$ and $\mathfrak{f}(z) \in \mathfrak{F}(z,\Theta(z,\rho,\varpi,\omega))$.

Let

$$B^0_{+\infty} = \{\varpi \in B_{+\infty} : \varpi_0 = 0\}.$$

For any $\varpi \in B^0_{+\infty}$, we have

$$\|\varpi\|_{+\infty} = \|\varpi_0\|_{\mathcal{B}} + \sup_{0\le x<+\infty}|\varpi(x)| = \sup_{0\le x<+\infty}|\varpi(x)|.$$

Thus $(B^0_{+\infty}, \|\cdot\|_{+\infty})$ is a Banach space.

We define the operator $\widetilde{\aleph} : B^0_{+\infty} \to B^0_{+\infty}$ by

$$\widetilde{\aleph}(\varpi)(x) = \left\{\begin{array}{rcl} \mathfrak{h} \in B^0_{+\infty} : \mathfrak{h}(x) &=& g(x,\Theta(x,\rho,\varpi,\omega)) - \mathcal{U}(t,0)g(0,\psi) + \int_0^x \mathcal{U}(x,y)\mathcal{A}(y)g(y,\Theta(x,\rho,\varpi,\omega))dy \\ &+& \int_0^x \mathcal{U}(x,y)\int_0^y \mathcal{I}(y,z)\mathfrak{f}(z)\ dz\ dy, \quad x \in \mathcal{J} \end{array}\right\}$$

where $\mathfrak{f} \in S_{\mathfrak{F},\varpi} = \{v \in L^1(\mathcal{J},\mathbb{E}) : v(x) \in \mathfrak{F}(z,\Theta(z,\rho,\varpi,\omega))$ a. e. $x \in \mathcal{J}\}$.

Obviously the operator $\aleph$ having a fixed-point is equivalent to $\widetilde{\aleph}$ having one, so prove it in the next steps.

Step 1 : Estimates of solutions. Let $\varpi \in B^0_{+\infty}$ be a possible fixed point of the operator $\widetilde{\aleph}$. Given $n \in \mathbb{N}$, then ϖ should be solution of the inclusion $\varpi \in \lambda\widetilde{\aleph}(\varpi)$ for some $\lambda \in (0,1)$ and there exists $\mathfrak{f} \in S_{\mathfrak{F},\varpi} \Leftrightarrow \mathfrak{f}(x) \in \mathfrak{F}(z,\Theta(x,\rho,\varpi,\omega))$ such that, for each $z \in [0,n]$ we have

$$\begin{aligned}|\varpi(x)| &\le \lambda\Big[|g(x,\Theta(x,\rho,\varpi,\omega))| + \|\mathcal{U}(t,0)\|_{B(\mathbb{E})}|g(0,\psi)| + \int_0^x \|\mathcal{U}(x,y)\|_{B(\mathbb{E})}|\mathcal{A}(y)g(y,\Theta(x,\rho,\varpi,\omega))|dy \\ &+ \int_0^x \|\mathcal{U}(x,y)\|_{B(\mathbb{E})}\int_0^y |\mathcal{I}(y,z)||\mathfrak{f}(z)|\ dz\ dy\Big].\end{aligned}$$

By $(H1),(H2),(H3)$, $(H5)$ and $(H6)$, we get

$$\begin{aligned}|\varpi(x)| &\le \|\mathcal{A}^{-1}(x)\|_{B(\mathbb{E})}|\mathcal{A}(x)g(x,\Theta(x,\rho,\varpi,\omega))| + \widehat{\mathcal{M}}\|\mathcal{A}^{-1}(0)\|_{B(\mathbb{E})}|\mathcal{A}(0)g(0,\psi)| \\ &+ \widehat{\mathcal{M}}\int_0^x |\mathcal{A}(y)g(y,\Theta(x,\rho,\varpi,\omega))|\ dy + \widehat{\mathcal{M}}\ n \sup_{x\in[0,n]}|\mathcal{I}(x)|\int_0^x |\mathfrak{f}(y)|\ dy, \\ &\le \overline{M}_0L(\|\Theta(x,\rho,\varpi,\omega)\|_{\mathcal{B}}+1) + \widehat{\mathcal{M}}\overline{M}_0L(\|\psi\|_{\mathcal{B}}+1) + \widehat{\mathcal{M}}L\int_0^x (\|\Theta(y,\rho,\varpi,\omega)\|_{\mathcal{B}}+1)\ dy \\ &+ \widehat{\mathcal{M}}\ n\ S_n\int_0^x p(y)\Psi(\|\Theta(y,\rho,\varpi,\omega)\|_{\mathcal{B}})\ dy.\end{aligned}$$

Hence, we obtain

$$\begin{aligned} |\varpi(x)| &\leq \overline{M}_0L(\widehat{\mathcal{M}}+1)+\widehat{\mathcal{M}}Ln+\widehat{\mathcal{M}}\overline{M}_0L\|\psi\|_{\mathcal{B}}+\overline{M}_0L\|\Theta(x,\rho,\varpi,\omega)\|_{\mathcal{B}} \\ &+ \widehat{\mathcal{M}}L\int_0^x\|\Theta(y,\rho,\varpi,\omega)\|_{\mathcal{B}}\,dy+\widehat{\mathcal{M}}\;n\;S_n\int_0^x p(y)\Psi(\|\Theta(y,\rho,\varpi,\omega)\|_{\mathcal{B}})\,dy. \end{aligned} \tag{3.3}$$

From the hypothesis (H_ψ), the axiom (A_1), the definitions of ϖ and ω, the Lemma 2.7 and the Proposition 2.8, we get for each $x\in[0,n]$

$$\begin{aligned} \|\Theta(x,\rho,\varpi,\omega)\|_{\mathcal{B}} &\leq \|\varpi_{\rho(x,\varpi_x+\omega_x)}\|_{\mathcal{B}}+\|\omega_{\rho(x,\varpi_x+\omega_x)}\|_{\mathcal{B}}, \\ &\leq \kappa_n(|\varpi(x)|+|\omega(x)|)+(\Pi_n+\mathcal{L}^\psi)(\|\varpi_0\|_{\mathcal{B}}+\|\omega_0\|_{\mathcal{B}}), \\ &\leq \kappa_n(|\varpi(x)|+\|\mathcal{U}(x,0)\|_{B(\mathbb{E})}|\psi(0)|)+(\Pi_n+\mathcal{L}^\psi)\|\psi\|_{\mathcal{B}}, \\ &\leq \kappa_n(|\varpi(x)|+\widehat{\mathcal{M}}|\psi(0)|)+(\Pi_n+\mathcal{L}^\psi)\|\psi\|_{\mathcal{B}}. \end{aligned}$$

Using (ii), we get

$$\|\Theta(x,\rho,\varpi,\omega)\|_{\mathcal{B}} \leq \kappa_n|\varpi(x)|+(\Pi_n+\mathcal{L}^\psi+\kappa_n\widehat{\mathcal{M}}\epsilon)\|\psi\|_{\mathcal{B}}.$$

Set $c_n:=(\Pi_n+\mathcal{L}^\psi+\kappa_n\widehat{\mathcal{M}}\epsilon)\|\psi\|_{\mathcal{B}}$. Then

$$\|\Theta(x,\rho,\varpi,\omega)\|_{\mathcal{B}}\leq\kappa_n|\varpi(x)|+c_n. \tag{3.4}$$

Then the inequality (3.3) becomes using the increasing character of Ψ

$$\begin{aligned} |\varpi(x)| &\leq \overline{M}_0L(\widehat{\mathcal{M}}+1)+\widehat{\mathcal{M}}Ln+\widehat{\mathcal{M}}\overline{M}_0L\|\psi\|_{\mathcal{B}}+\overline{M}_0L(\kappa_n|\varpi(x)|+c_n) \\ &+ \widehat{\mathcal{M}}L\int_0^x(\kappa_n|\varpi(y)|+c_n)\,dy+\widehat{\mathcal{M}}\;n\;S_n\int_0^x p(y)\;\Psi\left(\kappa_n|\varpi(y)|+c_n\right)\,dy. \end{aligned}$$

So

$$\begin{aligned} (1-\overline{M}_0L\kappa_n)|\varpi(x)| &\leq \overline{M}_0L(\widehat{\mathcal{M}}+1)+\widehat{\mathcal{M}}Ln+\overline{M}_0Lc_n+\widehat{\mathcal{M}}\overline{M}_0L\|\psi\|_{\mathcal{B}} \\ &+ \widehat{\mathcal{M}}L\int_0^x(\kappa_n|\varpi(y)|+c_n)\,dy+\widehat{\mathcal{M}}\;n\;S_n\int_0^x p(y)\;\Psi\left(\kappa_n|\varpi(y)|+c_n\right)\,dy. \end{aligned}$$

Set $\delta_n:=c_n+\dfrac{\kappa_n}{(1-\overline{M}_0L\kappa_n)}\left[\overline{M}_0L(\widehat{\mathcal{M}}+1)+\widehat{\mathcal{M}}Ln+\overline{M}_0Lc_n+\widehat{\mathcal{M}}\overline{M}_0L\|\psi\|_{\mathcal{B}}\right]$. Thus

$$\begin{aligned} \kappa_n|\varpi(x)|+c_n \leq \delta_n+\frac{\kappa_n\widehat{\mathcal{M}}}{(1-\overline{M}_0L\kappa_n)}\Bigg[&L\int_0^x(\kappa_n|\varpi(y)|+c_n)dy \\ &+nS_n\int_0^x p(y)\Psi\left(\kappa_n|\varpi(y)|+c_n\right)dy\Bigg]. \end{aligned} \tag{3.5}$$

Consider the function μ defined by

$$\mu(x)=\sup\;\{\kappa_n|\varpi(y)|+c_n\;:\;0\leq y\leq x\}\,,\quad 0\leq x\leq+\infty.$$

Let $x^\star \in [0,x]$ be such that

$$\mu(x) = \kappa_n|\zeta(x^\star)| + c_n.$$

From the inequality (3.5), we get for all $x \in [0,n]$

$$\mu(x) \quad \leq \quad \delta_n + \frac{\kappa_n\widehat{\mathcal{M}}}{(1-\overline{M}_0 L\kappa_n)}\left[L\int_0^x \mu(y)dy + nS_n\int_0^x p(y)\Psi\left(\mu(y)\right)dy\right].$$

Let us take the right-hand side of the above inequality as $\nu(x)$. Then, we have

$$\mu(x) \leq \nu(x) \quad \forall x \in [0,n].$$

From the definition of ν, we get

$$\nu(0) = \delta_n \quad \text{and} \quad \nu'(x) = \frac{\kappa_n\widehat{\mathcal{M}}}{(1-\overline{M}_0 L\kappa_n)}\left[L\mu(x) + nS_n p(x)\Psi\left(\mu(x)\right)\right] \quad \text{a.e. } x \in [0,n].$$

Using the increasing character of Ψ, we have

$$\nu'(x) \leq \frac{\kappa_n\widehat{\mathcal{M}}}{(1-\overline{M}_0 L\kappa_n)}\left[L\nu(x) + nS_n p(x)\Psi\left(\nu(x)\right)\right] \quad \text{a.e. } x \in [0,n].$$

Hence

$$\nu'(x) \leq \frac{\kappa_n\widehat{\mathcal{M}}}{(1-\overline{M}_0 L\kappa_n)}\max(L;nS_n p(x))\left[\nu(x) + \Psi\left(\nu(x)\right)\right] \quad \text{a.e. } x \in [0,n].$$

So, using (3.2) for each $x \in [0,n]$, we have

$$\begin{aligned}\int_{\delta_n}^{\nu(x)} \frac{dy}{y+\Psi(y)} \quad &\leq \quad \frac{\kappa_n\widehat{\mathcal{M}}}{(1-\overline{M}_0 L\kappa_n)}\int_0^x \max(L;nS_n p(y))\; dy,\\ &\leq \quad \frac{\kappa_n\widehat{\mathcal{M}}}{(1-\overline{M}_0 L\kappa_n)}\int_0^n \max(L;nS_n p(y))\; dy,\\ &< \quad \int_{\delta_n}^{+\infty} \frac{dy}{y+\Psi(y)}.\end{aligned}$$

Thus, for every $x \in [0,n]$, there exists a constant Λ_n such that $\nu(x) \leq \Lambda_n$ and hence $\mu(x) \leq \Lambda_n$. Since $\|\varpi\|_n \leq \mu(x)$, we have $\|\varpi\|_n \leq \Lambda_n$. Set

$$\mathfrak{U} = \left\{\ \varpi \in B^0_{+\infty} : \sup_{0\leq x\leq n} |\varpi(x)| \leq \Lambda_n + 1 \quad \text{for all} \;\; n \in \mathbb{N}\right\}.$$

Clearly, $\mathfrak{U}$ is an open subset of $B^0_{+\infty}$.

Step 2 : We shall show that $\widetilde{\aleph} : \overline{\mathfrak{U}} \to \mathcal{P}(B^0_{+\infty})$ is a multivalued contraction.

Let $\varpi, \overline{\varpi} \in B^0_{+\infty}$ and $\mathfrak{h} \in \widetilde{\aleph}(\varpi)$. Then there exists $\mathfrak{f}(x) \in \mathfrak{F}(x, \Theta(x,\rho,\varpi,\omega))$ such that for each $x \in [0,n]$

$$\begin{aligned} \mathfrak{h}(x) &= g(x,\Theta(x,\rho,\varpi,\omega)) - \mathcal{U}(t,0)g(0,\psi) + \int_0^x \mathcal{U}(x,y)\mathcal{A}(y)g(y,\Theta(x,\rho,\varpi,\omega))dy \\ &+ \int_0^x \mathcal{U}(x,y)\int_0^y \mathcal{I}(y,z)\mathfrak{f}(z)\ dz\ dy. \end{aligned}$$

From $(H7)$, we get

$$|\mathcal{A}(y)\ [g(y,\Theta(x,\rho,\varpi,\omega)) - g(y,\Theta(x,\rho,\overline{\varpi},\omega))]| \le L_*\ \|\Theta(x,\rho,\varpi,\omega) - \Theta(x,\rho,\overline{\varpi},\omega)\|_{\mathcal{B}}.$$

From $(H4)$, we have that

$$H_d\left(\mathfrak{F}\left(x,\Theta(x,\rho,\varpi,\omega)\right), \mathfrak{F}\left(\Theta(x,\rho,\overline{\varpi},\omega)\right)\right) \le l_n(x)\ \|\Theta(x,\rho,\varpi,\omega) - \Theta(x,\rho,\overline{\varpi},\omega)\|_{\mathcal{B}}.$$

Set $\Upsilon(x,\rho,\varpi,\omega,\overline{\varpi},\overline{\omega},\mathcal{B}) = \|\Theta(x,\rho,\varpi,\omega) - \Theta(x,\rho,\overline{\varpi},\omega)\|_{\mathcal{B}}$.
Hence, there is $\vartheta \in \mathfrak{F}\left(\Theta(x,\rho,\overline{\varpi},\omega)\right)$ such that for $x \in [0,n]$

$$|\mathfrak{f}(x) - \vartheta| \le l_n(x)\ \Upsilon(x,\rho,\varpi,\omega,\overline{\varpi},\overline{\omega},\mathcal{B}).$$

Consider $\mathcal{U}_\star : [0,n] \to \mathcal{P}(\mathbb{E})$, given by

$$\mathcal{U}_\star = \{\vartheta \in \mathbb{E} : |\mathfrak{f}(x) - \vartheta| \le l_n(x)\ \Upsilon(x,\rho,\varpi,\omega,\overline{\varpi},\overline{\omega},\mathcal{B})\}$$

Since the multivalued operator $\mathcal{V}(x) = \mathcal{U}_\star(x) \cap \mathfrak{F}\left(\Theta(x,\rho,\overline{\varpi},\omega)\right)$ is measurable (in [17] as in Proposition III.4), so there exists a function $\bar{\mathfrak{f}}(x)$, which is also a measurable selection for $\mathcal{V}$. So, for $\bar{\mathfrak{f}}(x) \in \mathfrak{F}\left(\Theta(x,\rho,\overline{\varpi},\omega)\right)$, we have for each $x \in [0,n]$

$$|\mathfrak{f}(x) - \bar{\mathfrak{f}}(x)| \le l_n(x)\ \Upsilon(x,\rho,\varpi,\omega,\overline{\varpi},\overline{\omega},\mathcal{B}). \tag{3.6}$$

Using (A_1), Lemma 2.7 and Proposition 2.8, we get for each $x \in [0,n]$

$$\begin{aligned} \Upsilon(x,\rho,\varpi,\omega,\overline{\varpi},\overline{\omega},\mathcal{B}) &\le \|\varpi_{\rho(x,\varpi_x+\omega_x)} - \overline{\varpi}_{\rho(x,\overline{\varpi}_x+\omega_x)}\|_{\mathcal{B}} + \|\omega_{\rho(x,\varpi_x+\omega_x)} - \omega_{\rho(x,\overline{\varpi}_x+\omega_x)}\|_{\mathcal{B}}, \\ &\le \kappa(x)|\varpi(x) - \overline{\varpi}(x)| + \left(\Pi(x) + \mathcal{L}^\psi(x)\right)\|\varpi_0 - \overline{\varpi}_0\|_{\mathcal{B}} \\ &+ \kappa(x)|\omega(x) - \overline{\omega}(x)| + \left(\Pi(x) + \mathcal{L}^\psi(x)\right)\|\omega_0 - \overline{\omega}_0\|_{\mathcal{B}}. \end{aligned}$$

By the definition of $B_{+\infty}$ and $B^0_{+\infty}$, we have the fact that $\varpi_0 = \overline{\varpi}_0 = 0$, $\omega_0 = \overline{\omega}_0 = \psi$ and $\omega(x) = \overline{\omega}(x) = \mathcal{U}(x,0)\psi(0)$. Then we get

$$\Upsilon(x,\rho,\varpi,\omega,\overline{\varpi},\overline{\omega},\mathcal{B}) \le \kappa_n\ |\varpi(x) - \overline{\varpi}(x)|. \tag{3.7}$$

Let us define, for each $x \in [0,n]$

$$\begin{aligned} \overline{\mathfrak{h}}(x) &= g(x,\Theta(x,\rho,\overline{\varpi},\omega)) - \mathcal{U}(t,0)g(0,\psi) + \int_0^x \mathcal{U}(x,y)\mathcal{A}(y)g(y,\Theta(x,\rho,\overline{\varpi},\omega))dy \\ &+ \int_0^x \mathcal{U}(x,y)\int_0^y \mathcal{I}(y,z)\bar{\mathfrak{f}}(z)\ dz\ dy. \end{aligned}$$

Then, by $(H1)$, $(H2)$, $(H5)$, $(H7)$ and the inequality (3.6), we have

$$\begin{aligned}
|\mathfrak{h}(x)-\overline{\mathfrak{h}}(x)| &\leq \|\mathcal{A}^{-1}(x)\|_{B(\mathbb{E})}|\mathcal{A}(x)[g(x,\Theta(x,\rho,\varpi,\omega))-g(x,\Theta(x,\rho,\overline{\varpi},\omega))]| \\
&+ \int_0^x \|\mathcal{U}(x,y)\|_{B(E)}|\mathcal{A}(y)[g(y,\Theta(x,\rho,\varpi,\omega))-g(y,\Theta(x,\rho,\overline{\varpi},\omega))]|dy \\
&+ \int_0^x \|\mathcal{U}(x,y)\|_{B(E)}\int_0^y |\mathcal{I}(y,z)||\mathfrak{f}(z)-\overline{\mathfrak{f}}(z)|\ dz\ dy, \\
&\leq \overline{M}_0 L_*\ \Upsilon(x,\rho,\varpi,\omega,\overline{\varpi},\overline{\omega},\mathcal{B}) + \int_0^x \widehat{\mathcal{M}} L_*\ \Upsilon(y,\rho,\varpi,\omega,\overline{\varpi},\overline{\omega},\mathcal{B})dy \\
&+ \int_0^x \widehat{\mathcal{M}}\ n \sup_{y\in[0,n]}|\mathcal{I}(y)|\ l_n(y)\ \Upsilon(y,\rho,\varpi,\omega,\overline{\varpi},\overline{\omega},\mathcal{B})\ dy.
\end{aligned}$$

From the inequality (3.7) and by the hypothesis $(H4)$, we get

$$\begin{aligned}
|\mathfrak{h}(x)-\overline{\mathfrak{h}}(x)| &\leq \overline{M}_0 L_*\kappa_n\ |\varpi(x)-\overline{\varpi}(x)| + \int_0^x \widehat{\mathcal{M}} L_*\kappa_n\ |\varpi(y)-\overline{\varpi}(y)|\ dy \\
&+ \int_0^x \widehat{\mathcal{M}} n S_n \kappa_n l_n(y)\ |\varpi(y)-\overline{\varpi}(y)|\ dy, \\
&\leq \overline{M}_0 L_*\kappa_n|\varpi(x)-\overline{\varpi}(x)| + \int_0^x \widehat{\mathcal{M}}\kappa_n\,[L_*+nS_n l_n(y)]\,|\varpi(y)-\overline{\varpi}(y)|dy.
\end{aligned}$$

Let us take here $L_n^*(x)=\displaystyle\int_0^x \bar{l}_n(y)\ dy$ where $\bar{l}_n(x):=\widehat{\mathcal{M}}\kappa_n\,[L_*+nS_n l_n(x)]$ and l_n is the function from $(H4)$, we have

$$\begin{aligned}
|\mathfrak{h}(x)-\overline{\mathfrak{h}}(x)| &\leq \overline{M}_0 L_*\kappa_n\ |\varpi(x)-\overline{\varpi}(x)| + \int_0^x \bar{l}_n(y)\ |\varpi(y)-\overline{\varpi}(y)|\ dy, \\
&\leq \overline{M}_0 L_*\kappa_n\ \ e^{\tau\ L_n^*(x)}\ ]\ [e^{-\tau\ L_n^*(x)}|\varpi(x)-\overline{\varpi}(x)|] \\
&+ \int_0^x [\bar{l}_n(y)\ e^{\tau\ L_n^*(y)}\ ]\ [e^{-\tau\ L_n^*(y)}\ |\varpi(y)-\overline{\varpi}(y)|]\ dy.
\end{aligned}$$

The definition of the seminorms leads to

$$\begin{aligned}
|\mathfrak{h}(x)-\overline{\mathfrak{h}}(x)| &\leq \overline{M}_0 L_*\kappa_n\ \ e^{\tau\ L_n^*(x)}\ ]\ \|\varpi-\overline{\varpi}\|_n + \int_0^x \left[\frac{e^{\tau\ L_n^*(y)}}{\tau}\right]'\ dy\ \|\varpi-\overline{\varpi}\|_n, \\
&\leq \left[M_0 L_*\kappa_n + \frac{1}{\tau}\right]\ e^{\tau\ L_n^*(x)}\ \|\varpi-\overline{\varpi}\|_n.
\end{aligned}$$

Therefore,

$$\|\mathfrak{h}-\overline{\mathfrak{h}}\|_n \leq \left[M_0 L_*\kappa_n + \frac{1}{\tau}\right]\|\varpi-\overline{\varpi}\|_n.$$

By interchanging ϖ and $\overline{\varpi}$ roles, we obtain analogously

$$H_d\left(\widetilde{\aleph}(\varpi),\widetilde{\aleph}(\overline{\varpi})\right) \leq \left[M_0 L_*\kappa_n + \frac{1}{\tau}\right]\|\varpi-\overline{\varpi}\|_n.$$

Since,

$$M_0 L_* \kappa_n + \frac{1}{\tau} < 1,$$

so the multivalued operator $\widetilde{\aleph}$ is a contraction for all $n \in \mathbb{N}$.

Step 3 : $\widetilde{\aleph}$ is an admissible multivalued operator.

Let $\varpi \in B^0_{+\infty}$. Set, for every $n \in \mathbb{N}$, the space

$$B^0_n := \{u : (-\infty, n] \to E : u|_{[0,n]} \in C([0,n], E), \quad u_0 \in \mathcal{B}\}$$

and let us consider the multivalued operator $\widetilde{\aleph} : B^0_n \to \mathcal{P}_{cl}(B^0_n)$ defined by:

$$\widetilde{\aleph}(\varpi)(x) = \left\{ \begin{array}{lcl} \mathfrak{h} \in B^0_n : \mathfrak{h}(x) & = & g(x, \Theta(x,\rho,\varpi,\omega)) - \mathcal{U}(t,0)g(0,\psi) + \int_0^x \mathcal{U}(x,y)\mathcal{A}(y)g(y,\Theta(x,\rho,\varpi,\omega))dy \\ & + & \int_0^x \mathcal{U}(x,y) \int_0^y \mathcal{I}(y,z)\mathfrak{f}(z)\; dz\; dy, \quad x \in [0,n] \end{array} \right\}$$

where $\mathfrak{f} \in S^n_{\mathfrak{F},\varpi} = \{v \in L^1([0,n], \mathbb{E}) : v(x) \in \mathfrak{F}(z, \Theta(z,\rho,\varpi,\omega))$ a. e. $x \in [0,n]\}$.

By the hypotheses $(H1)$, $(H3)$ and since $\mathfrak{F}$ is a multivalued map with compact values, we can show here that for every $\varpi \in B^0_n$, $\widetilde{\aleph}(\varpi) \in \mathcal{P}_{cl}(B^0_n)$ and there exists $\varpi_\star \in B^0_n$ such that $\varpi_\star \in \widetilde{\aleph}(\varpi_\star)$. Let $\mathfrak{h} \in B^0_n$, $\overline{\varpi} \in \overline{\mathfrak{U}}$ and $\alpha > 0$. Assume that $\varpi_\star \in \widetilde{\aleph}(\overline{\varpi})$, then we have

$$\begin{array}{lcl} |\overline{\varpi}(x) - \varpi_\star(x)| & \leq & |\overline{\varpi}(x) - \mathfrak{h}(x)| + |\varpi_\star(x) - \mathfrak{h}(x)| \\ & \leq & e^{\tau\; L^*_n(x)} \left[\|\overline{\varpi} - \widetilde{\aleph}(\overline{\varpi})\|_n + \|\varpi_\star - \mathfrak{h}\|_n\right]. \end{array}$$

Since $\mathfrak{h}$ is arbitrary, we may suppose that $\mathfrak{h} \in B(\varpi_\star, \alpha) = \{\mathfrak{h} \in B^0_n : \|\mathfrak{h} - \varpi_\star\|_n \leq \alpha\}$.

Therefore,

$$\|\overline{\varpi} - \varpi_\star\|_n \leq \|\overline{\varpi} - \aleph(\overline{\varpi})\|_n + \alpha.$$

If ϖ is not in $\aleph\overline{\varpi})$, then $\|\varpi_\star - \widetilde{\aleph}(\overline{\varpi})\| \neq 0$. Since $\widetilde{\aleph}(\overline{\varpi})$ is compact, there exists $\omega \in \widetilde{\aleph}(\overline{\varpi})$ such that $\|\varpi_\star - \widetilde{\aleph}(\overline{\varpi})\| = \|\varpi_\star - \omega\|$. Then we have

$$\begin{array}{lcl} |\overline{\varpi}(x) - \omega(x)| & \leq & |\overline{\varpi}(x) - \mathfrak{h}(x)| + |\omega(x) - \mathfrak{h}(x)| \\ & \leq & e^{\tau\; L^*_n(x)} \left[\|\overline{\varpi} - \widetilde{\aleph}(\overline{\varpi})\|_n + \|\omega - \mathfrak{h}\|_n\right]. \end{array}$$

Thus,

$$\|\overline{\varpi} - \omega\|_n \leq \|\overline{\varpi} - \widetilde{\aleph}(\overline{\varpi})\|_n + \alpha.$$

So, $\widetilde{\aleph}$ is an admissible multivalued operator.

Finally, the three steps imply that $\widetilde{\aleph}$ is an admissible multivalued contraction and there is no $\varpi \in \partial\mathfrak{U}$ such that $\varpi \in \lambda\widetilde{\aleph}(\varpi)$ for some $\lambda \in (0,1)$ from the choice of $\mathfrak{U}$.

Then the statement $(Fr2)$ in Theorem 2.12 does not hold and the nonlinear alternative of Frigon implies only the fact that the statement $(Fr1)$ is verified, we deduce so that the multivalued operator $\widetilde{\aleph}$ has a fixed-point $\varpi^{\star}$.

So $u^{\star}(x) = \varpi^{\star}(x) + \omega(x)$, $x \in (-\infty, +\infty)$ is a fixed point of the operator $\aleph$, which is a mild solution of problem (1.1) − (1.2). □

4. Example

To illustrate the previous results, we consider the neutral functional differential inclusion

$$\begin{cases} \dfrac{\partial}{\partial x}\left[u(x,\xi) - \displaystyle\int_{-\infty}^{0} a_1(y-x)u\left(y-\rho_1(x)\rho_2\left(\int_0^{\pi} a_2(z)|u(z,y)|^2 dz\right),\xi\right)dy\right] \in \dfrac{\partial^2 u(x,\xi)}{\partial \xi^2} \\ +a_0(x,\xi)u(x,\xi) + \displaystyle\int_{-\infty}^{x} \eta(x,y)\int_{-\infty}^{0} \mathcal{D}\left(y-x, u\left(z-\rho_1(y)\rho_2\left(\int_0^{\pi} a_2(t)|u(y,x)|^2 dt\right),\xi\right)\right) dydz, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a. e. } x \geq 0,\ \xi \in [0,\pi], \\ u(x,0) = u(x,\pi) = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \geq 0, \\ u(t,\xi) = u_0(t,\xi), \qquad\qquad\qquad\qquad\qquad\qquad -\infty < t \leq 0,\ \xi \in [0,\pi], \end{cases} \tag{4.1}$$

where $a_0(x,\xi)$ is a continuous function and is uniformly Hölder continuous in x; $a_1 : \mathbb{R}_- \to \mathbb{R}$ and $a_2 : [0,\pi] \to \mathbb{R}$, $\rho_i : [0,+\infty[\to \mathbb{R}$ for $i = 1,2$ and $\eta : \mathbb{R}_+^2 \to \mathbb{R}$ are given continuous functions and $\mathcal{D} : \mathbb{R}^+ \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued application with compact and convex values.

For this system, we are going to consider the space $\mathbb{E} = L^2([0,\pi],\mathbb{R})$ and the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{E} \to \mathbb{E}$ given by $\mathcal{A}w = w''$ with

$$D(\mathcal{A}) := \{\ w \in \mathbb{E}\ : w'' \in \mathbb{E},\ w(0) = w(\pi) = 0\ \}.$$

Thus $\mathcal{A}$ is the infinitesimal generator of an analytic semigroup $\{T(x)\}_{x\geq 0}$ on $\mathbb{E}$. Hence, $\mathcal{A}$ has a discrete spectrum with $-n^2$ eigenvalues, $n \in \mathbb{N}$ and the corresponding eigenfunctions are defined by

$$y_n(\xi) = \sqrt{\frac{2}{\pi}}\sin(n\xi)$$

are normalized.

In addition, $\{y_n : n \in \mathbb{N}\}$ is an orthonormal basis of $\mathbb{E}$ and

$$T(x)\omega = \sum_{n=1}^{\infty} e^{-n^2 x}(\omega, y_n)y_n \quad \text{for } \omega \in \mathbb{E} \quad \text{and } x \geq 0.$$

From this representation, then $T(x)$ is compact for every $x > 0$ and that $\|T(x)\| \leq e^{-x}$ for every $x \geq 0$.

Define the operator $\mathcal{A}(x) : D(\mathcal{A}) \subset \mathbb{E} \to \mathbb{E}$ on $D(\mathcal{A})$ by

$$\mathcal{A}(x)\omega(\xi) = \mathcal{A}\omega(\xi) + a_0(x,\xi)\omega(\xi).$$

Then since $a_0(\cdot)$ is continuous and that $a_0(x,\xi) \leq -\delta_0$ $(\delta_0 > 0)$ for every $x \in \mathbb{R}$, $\xi \in [0,\pi]$, it follows that the system

$$u'(x) = \mathcal{A}(x)u(x) \quad x \geq y,$$

$$u(y) = x \in \mathbb{E},$$

has an associated evolution family given by

$$\mathcal{U}(x,y)\omega(\xi) = \left[T(x-y)\exp\left(\int_y^x a_0(z,\xi)dz\right)\omega\right](\xi).$$

From this expression, it follows that $\mathcal{U}(x,y)$ is a compact linear operator and that

$$\|\mathcal{U}(x,y)\| \leq e^{-(1+\delta_0)(x-y)} \text{ for every } (x,y) \in \Lambda.$$

Theorem 4.1 *Let $\mathcal{B} = BUC(\mathbb{R}_-;\mathbb{E})$ and $\psi \in \mathcal{B}$. Assume that the condition (H_ψ) holds. Suppose that the functions $a_1 : \mathbb{R}_- \to \mathbb{R}$, $a_2 : [0,\pi] \to \mathbb{R}$, $\rho_i : \mathbb{R}_+ \longrightarrow \mathbb{R}$ for $i = 1,2$ and $\eta : \mathbb{R}_+^2 \to \mathbb{R}$ are given continuous functions and for $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued application with compact and convex values. Then there exists a mild solution of (4.1) on $]-\infty,+\infty[$.*

Proof. From the assumptions, we have that

$$\mathcal{I}(x,\psi) = \eta(x,\psi),$$

$$\mathfrak{F}(x,\psi)(\xi) = \int_{-\infty}^0 \mathcal{D}(y-x)\psi(y,\xi)dy,$$

$$g(x,\psi)(\xi) = \int_{-\infty}^0 a_1(y-x)\psi(y,\xi)dy$$

and

$$\rho(y,\psi) = y - \rho_1(y)\rho_2\left(\int_0^\pi a_2(t)|\psi(0,\xi)|^2 dt\right)$$

are defined functions, with it possible to transform system (4.1) into our abstract system (1.1)−(1.2). Moreover, the functions ρ_i are bounded and linear. So, we deduce the existence of mild solutions directly from application of Theorem 3.2 for a suitable given multifunction with compact and convex values $\mathfrak{F}$.
From Remark 2.6, we have the next result:

Corollary 4.2 *Let $\psi \in \mathcal{B}$ be continuous and bounded. Then there exists a mild solution of (4.1) on $\mathbb{R}$.*

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