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The Hewitt realcompactification of an orbit space

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Abstract: In this paper, we show that the statement in the study of Srivastava (1987) holds also for the Hewitt realcompactification. The mentioned statement showed that when the action of a finite topological group on a Tychonoff space is given, the Stone-Čech compactification of the orbit space of the action is the orbit space of the Stone-Čech compactification of the space. As an application, we show that Srivastava's result can be obtained using the main theorem of the present study.

Key words: Realcompactification, orbit space, real maximal ideal

1. Introduction

Let G be a topological group and X be a Hausdorff topological space. If the continuous map $\theta : G \times X \to X$ holds the following statements, then θ is called an action of G on X.

1. $\theta(e, x) = x$ for all $x \in X$, where e is the identity of G.
2. $\theta(g, \theta(h, x)) = \theta(gh, x)$ for all g, $h \in G$ and $x \in X$.

The space X, together with a given action θ of G is called a G-space. We shall use notation gx for $\theta(g, x)$. The space $X/G = \{G(x) : x \in X\}$ endowed with the quotient topology relative to π is called the orbit space of X, where π is the orbit map from X to X/G and $G(x) = \{gx : g \in G\}$ is orbit of x. A subspace A of a G-space X is called invariant, if $\theta(G \times A) = A$. A map f from a G-space X to a G-space Y is called G-equivariant if $f(gx) = gf(x)$ for all $g \in G$, $x \in X$.

Tychonoff [17] proved that a Tychonoff space could be imbedded in a compact Hausdorff space. In 1937, devoloping an idea of Tychonoff, Stone [13] and Čech [4] indepentdently introduced the existence and uniquness (up to homeomorphism) of the Stone–Čech compactification of a Tychonoff space. In 1938, a more general construction was given by Wallman [15] applied to any T_1 space X and produced the Stone–Čech compactification of X whenever X is T_4. Stone's original construction was simplified by Gelfand and Shilov [6] in 1941.

It is well known that every continuous map of a Tychonoff space into $\mathbb{R}$ may not be extended to a continuous map of the Stone–Čech compactification of the the space into $\mathbb{R}$. This leads to the notion of the

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(Hewitt) realcompactification of a Tychonoff space. If X is a Tychonoff space such that realcompactification of X is X, then X is said to be realcompact. Realcompact spaces were introduced by Hewitt [10].

In the theory of transformation groups, the structure of the orbit space plays an important role. Srivastava [12] showed that $\beta(X/G)$ is homeomorphic to $\beta(X)/G$ for a finite group G. Note that here, βX and $\beta(X/G)$ are the Stone–Čech compactifications of X and X/G, respectively.

In this paper, we show that the Hewitt realcompactification of the orbit space of X is the orbit space of the Hewitt realcompactification of X. Among different constructions of the Stone–Čech or the Hewitt realcompactifications, we will use the construction with maximal ideals or real maximal ideals of the ring of all real-valued contiuous functions on a Tychonoff space.

2. Preliminaries

We recall some basic notions and fundamental knowledge about continuous functions for more detail, see [8, 10, 14]. By a mapping we always mean a continuous function.

The set $C(X)$ of all real-valued continuos functions on a topological space X has an algebra structure under the pointwise operation, that is for each $f,\ g \in C(X)$ and $c \in \mathbb{R}$,

$$(f+g)(x) = f(x) + g(x),\ (fg)(x) = f(x)g(x), \text{ and } (cf)(x) = cf(x)$$

We shall say that a subspace S of X is C-embedded (C^*-embedded) in X if every (bounded) function in $C(S)$ ($C^*(S)$) can be extended to a function in $C(X)$ ($C^*(X)$).

Let f be a element of $C(X)$. The set $Z(f) = \{x \in X : f(x) = 0\}$ will be called the zero-set of f. For $A \subseteq C(X)$, we will write $Z[A]$ to designate the family of the zero-sets $\{Z(f) : f \in A\}$. Furthermore, the family $Z[C(X)]$ of all zero-sets in X will be denoted, for simplicity, by $Z(X)$.

Definition 2.1 *[8] A nonempty subfamily $\mathcal{F}$ of $Z(X)$ is called a z-filter on X provided that*

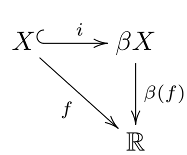
i) $\emptyset \notin \mathcal{F}$

ii) *if $Z_1,\ Z_2 \in \mathcal{F}$, then $Z_1 \cap Z_2 \in \mathcal{F}$*

iii) *if $Z \in \mathcal{F}$, $Z' \in Z(X)$, and $Z' \supset Z$, then $Z' \in \mathcal{F}$*

The collection of all z-filters on X, denoted by $F(X)$, is partially ordered by set inclusion. It is said that a proper z-filter in $F(X)$ is a z-ultrafilter in case it is maximal in $F(X)$. Since there is a bijective map between the set of all (maximal) ideals in $C(X)$ and the set of all z-(ultra) filters [8, Theorem 7.2], we can classifying maximal ideals in $C(X)$ using some definitions for ultrafilters. Therefore, if $\cap Z[I]$ is nonempty for an ideal I in $C(X)$, then we call I a fixed ideal; otherwise I is a free ideal.

Definition 2.2 *[14] Let X be a Tychonoff space. For every bounded, real valued continuous function, there is a unique compact space βX such that the following diagram is commutative*



where i is an imbedding with $i(X)$ dense in βX. βX is called as the Stone–Čech compactification of X.

Any element f of $C(X)$ is a mapping of X into $\alpha\mathbb{R} = \mathbb{R} \cup \{\infty\}$, the one point compactification of $\mathbb{R}$, and thus has an extension f^{α} which maps βX into $\alpha\mathbb{R}$.

$$\begin{array}{ccc} X & \hookrightarrow & \beta X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{\alpha}} \\ \mathbb{R} & \longrightarrow & \alpha\mathbb{R} \end{array}$$

Definition 2.3 *[14] If f is unbounded, there will be a point in $\beta X \backslash X$ at which f^{α} will take the value ∞. For each map f in $C(X)$, we define*

$$\nu_f X = \beta X \backslash \{p \in \beta X : f^{\alpha}(p) = \infty\}$$

Thus, $\nu_f X$ is the set of points of βX at which f^{α} is finite, and we will call $\nu_f X$ the set of real points of f. Let νX the subspace of βX consisting of points which are real points for every f in $C(X)$, i.e.

$$\nu X = \cap \{\nu_f X : f \in C(X)\}$$

The space νX is called the (Hewitt) realcompactification of X. A space X is said to be realcompact if $X = \nu X$, i.e. the only points which are real points for every f in $C(X)$ are the points of X itself. X is called pseudocompact if $\nu X = \beta X$.

Remark 2.4 *It is immediate that the subspace νX of βX is the largest subspace of βX to which every member of $C(X)$ can be extended without any extension taking on the value ∞. The extension of f to νX is denoted by $\nu(f)$ which is the restriction $\beta(f)|_{\nu X}$ [14]. Any continuous map $f : X \to Y$ induces the following commutative diagram.*

$$\begin{array}{ccc} X & \hookrightarrow & \nu X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \nu(f)} \\ Y & \hookrightarrow & \nu Y \end{array}$$

It is well known that [8, Corollary 8.5]

a) νX is the largest subspace of βX in which X is C-embedded.

b) νX is the smallest realcompact space between X and βX.

Now let $\mathcal{M} = \mathcal{M}(X)$ denote the set of all maximal ideals in $C(X)$. We can make $\mathcal{M}$ into a topological space by taking, as a base for the closed sets, all sets of form

$$\mathcal{S}(f) = \{M \in \mathcal{M} : f \in M\}, \, f \in C(X)$$

The topology thus defined is called the Stone topology on $\mathcal{M}$. The resultant topological space $\mathcal{M}$ is called the Structure space of the ring $C(X)$. It turns out that $\mathcal{M}$ is a compact Hausdorff space. Gelfand and Kolmogoroff showed that the maximal ideals of $C(X)$ are in one-to-one correspondence with the points of βX. It is worth noting that if $\mathcal{M}$ is topologized as above, then $\mathcal{M}$ is homeomorphic to βX [10, Theorem 46].

Theorem 2.5 (Gelfand–Kolmogoroff) *[7, 8] For every point p of βX, the set*

$$M^p = \{f \in C(X) : p \in Cl_{\beta X} Z(f)\}$$

is a maximal ideal of $C(X)$. Conversely, for every maximal ideal M of $C(X)$, there is a unique $p \in \beta X$ such that $M = M^p$. If $p \in X$, then M^p is the fixed ideal $M_p = \{f \in C(X) : f(p) = 0\}$; otherwise M^p is free.

It is well known that for each fixed maximal ideal M in $C(X)$, the quotient ring $C(X)/M$ is isomorphic to the real field $\mathbb{R}$. Note that for each maximal ideal M in $C(X)$, the quotient ring $C(X)/M$ always contains an isomorphic copy of $\mathbb{R}$ [10, 16]. Now we can give the following definition.

Definition 2.6 *[10] A maximal ideal M in $C(X)$ is said to be real in case the quotient ring $C(X)/M$ is isomorphic to $\mathbb{R}$, otherwise M is said to be hyperreal.*

We close this section with some fundamental knowledge of orbit space which will be need to be able to determine realcompactification of orbit space.

Lemma 2.7 *[2, Theorem 3.1] If X is a Hausdorff G-space with G compact, then*

1. *X/G is Hausdorff.*
2. *$\pi : X \to X/G$ is open and closed.*
3. *$\pi : X \to X/G$ is proper (π^{-1}(compact) is compact).*

Let X and Y be topological spaces and let f be a map from X to Y that is continuous, closed, surjective and $f^{-1}(y)$ is compact relative to X for each y in Y, then f is called as perfect map. Hence the orbit map is perfect.

Proposition 2.8 *[11] If X is a completely regular G-space, then the orbit space X/G is completely regular.*

Note that, more generally, if X is a completely regular and $f : X \to Y$ is a closed, open surjective mapping, then Y is also completely regular [3].

Lemma 2.9 *[5, 3.11.G] If there exists a perfect open mapping $f : X \to Y$ of a realcompact space X onto a Tychonoff space Y, then Y is realcompact.*

3. Main results

From now on, we shall consider βX as the space of all maximal ideals of $C(X)$. Since we mentioned that we would prefer the construction of the Stone–Čech compactification using maximal ideals, now we will prove the following proposition differently from Srivastava's method [12].

Proposition 3.1 *If G is a finite topological group and X is a Tychonoff G-space, then we can extend this action on βX.*

Proof Define $\varphi : G \times \beta X \to \beta X$, $\varphi(g, M) = g^{-1}M$, where $g^{-1}M = \{g^{-1}f : f \in M\}$ and $g^{-1}f : X \to \mathbb{R}$, $\left(g^{-1}f\right)(x) = f\left(g^{-1}x\right)$. It is trivial that $\varphi(e, M) = M$, and $\varphi(g_1, \varphi(g_2, M)) = \varphi(g_1 g_2, M)$. It is sufficient to show that the action is continuous. Since the collection $\{\mathcal{S}(f) : f \in C(X)\}$ is a base for the closed sets, then

$$\begin{aligned}\varphi^{-1}(\mathcal{S}(f)) &= \left\{(g, M) : \varphi(g, M) = g^{-1}M \in \mathcal{S}(f)\right\} \\ &= \left\{(g, M) : f \in g^{-1}M\right\} \\ &= \{(g, M) : gf \in M\} = \bigcup_{g \in G} \{g\} \times \{\mathcal{S}(gf)\}\end{aligned}$$

is the finite union of closed sets, and it is closed, so the action is continuos. Now we will show that the restriction on $G \times X$ of the action of G on βX is the action of G on X. Let $g \in G$ and $p \in X$. Since the elements of X are in one-to-one correspondence with the fixed maximal ideal M_p, we shall show that $g^{-1}M_p = M_{gp}$. If $g^{-1}f \in g^{-1}M_p$, then $\left(g^{-1}f\right)(gp) = f\left(g^{-1}gp\right) = f(p) = 0$. So $g^{-1}f \in M_{gp}$. On the contrary, if $f \in M_{gp}$, then $(gf)(p) = f(gp) = 0 \Rightarrow gf \in M_p$. Hence $f = g^{-1}(gf) \in g^{-1}M_p$. □

Remark 3.2 *It is shown [10] that the realcompact space νX is the family of all real maximal ideals of $C(X)$. Now, we will show that $\nu(X)$ is a G-invariant subspace. Suppose that $g \in G$ and $M \in \nu X$, i.e. M is a real maximal ideal of $C(X)$. Since M is real, then there is an isomorphism $\Phi : C(X)/M \to \mathbb{R}$. Therefore $\bar{\Phi} : C(X)/g^{-1}M \to \mathbb{R}$, $f + g^{-1}M \to \Phi(gf + M)$ is also an isomorphism, so $g^{-1}M$ is also real. Thus νX is a G-invariant subspace of βX.*

Remark 3.3 *Now, we shall show that any equivariant map induces equivariant map on realcompact spaces. That is, if X and Y are G-spaces and $f : X \to Y$ is a equivariant map, then $\upsilon(f) : \upsilon X \to \upsilon Y$ is also equivariant map. Let $M^p \in \upsilon X \subset \beta X$. Since X is dense in βX, then there exists a net $(M_i^p)_{i \in I}$ in X which convergences to M^p. Therefore we have $\upsilon(f)(gM^p) = \lim f(gM_i^p) = g \lim f(M_i^p) = g\upsilon(f)(M^p)$, which shows that $\upsilon(f) : \upsilon X \to \upsilon Y$ is equivariant map.*

Theorem 3.4 *Let G be a finite topological group and X be a Tychonoff G-space, then $\nu X/G$ is homeomorphic to $\nu(X/G)$, that is, $\nu X/G \approx \nu(X/G)$.*

Proof Since ν is functorial, the orbit map $\pi_X : X \to X/G$ induces the map $\nu(\pi_X) : \nu X \to \nu(X/G)$. Define $\varphi : (\nu X)/G \to \nu(X/G)$, $G(M) \to \nu(\pi_X)(M)$ for $M \in \nu(X)$.

φ is well-defined:

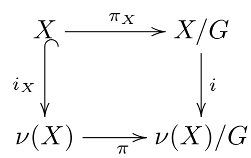
Firstly, let show that the extended action of G on $\nu(X/G)$ is trivial. Let $g \in G$ and $M \in \nu(X/G)$, where $\nu(X/G) = \{M \subset C(X/G) : M \text{ is real maximal ideal}\}$. Then $gM = \{gf : f \in M\}$ and $gf : X/G \to \mathbb{R}$, $(gf)(G(P)) = f(gG(P)) = f(G(P)) \Rightarrow gf = f \Rightarrow gM = M$.

If $G(M) = G(N)$, then $M = gN$ for some $g \in G$. Since $\nu(\pi_X)$ is G-equivariant and the action of G on $\nu(X/G)$ is trivial, then we have that $\nu(\pi_X)(M) = \nu(\pi_X)(gN) = g\nu(\pi_X)(N) = \nu(\pi_X)(N)$.

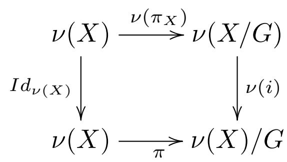
φ is injective:

Suppose that $i : X/G \to (\nu X)/G$ and $i_X : X \to \nu(X)$ are the inclusion map, $\pi_X : X \to X/G$ and

$\pi : \nu X \to (\nu X)/G$ are the orbit maps. Consider the following commutative diagram.



Since the orbit space of any realcompact space is also realcompact by Lemma 2.9, $(\nu X)/G$ is realcompact space, $\nu(\pi) = \pi$. And we have $\nu(i_X) = Id_{\nu(X)}$. From the functorial property of ν, we have that the following commutative diagram.



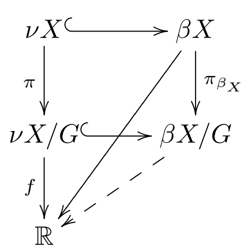
If $\varphi(G(M)) = \varphi(G(N))$, then $\nu(\pi_X)(M) = \nu(\pi_X)(N)$. From the above diagram, $M = gN$ for some $g \in G$. We have $G(M) = G(N)$.

Since φ is injective, we can consider $(\nu X)/G \subset \nu(X/G)$. Since $X/G \subset (\nu X)/G \subset \nu(X/G) \subset \beta(X/G)$, the orbit space $(\nu X)/G$ is realcompact space, and $\nu(X/G)$ is the smallest realcompact space between X/G and $\beta(X/G)$, we have that $(\nu X)/G = \nu(X/G)$. □

Now, as a result of this theorem, we obtain Srivastava's theorem [12].

Theorem 3.5 *Let G be a finite topological group and X be a Tychonoff G-space. The Stone–Čech compactification of the orbit space is the orbit space of the Stone–Čech compactification of X, that is, $\beta(X/G) = (\beta X)/G$.*

Proof Since $\nu X/G \approx \nu(X/G)$, $\beta(\nu X/G) = \beta(\nu(X/G)) = \beta(X/G)$. It is sufficient to show that $\beta(\nu X/G) = \beta X/G$. For this, let show that $\nu X/G$ is C^*-embeded in $\beta X/G$. Suppose that $f : \nu X/G \to \mathbb{R}$ any bounded countinuous function. Consider the next diagram.



Since there is a unique Stone extension $\beta(f\pi) : \beta X \to \mathbb{R}$ of $f\pi$, and $\beta(f\pi)$ is constant on orbits, then it induces the map $\beta X/G \to \mathbb{R}$ which commutes the diagram. Thus $\nu X/G$ is C^*-embeded in $\beta X/G$. □

In [1], Blair and Van Douwen generalized the concept of realcompactness by defining a space X to be nearly realcompact if $\beta X - \nu X$ is dense in $\beta X - X$; that is, X nearly νX. Clearly every realcompact space has this property.

Now, the orbit space of any nearly realcompact space by finite group is also nearly realcompact space.

Corollary 3.6 *Let G be a finite group and X be a Tychonoff G-space. If X is a nearly realcompact space, then the orbit space X/G is also nearly realcompact.*

Proof Suppose that X is a nearly realcompact space. Hence $\beta X - \nu X$ is dense in $\beta X - X$. Then $(\beta X - \nu X)/G = (\beta X)/G - (\nu X)/G$ is also dense in $(\beta X - X)/G = (\beta X)/G - X/G$. Since $(\beta X)/G = \beta(X/G)$ and $(\nu X)/G = \nu(X/G)$, then $\beta(X/G) - \nu(X/G)$ is dense in $\beta(X/G) - X/G$, which proves the claim. □

In [9], Henriksen and Rayburn defined a space X to be nearly pseudocompact if $\nu X - X$ is dense in $\beta X - X$; that is, νX nearly βX. Obviously, every pseudocompact space is nearly pseudocompact. The following corollary can be proved in the same way as the above corollary.

Corollary 3.7 *Let G be a finite group and X be a Tychonoff G-space. If X is a nearly pseudocompact space, then the orbit space X/G is also nearly pseudocompact.*

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