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New classes of Catalan-type numbers and polynomials with their applications related to p-adic integrals and computational algorithms

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Abstract: The aim of this paper is to construct generating functions for new classes of Catalan-type numbers and polynomials. Using these functions and their functional equations, we give various new identities and relations involving these numbers and polynomials, the Bernoulli numbers and polynomials, the Stirling numbers of the second kind, the Catalan numbers and other classes of special numbers, polynomials and functions. Some infinite series representations, including the Catalan-type numbers and combinatorial numbers, are investigated. Moreover, some recurrence relations and computational algorithms for these numbers and polynomials are provided. By implementing these algorithms in the Python programming language, we illustrate the Catalan-type numbers and polynomials with their plots under the special conditions. We also give some derivative formulas for these polynomials. Applying the Riemann integral, contour integral, Volkenborn (bosonic p-adic) integral and fermionic p-adic integral to these polynomials, we also derive some integral formulas. With the help of these integral formulas, we give some identities and relations associated with some classes of special numbers and also the Cauchy-type numbers.

Key words: Generating function, Bernoulli polynomials, Stirling numbers, Catalan numbers, partial differential equations, computational algorithms, p-adic integral

1. Introduction

Special numbers and polynomials are among the most commonly used tools in applied mathematics, in combinatorial probability, in mathematical physics, and in mathematical analysis. In recent years, some researchers have studied these families of combinatorial numbers and polynomials and used their generating functions to present a variety of relations involving those numbers and polynomials (cf. [2]–[35]). The main object of this paper is to construct generating functions for Catalan-type combinatorial numbers and polynomials, give computational algorithms with plots, and make applications of these polynomials. Then, many properties of these numbers and polynomials are provided with the help of p-adic integral and generating function methods. There are very interesting applications of the generating functions for the Catalan-type numbers and polynomials. Some of these are given in this paper. Some remarks and observations on the results of this paper among generating functions, computational algorithms, integral representations including the Riemann integral, contour integral and p-adic integrals, and infinite series representation for the Catalan-type numbers and polynomials are given.

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The notations and definitions related to some known numbers and polynomials and used to derive main results of this article are briefly summarized below. This summary involves the definitions, relations and formulas such as the Bernoulli numbers and polynomials, the Apostol-type numbers and polynomials, the Stirling numbers, the Catalan numbers and combinatorial numbers with their generating functions.

The following notations and definitions are used throughout this paper.

The Bernoulli polynomials $B_n(x)$ and the Bernoulli numbers B_n are given respectively by

$$B_P(x,t) = e^{xt}\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \tag{1.1}$$

where

$$B_N(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}, \tag{1.2}$$

so that $B_n = B_n(0)$ and $B_{2n+1} = 0$ for $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ (cf. [3]–[34]).

Let $v \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Next, we note that the Stirling numbers, $S_1(a, v)$, of the first kind are defined by

$$(\ln(z+1))^v = \sum_{a=0}^{\infty} v! S_1(a, v)\frac{z^a}{a!}, \tag{1.3}$$

so that these numbers are also given by

$$(z)_a = \sum_{v=0}^{a} S_1(a, v)\, z^v, \tag{1.4}$$

where $(z)_a = z(z-1)\cdots(z-a+1)$ with $(z)_0 = 1$ (*cf.* [2]-[34]).

The Stirling numbers, $S_2(a, v)$, of the second kind are given by

$$(e^z - 1)^v = \sum_{a=0}^{\infty} v! S_2(a, v)\frac{z^a}{a!}, \tag{1.5}$$

where $v \in \mathbb{N}_0$ (cf. [2], [7], [29], [34]). Similary, another relation between these numbers and $(z)_a$ is given by

$$z^a = \sum_{k=0}^{a} (z)_k\, S_2(a, k) \tag{1.6}$$

(cf. [2]–[34]). The Stirling numbers came also into existence in many types of not only enumeration problems, but also ODEs problems such as differential operators, the Bell numbers and polynomials, combination lock, the Poisson distribution, the Stirling and binomial transforms, and etc. (cf. [7], [2], [29], [34]).

The Catalan numbers, C_n, are defined by

$$1 - (1-4t)^{\frac{1}{2}} = 2\sum_{n=0}^{\infty} C_n t^{n+1}$$

where

$$C_n = \frac{(2n)!}{(n+1)!n!} \tag{1.7}$$

(cf. [2], [20, pp. 109–110]). The Catalan numbers are came up as a solution of many types of problems: the Euler's polygon problem, the Ballot problems, the Dyck path, and other kinds of enumeration problems (cf. [2], [5, pp. 96–106], [20, pp. 109–110]).

The Cauchy numbers, $b_l(0)$, are given by

$$b_l(0) = \int_0^1 (x)_l \, dx \tag{1.8}$$

and

$$f(z) = \frac{z}{\ln(z+1)} = \sum_{l=0}^{\infty} b_l(0) \frac{z^l}{l!} \tag{1.9}$$

(cf. [27, p. 116], [18], [22], [26]). Using (1.9), we easily have

$$b_l(0) = \sum_{a=0}^{l} \frac{S_1(l,a)}{a+1}, \tag{1.10}$$

(cf. [4, p. 294], [22, p. 1908], [27, p. 114]).

Let $v \in \mathbb{N}$ and θ be a real or complex number. The numbers $Y_n(\theta)$ of order v and the polynomials $Y_n(x;\theta)$ of order v are defined respectively by the following generating functions:

$$\frac{2^v}{(\theta(1+\theta z)-1)^v} = \sum_{m=0}^{\infty} Y_m^{(v)}(\theta) \frac{z^m}{m!} \tag{1.11}$$

and

$$\frac{2^v (1+\theta z)^x}{(\theta(1+\theta z)-1)^v} = \sum_{m=0}^{\infty} Y_m^{(v)}(x;\theta) \frac{z^m}{m!} \tag{1.12}$$

(cf. [21]). If we set $v=1$ in (1.11) and (1.12), we have

$$Y_m(\theta) = Y_m^{(1)}(\theta),$$

and

$$Y_m(x;\theta) = Y_m^{(1)}(x;\theta),$$

which are related to the Bernoulli-type numbers, the Fibonacci numbers, the Stirling numbers, Euler-type numbers, and etc. Note that these numbers have many applications in enumerative combinatorics and probability (for details, see [32], [35], [21]).

We end this section by providing the outline of this paper. In Section 2, we construct generating functions for new classes of Catalan-type numbers and polynomials related to the Catalan numbers, and other kinds of numbers and polynomials. In Section 3, in order to evaluate numerical values of the Catalan-type numbers and polynomials, we not only give recurrence relations, but also computational algorithms. In Section 4, using PDEs of the generating functions for the Catalan-type polynomials, some derivative formulas and identities are derived. In Section 5, applying integrals of the Riemann, the contour and the p-adic to the Catalan-type polynomials some formulas including Cauchy-type numbers are also derived. In Section 6, using generating function of the Catalan-type numbers with its functional equations, we derive various new identities and relations for the Catalan-type polynomials, the Bernoulli numbers and polynomials, the Stirling numbers, and the Catalan numbers. Finally, some infinite series representations related to the Catalan-type numbers are given.

2. New classes of Catalan-type numbers and polynomials

Here, generating functions for new classes of Catalan-type numbers and polynomials are constructed. In order to give this construction, we need to present the following theorem:

Theorem 2.1 *A power series given by*

$$F_V(t,\lambda) = \sum_{n=0}^{\infty} V_n(\lambda)\, t^n$$

converges when $0 < \left| \frac{\lambda^2 t}{(\lambda-1)^2} \right| \leq \frac{1}{8}$, *and*

$$F_V(t,\lambda) = \frac{1-\lambda+\sqrt{(\lambda-1)^2+8\lambda^2 t}}{2\lambda^2 t}. \tag{2.1}$$

In particular,

$$V_n(\lambda) = (-1)^n C_n \frac{2^{n+1}\lambda^{2n}}{(\lambda-1)^{2n+1}}, \tag{2.2}$$

where $n \geq 0$.

Proof We first consider the following set:

$$D = \left\{ t : |t| < \frac{(\lambda-1)^2}{8\lambda^2} \right\}.$$

For $t \in D$, by applying binomial theorem, we get

$$\left(1-4\left(-2\lambda^2(\lambda-1)^{-2}t\right)\right)^{\frac{1}{2}} = \sum_{l=0}^{\infty} \binom{\frac{1}{2}}{l} \left(8\lambda^2(\lambda-1)^{-2}t\right)^l .$$

After some elementary calculations on RHS of the above equation, we have

$$\left(1-4\left(-2\lambda^2(\lambda-1)^{-2}t\right)\right)^{\frac{1}{2}} - 1 = \sum_{l=0}^{\infty} (-1)^{l+1} \frac{2^l(2l)!\lambda^{2l+2}}{(l+1)(l!)^2(\lambda-1)^{2l+2}} t^{l+1}$$

which yields

$$\frac{1-\lambda+\sqrt{(\lambda-1)^2+8\lambda^2 t}}{2\lambda^2 t} = \sum_{n=0}^{\infty} (-1)^n C_n \frac{2^{n+1}\lambda^{2n}}{(\lambda-1)^{2n+1}} t^n. \tag{2.3}$$

Thus, the desired result is obtained. □

Note that the series on the RHS of Equation (2.3) converges when

$$t = -\frac{(\lambda - 1)^2}{8\lambda^2},$$

and it converges absolutely and uniformly on $\bar{D}$ which is a closed subset of D. Consequently, we have the following summary on the behavior of the function $F_V(t, \lambda)$ on the sets D and $\bar{D}$, respectively:

1) $F_V(t, \lambda)$ is analytic in D.

2) $F_V(t, \lambda)$ is continuous on $\bar{D}$.

3) $F_V(t, \lambda)$ is a generating function for so-called Catalan-type numbers $V_n(\lambda)$.

We now define generating function for so-called Catalan-type polynomials, $V_n(x; \lambda)$, as follows:

$$F_V(z, x, \lambda) = F_V(z, \lambda)(1+z)^{\frac{x}{2}} = \sum_{n=0}^{\infty} V_n(x; \lambda) z^n. \tag{2.4}$$

Theorem 2.2

$$V_m(x; \lambda) = \sum_{j=0}^{m} \frac{\left(\frac{x}{2}\right)_j}{j!} V_{m-j}(\lambda), \tag{2.5}$$

where $m \in \mathbb{N}_0$.

Proof Applying the binomial theorem to (2.4) and using (2.1), we get

$$\sum_{m=0}^{\infty} V_m(x; \lambda) z^m = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)_m}{m!} z^m \sum_{m=0}^{\infty} V_m(\lambda) z^m.$$

Thus

$$\sum_{m=0}^{\infty} V_m(x; \lambda) z^m = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \frac{\left(\frac{x}{2}\right)_j}{j!} V_{m-j}(\lambda) z^m.$$

Comparing the coefficients z^m on the both sides of the equation just above, we get the desired result. □

Theorem 2.3 *Let* $n \in \mathbb{N}_0$. *Then we have*

$$V_n(x; \lambda) = \sum_{j=0}^{n} \sum_{c=0}^{j} \frac{V_{n-j}(\lambda) S_1(j, c)}{2^c j!} x^c. \tag{2.6}$$

Proof Replacing z by $\frac{x}{2}$ in (1.4), we have

$$\left(\frac{x}{2}\right)_j = \sum_{c=0}^{j} \left(\frac{x}{2}\right)^c S_1(j, c). \tag{2.7}$$

Combining the equation just above with (2.5), the desired result is obtained. □

Notice that the formula (2.5) allows us to compute the values of the polynomials $V_n(x;\lambda)$ by using the falling factorial polynomials, while the formula (2.6) provides easier computation of these polynomials directly with the numbers $S_1(n,k)$.

By combining (2.5) with (2.6), we get the following corollary:

Corollary 2.4

$$\sum_{j=0}^{n} \frac{\left(\frac{x}{2}\right)_j}{j!} V_{n-j}(\lambda) = \sum_{j=0}^{n}\sum_{c=0}^{j} \frac{V_{n-j}(\lambda) S_1(j,c)}{2^c j!} x^c$$

where $n \in \mathbb{N}_0$.

Remark 2.5 *Recently, in [6], Cossali introduced the generating functions for the polynomials $j_n(y)$ as follows:*

$$J(x,y) = \frac{(1-yx) + \sqrt{(1-yx)^2 - 4x}}{2x} = \sum_{n=0}^{\infty} j_n(y) x^n,$$

cf. [6]). *Substituting $xy = \lambda$ (for instance $x = -2\lambda^2 t$ and $y = -\frac{1}{2\lambda t}$) into the above equation, we have*

$$\begin{aligned} J\left(-2\lambda^2 t, -\frac{1}{2\lambda t}\right) &= -\frac{1}{2}\left(\frac{1-\lambda+\sqrt{(\lambda-1)^2+8\lambda^2 t}}{2\lambda^2 t}\right) \\ &= \sum_{n=0}^{\infty} j_n\left(-\frac{1}{2\lambda t}\right) x^n. \end{aligned}$$

We should note here that even though the generating function given in Remark 2.5 is obtained after the change of variables, it can be seen that there is no relation between the numbers $V_n(\lambda)$ and the expression $j_n\left(-\frac{1}{2\lambda t}\right)$, since the expression $j_n\left(-\frac{1}{2\lambda t}\right)$ does not satisfy a polynomial property.

3. Computational algorithms and plots of the Catalan-type numbers and polynomials

Here, recurrence relations and computational algorithms are given in order to evaluate numerical values of the Catalan-type numbers and polynomials. Implementing these algorithms in Python programming language (Python version 2.7.10)* on a Python IDE (JetBrains PyCharm Community Edition 5.0.3)† with the library Matplotlib [12] of 2D plotting and the library NumPy [23] of scientific computing, we illustrate the polynomials $V_n(x;\lambda)$ with their plots under the special conditions.

Theorem 3.1 *Let $V_0(\lambda) = \frac{2}{\lambda-1}$. For $n \in \mathbb{N}$, we have*

$$V_n(\lambda) = \frac{n\lambda^2}{1-\lambda}\sum_{j=0}^{n-1} V_j(\lambda) V_{n-j-1}(\lambda).$$

*Python Software Foundation (2020). Python Language Reference, Version 2.7.10 [online]. Website https://www.python.org/downloads/release/python-2710/ [accessed 00 Month Year].

†JetBrains (2020). JetBrains PyCharm Community Edition 5.0.3. The Python IDE for Professional Developers [online]. Website https://www.jetbrains.com/pycharm/ [accessed 00 Month Year].

Proof The following functional equation is obtained with the aid of (2.1):

$$tF_V^2\left(t,\lambda\right)+\frac{\lambda-1}{\lambda^2}F_V\left(t,\lambda\right)=\frac{2}{\lambda^2}. \tag{3.1}$$

By using (3.1), we get

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n-1}V_j\left(\lambda\right)V_{n-j-1}\left(\lambda\right)nt^n+\frac{\lambda-1}{\lambda^2}\sum_{n=0}^{\infty}V_n\left(\lambda\right)t^n=\frac{2}{\lambda^2}.$$

Making some calculations, the assertion of Theorem 3.1 is obtained. □

By using Theorem 3.1, we give a computation algorithm, given by Algorithm 1, consisting of the procedure `V_CATALAN_TYPE_NUM` which returns the values of the numbers $V_n(\lambda)$.

Algorithm 1 Let n be a nonnegative integer and $\lambda\in\mathbb{C}$. This algorithm will return the numbers $V_n(\lambda)$, recursively.

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procedure `V_CATALAN_TYPE_NUM`(n: nonnegative integer, λ)
  Begin
  Local variable j : nonnegative integer
  if $n=0$ **then**
    return $2/\left(\lambda-1\right)$
  else
    return $\left(\left(n*\texttt{power}\ (\lambda,2)\right)/\left(1-\lambda\right)\right)*\text{sum}\Big($`V_CATALAN_TYPE_NUM`$(j,\lambda)*$
   `V_CATALAN_TYPE_NUM`$\left(n-j-1,\lambda\right),j,0,n-1\Big)$
  end if
end procedure

By using Algorithm 1, we compute some values of the numbers $V_n(\lambda)$ as follows:

$$V_0(\lambda)=2\left(\lambda-1\right)^{-1},\quad V_1(\lambda)=-4\lambda^2(\lambda-1)^{-3},$$
$$V_2(\lambda)=16\lambda^4(\lambda-1)^{-5},\quad V_3(\lambda)=-80\lambda^6(\lambda-1)^{-7},$$

and so on.

By the following Algorithm 2, we also give `V_CATALAN_TYPE_POLY` procedure for calculating values of the polynomials $V_n(x;\lambda)$.

Algorithm 2 Let n be a nonnegative integer and $\lambda\in\mathbb{C}$. By using (2.5), this algorithm will return the polynomials $V_n(x;\lambda)$ with the help of `V_CATALAN_TYPE_NUM` procedure given by Algorithm 1.

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procedure `V_CATALAN_TYPE_POLY`(n: nonnegative integer, x, λ)
  Global variable $Vpoly\leftarrow 0$
  Local variable j : nonnegative integer
  for $j=0;\ j\leq n;\ j=j+1$ **do**
    $Vpoly\leftarrow Vpoly+\left(\texttt{Falling\_Fact}\left(x/2,j\right)/\texttt{Fact}\left(j\right)\right)*$`V_CATALAN_TYPE_NUM`$(n-j,\lambda)$
  end for
  return $Vpoly$
end procedure

Using Algorithm 2, we also compute some values of the polynomials $V_n(x;\lambda)$ as follows:

$$\begin{aligned} V_0(x;\lambda) &= 2(\lambda-1)^{-1}, \\ V_1(x;\lambda) &= (\lambda-1)^{-1}x - 4\lambda^2(\lambda-1)^{-3}, \\ V_2(x;\lambda) &= x^2(\lambda-1)^{-1} - \left(4\lambda^2-4\lambda+2\right)(\lambda-1)^{-3}x + 16\lambda^4(\lambda-1)^{-5}, \end{aligned}$$

and so on. Next, by implementing the above algorithms in Python, some plots of the Catalan-type polynomials are illustrated under the special conditions. The curves in Figures 1 and 2 provide considerable information to analyse some characteristics of these polynomials.

4. Some derivative formulas and identities of the Catalan-type polynomials derived from PDEs

Here, using PDEs of (2.4), some derivative formulas and identities are derived.

Theorem 4.1

$$\frac{\partial}{\partial x}\{V_m(x;\lambda)\} = \frac{1}{2}\sum_{a=1}^{m}\frac{(-1)^{a-1}V_{m-a}(x;\lambda)}{a} \tag{4.1}$$

where $m \in \mathbb{N}$.

Proof Differentiating both side of (2.4), the following equation is obtained:

$$\frac{\partial}{\partial x}\{F_V(z,x;\lambda)\} = \frac{\ln(z+1)}{2}F_V(z,x;\lambda). \tag{4.2}$$

Combining (2.4) and series representation for the function $\ln(z+1)$ with (4.2), we have

$$\sum_{m=0}^{\infty}\frac{\partial}{\partial x}\{V_m(x;\lambda)\}z^m = \frac{1}{2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}z^n\sum_{m=0}^{\infty}V_m(x;\lambda)z^m.$$

Therefore

$$\sum_{m=0}^{\infty}\frac{\partial}{\partial x}\{V_m(x;\lambda)\}z^m = \frac{1}{2}\sum_{m=0}^{\infty}\sum_{a=1}^{m}\frac{(-1)^{a-1}V_{m-a}(x;\lambda)}{a}z^m.$$

Making some calculations, the desired result is obtained. □

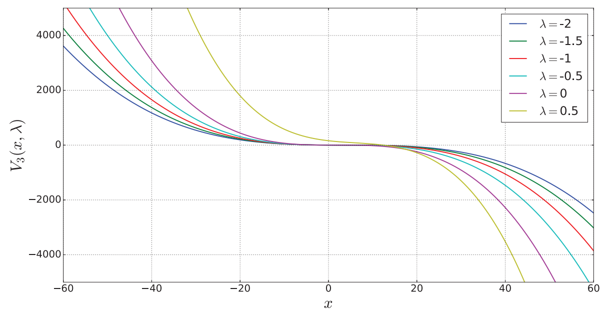
Theorem 4.2

$$\frac{\partial}{\partial x}\{V_m(x;\lambda)\} = 2\sum_{a=1}^{m}\sum_{c=0}^{a-1}\frac{\left(\frac{x}{2}\right)_a V_{m-a}(\lambda)}{a!\,(x-2c)} \tag{4.3}$$

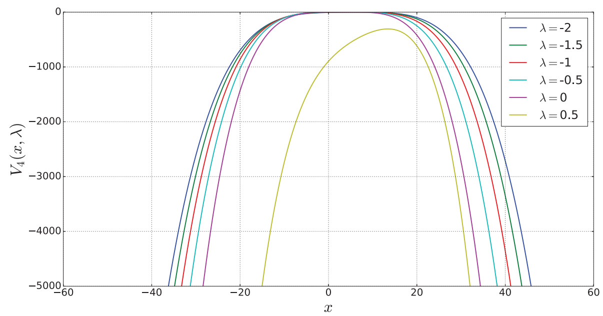
where $m \in \mathbb{N}$.

Proof Applying the operator $\frac{\partial}{\partial x}$ to (2.5) yields

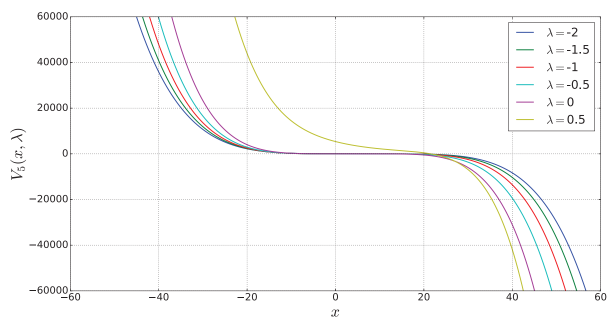
$$\frac{\partial}{\partial x}\{V_m(x;\lambda)\} = \sum_{a=0}^{m}\frac{V_{m-a}(\lambda)}{a!}\frac{d}{dx}\left\{\left(\frac{x}{2}\right)_a\right\}. \tag{4.4}$$



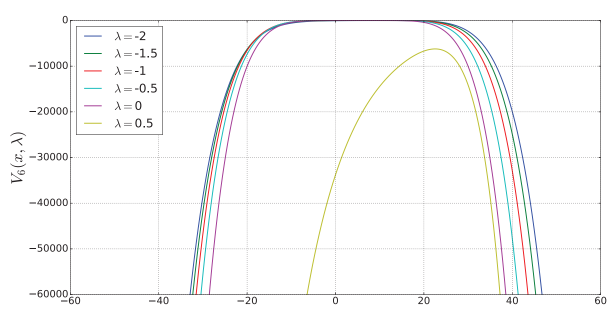
(a) n = 3 and x ∈ [− 60, 60].



(b) n = 4 and x ∈ [− 60, 60].

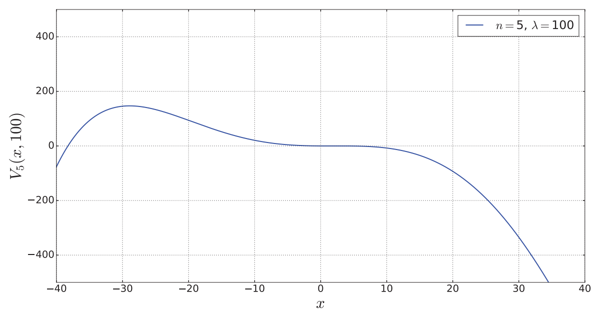


(c) n = 5 and x ∈ [− 60, 60].

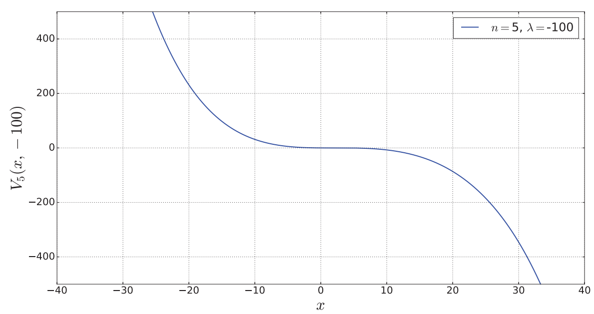


(d) n = 6 and x ∈ [− 60, 60].

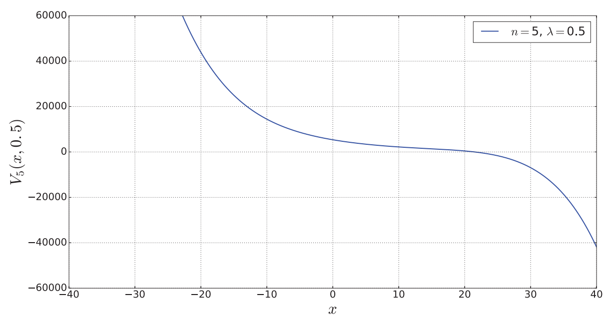
Figure 1. For $\lambda \in \{-2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}\}$, some plots of the polynomials $V_n(x;\lambda)$ in the cases: (a) $n = 3$ and $x \in [-60, 60]$; (b) $n = 4$ and $x \in [-60, 60]$; (c) $n = 5$ and $x \in [-60, 60]$; (d) $n = 6$ and $x \in [-60, 60]$.



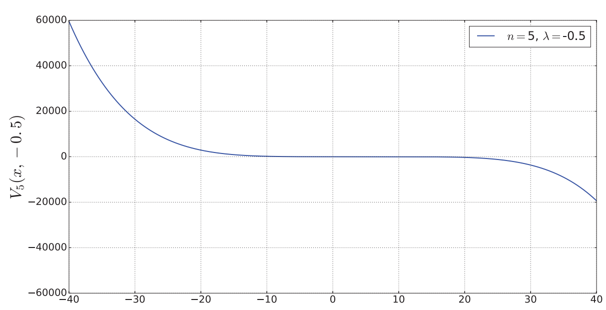
(a) n = 5 and λ = 100.



(b) n = 5 and λ = − 100.



(c) n = 5 and λ = $\frac{1}{2}$.



(d) n = 5 and λ = − $\frac{1}{2}$.

Figure 2. For $x \in [-40, 40]$, some plots of the polynomials $V_n(x;\lambda)$ in the cases: (a) $n = 5$ and $\lambda = 100$; (b) $n = 5$ and $\lambda = -100$; (c) $n = 5$ and $\lambda = \frac{1}{2}$; (d) $n = 5$ and $\lambda = -\frac{1}{2}$.

Substituting

$$\frac{d}{dx}\left\{(x)_a\right\} = \begin{cases} (x)_a \sum\limits_{c=0}^{a-1} \frac{1}{x-c} & \text{if} \quad a \in \mathbb{N} \\ 0 & \text{if} \quad a = 0, \end{cases} \tag{4.5}$$

(cf. [25, Eq. (9)]) into (4.4), the desired result is obtained. □

Theorem 4.3

$$\frac{\partial^k}{\partial x^k}\{V_m(x;\lambda)\} = \frac{k!}{2^k}\sum_{a=k}^{m}\frac{S_1(a,k)V_{m-a}(x;\lambda)}{a!} \tag{4.6}$$

where $m, k \in \mathbb{N}$*.*

Proof Taking kth derivative of (2.4) yields

$$\frac{\partial^k}{\partial x^k}\{F_V(z,x,\lambda)\} = \frac{(\ln(z+1))^k}{2^k}F_V(z,x,\lambda). \tag{4.7}$$

Combining (2.4) and (1.3) with (4.7) yields

$$\sum_{m=0}^{\infty}\frac{\partial^k}{\partial x^k}\{V_m(x;\lambda)\}z^m = k!2^{-k}\sum_{m=k}^{\infty}S_1(m,k)\frac{z^m}{m!}\sum_{m=0}^{\infty}V_m(x;\lambda)z^m.$$

Hence

$$\sum_{m=0}^{\infty}\frac{\partial^k}{\partial x^k}\{V_m(x;\lambda)\}z^m = k!2^{-k}\sum_{m=0}^{\infty}\sum_{a=k}^{m}\frac{S_1(a,k)V_{m-a}(x;\lambda)}{a!}z^m.$$

Making some calculations, the desired result is obtained. □

Remark 4.4 *When* $k = 1$*,* (4.6) *yields*

$$\frac{\partial}{\partial x}\{V_m(x;\lambda)\} = \frac{1}{2}\sum_{a=1}^{m}\frac{S_1(a,1)V_{m-a}(x;\lambda)}{a!}. \tag{4.8}$$

Combining (1.3) *with* (4.8) *and*

$$S_1(a,1) = (-1)^{1+a}(a-1)!, \tag{4.9}$$

we also get (4.1)*.*

5. Integral formulas for the Catalan-type polynomials

Here, we apply integrals of the Riemann, the contour and the p-adic to the Catalan-type polynomials in order to derive some formulas.

5.1. Riemann integral formulas for the polynomials $V_n(x;\lambda)$

Here, we give Riemann integral formula for the polynomials $V_n(x;\lambda)$.

Theorem 5.1

$$\int_0^1 V_{m-1}(x;\lambda)\,dx = 2\sum_{a=0}^{m}\frac{b_{m-a}(0)}{(m-a)!}\left(V_a(1;\lambda)-V_a(\lambda)\right) \tag{5.1}$$

where $m\in\mathbb{N}$.

Proof Integrating (2.4), we get

$$\int_0^1 F_V(z,x,\lambda)\,dx = \frac{1-\lambda+\sqrt{(\lambda-1)^2+8\lambda^2 z}}{2\lambda^2 z}\int_0^1 (1+z)^{\frac{x}{2}}\,dx$$

which yields

$$\int_0^1 F_V(z,x,\lambda)\,dx = \frac{1-\lambda+\sqrt{(\lambda-1)^2+8\lambda^2 z}}{2\lambda^2 z}\left(\frac{2\left(\sqrt{z+1}-1\right)}{\ln(z+1)}\right).$$

Thus, we get

$$\int_0^1 F_V(z,x,\lambda)\,dx = \frac{2}{z}f(z)\left(F_V(z,1,\lambda)-F_V(z,\lambda)\right). \tag{5.2}$$

Combining (5.2) with (2.4) and (1.9) yields

$$\sum_{m=0}^{\infty} z^{m+1}\int_0^1 V_m(x;\lambda)\,dx = 2\sum_{m=0}^{\infty} b_m(0)\frac{z^m}{m!}\sum_{m=0}^{\infty}\left(V_m(1;\lambda)-V_m(\lambda)\right)z^m.$$

Therefore

$$\sum_{m=0}^{\infty} z^{m+1}\int_0^1 V_m(x;\lambda)\,dx = 2\sum_{m=0}^{\infty}\sum_{a=0}^{m}\frac{b_{m-a}(0)}{(m-a)!}\left(V_a(1;\lambda)-V_a(\lambda)\right)z^m.$$

Making some calculations, the desired result is obtained. □

Remark 5.2 *Integrating (2.5) yields another Riemann integral of the polynomials* $V_m(x;\lambda)$ *as follows:*

$$\int_0^1 V_m(x;\lambda)\,dx = \sum_{j=0}^{m} V_{m-j}(\lambda)\,k_j, \tag{5.3}$$

where the numbers k_n *is given by*

$$k_n = \int_0^1 \binom{\frac{x}{2}}{n}dx, \tag{5.4}$$

which are so-called Bernoulli-type numbers of the second kind (or Cauchy-type numbers). Using (5.4), some values of the numbers k_n *are given by*

$$k_0 = 1,\quad k_1 = \frac{1}{4},\quad k_2 = -\frac{1}{3},\quad k_3 = \frac{27}{16},$$

and so on.

5.2. Contour integral representation of the numbers $V_n(\lambda)$

Here, we give a contour integral representation of the numbers $V_n(\lambda)$.

Binomial coefficients are computed by following the Cauchy integral formula and the residue theorem:

$$\int_{\Omega} \frac{(1+w)^k}{w^{j+1}} \mathrm{d}w = 2\pi i \binom{k}{j} \tag{5.5}$$

where Ω is a simple closed contour surrounding the origin (*cf.* [1]).

By replacing k by $2k$ and j by k in (5.5), and then combining the final equation with (1.7) and (2.2), we obtain a contour integral representation of the numbers $V_n(\lambda)$ by the following theorem:

Theorem 5.3

$$V_k(\lambda) = \frac{i^{2k-1}\left(2\lambda^2\right)^k}{\pi(\lambda-1)^{2k+1}} \int_{\Omega} \frac{(1+w)^{2k}}{w^{k+1}} \mathrm{d}w, \tag{5.6}$$

where $k \in \mathbb{N}_0$ *and* $w \in \mathbb{C}$.

5.3. p-adic integrals of the polynomials $V_n(x;\lambda)$

Here, by applying the Volkenborn integral and the p-adic fermionic integral to the polynomials $V_n(x;\lambda)$ on the set of p-adic integers $\mathbb{Z}_p$, some formulas are derived.

Theorem 5.4

$$\int_{\mathbb{Z}_p} V_m(x;\lambda)\, d\mu_1(x) = \sum_{a=0}^{m} \sum_{k=0}^{a} \frac{V_{m-a}(\lambda)\, S_1(a,k) B_k}{a! 2^k},$$

where $m \in \mathbb{N}_0$.

Theorem 5.5

$$\int_{\mathbb{Z}_p} V_m(x;\lambda)\, d\mu_{-1}(x) = \sum_{a=0}^{m} \sum_{k=0}^{a} \frac{V_{m-a}(\lambda)\, S_1(a,k) E_k}{a! 2^k},$$

where $m \in \mathbb{N}_0$.

The following definitions and notations are used in order to prove Theorems 5.4 and 5.5.

Let $\mu_1(x)$ be the Haar distribution. The Volkenborn (p-adic bosonic) integral of $h(x)$, which is uniformly differentiable function on $\mathbb{Z}_p$, is defined by

$$\int_{\mathbb{Z}_p} h(x)\, d\mu_1(x) = \lim_{M\to\infty} p^{-M} \sum_{l=0}^{p^M-1} h(l), \tag{5.7}$$

(cf. [28]; see also [14], [15], [17], [33]).

In [8, Corollary 3, p. 5], Dolgy et al. defined the following numbers:

$$d_n = \frac{(-1)^n 2^{2n}}{n!} \int_{\mathbb{Z}_p} \left(\frac{x}{2}\right)_n d\mu_1(x) \tag{5.8}$$

and

$$\int_{\mathbb{Z}_p} \left(\frac{x}{2}\right)_n d\mu_1(x) = \sum_{k=0}^{n} 2^{-k} S_1(n,k) B_k, \tag{5.9}$$

(cf. [8], [19]).

Let $\mu_{-1}(x) = (-1)^x$. The p-adic fermionic integral of $h(x)$, which is uniformly differentiable function on $\mathbb{Z}_p$, is defined by

$$\int_{\mathbb{Z}_p} h(x)\, d\mu_{-1}(x) = \lim_{M\to\infty} \sum_{l=0}^{p^M-1} (-1)^l h(l) \tag{5.10}$$

(cf. [14], [15]).

In [16, Theorem 3, p.497], Kim gave the following formula:

$$C_n = \frac{(-1)^n 2^{2n}}{n!} \int_{\mathbb{Z}_p} \left(\frac{x}{2}\right)_n d\mu_{-1}(x), \tag{5.11}$$

and

$$\int_{\mathbb{Z}_p} \left(\frac{x}{2}\right)_n d\mu_{-1}(x) = \sum_{k=0}^{n} 2^{-k} S_1(n,k) E_k, \tag{5.12}$$

(*cf.* [16], [31], [19]).

For more examples and details about p-adic integration techniques, see Schikhof [28], and also [33].

In the light of the above p-adic integral representations, we can now give the proof of Theorems 5.4 and 5.5.

Proof [Proof of Theorem 5.4] Applying (5.7) to (2.5) yields

$$\int_{\mathbb{Z}_p} V_m(x;\lambda)\, d\mu_1(x) = \sum_{a=0}^{m} \frac{V_{m-a}(\lambda)}{a!} \int_{\mathbb{Z}_p} \left(\frac{x}{2}\right)_a d\mu_1(x). \tag{5.13}$$

Combining (5.13) with (5.9) yields the desired result. □

Proof [Proof of Theorem 5.5] Applying (5.10) to (2.5) yields

$$\int_{\mathbb{Z}_p} V_m(x;\lambda)\, d\mu_{-1}(x) = \sum_{a=0}^{m} \frac{V_{m-a}(\lambda)}{a!} \int_{\mathbb{Z}_p} \left(\frac{x}{2}\right)_a d\mu_{-1}(x). \tag{5.14}$$

Combining (5.14) with (5.12) yields the desired result. □

Combining (5.13) with (5.8) yields a relation between the polynomials $V_n(x;\lambda)$ and the numbers d_n by the following corollary:

Corollary 5.6

$$\int_{\mathbb{Z}_p} V_m(x;\lambda)\, d\mu_1(x) = \sum_{a=0}^{m} (-1)^a 2^{-2a} V_{m-a}(\lambda)\, d_a. \tag{5.15}$$

where $m \in \mathbb{N}_0$.

On the other hand, combining (5.14) with (5.11) yields the following result:

Corollary 5.7

$$\int_{\mathbb{Z}_p} V_m(x;\lambda)\, d\mu_{-1}(x) = \sum_{a=0}^{m} (-1)^a 2^{-2a} V_{m-a}(\lambda)\, C_a. \tag{5.16}$$

where $m \in \mathbb{N}_0$.

6. Further formulas and identities

Here, using generating function and functional equations, further formulas and identities are derived. These results include the numbers $V_n(\lambda)$, the numbers $Y_n^{(k)}(\lambda)$, the numbers B_n, the polynomials $B_n(x)$, the numbers $S_1(n,k)$, the numbers $S_2(n,k)$ and the numbers C_n. We also provide some infinite series representations including the Catalan-type numbers.

The numbers $V_m(\lambda)$ and the numbers $Y_m^{(m+1)}(\lambda)$ have the following relationship:

Theorem 6.1

$$Y_m^{(m+1)}(\lambda) = (m+1)! V_m(\lambda), \tag{6.1}$$

where $m \in \mathbb{N}_0$.

Proof Using (1.11) and (2.1), and also some elemantary calculations yields the desired result. □

Theorem 6.2

$$\sum_{n=0}^{a-1} n! \left(2^{-3}\lambda^{-2}(1-\lambda)^2\right)^n V_n(\lambda)\, S_2(a-1,n) = \frac{4}{(\lambda-1)\,a}\left(B_a\left(\frac{1}{2}\right) - B_a\right), \tag{6.2}$$

where $a \in \mathbb{N}$.

Proof Assuming $\lambda > 1$, we set the following functional equation:

$$z(\lambda-1) F_V\left(2^{-3}\lambda^{-2}(1-\lambda)^2(e^z-1), \lambda\right) = 4B_P\left(z, \frac{1}{2}\right) - 4B_N(z). \tag{6.3}$$

Combining (6.3) with (1.2), (1.1) and (2.1) yields

$$z\sum_{n=0}^{\infty} V_n(\lambda)\left(2^{-3}\lambda^{-2}(1-\lambda)^2(e^z-1)\right)^n = \frac{4}{\lambda-1}\sum_{a=0}^{\infty}\left(B_a\left(\frac{1}{2}\right) - B_a\right)\frac{z^a}{a!}.$$

Thus, with the aid of (1.5), we have

$$z\sum_{a=0}^{\infty}\sum_{n=0}^{\infty}n!\left(2^{-3}\lambda^{-2}\left(1-\lambda\right)^{2}\right)^{n}V_{n}\left(\lambda\right)S_{2}(a,n)\frac{z^{a}}{a!}$$
$$=\frac{4}{\lambda-1}\sum_{a=0}^{\infty}\left(B_{a}\left(\frac{1}{2}\right)-B_{a}\right)\frac{z^{a}}{a!}.$$

Since $S_2(a,n)=0$ when $a<n$, we have

$$\sum_{a=0}^{\infty}a\sum_{n=0}^{a-1}n!\left(2^{-3}\lambda^{-2}\left(1-\lambda\right)^{2}\right)^{n}V_{n}\left(\lambda\right)S_{2}(a-1,n)\frac{z^{a}}{a!}$$
$$=\frac{4}{\lambda-1}\sum_{a=0}^{\infty}\left(B_{a}\left(\frac{1}{2}\right)-B_{a}\right)\frac{z^{a}}{a!}.$$

Making some calculations, the desired result is obtained. □

Combining the following well-known identity (cf. [7], [34]):

$$B_{a}\left(\frac{1}{2}\right)=\left(2^{1-a}-1\right)B_{a}$$

with (6.2), we derive a computation formula for the Bernoulli numbers:

Theorem 6.3

$$B_{a}=\frac{a}{2^{3}-2^{3-a}}\sum_{n=0}^{a-1}\frac{\left(1-\lambda\right)^{2n+1}n!V_{n}\left(\lambda\right)S_{2}(a-1,n)}{\left(8\lambda^{2}\right)^{n}},$$

where $a\in\mathbb{N}$.

By using (2.2), the numbers C_n and the numbers $V_n(\lambda)$ have the following relationship:

Corollary 6.4

$$C_{n}=(-1)^{n}\frac{\left(\lambda-1\right)^{2n+1}V_{n}\left(\lambda\right)}{2^{n+1}\lambda^{2n}},$$

where $n\in\mathbb{N}_0$.

6.1. Infinite series representations for the Catalan-type numbers

Here, we define new functions by using some infinite series representation for the Catalan-type numbers and combinatorial numbers. Applications of these functions involve some generating functions for the Bernoulli, the Euler and the Genocchi polynomials. We end this section by presenting another series representation related to the Fibonacci-type polynomials in two variables.

Let $\theta\in\mathbb{C}$. Then we set

$$g(\theta)=\sum_{m=0}^{\infty}\frac{(m+1)\,V_{m}\left(\theta\right)}{Y_{m}^{(m+1)}\left(\theta\right)}\theta^{m}.$$

By using the function $g(\theta)$, we define the following function:

$$g(\theta,x)=\frac{\theta g(x\theta)}{g(\theta)-1}$$

which is a generating function of the polynomials $B_n(x)$ for $|\theta|<2\pi$. Similary, the function $g(\theta,x)$ gives the following functions

$$\frac{2g(x\theta)}{g(\theta)+1},\quad \frac{2g(x\theta)}{g(\theta)+g(-\theta)},\quad \text{and}\quad \frac{2\theta g(x\theta)}{g(\theta)+1}$$

which give generating functions of the Euler and the Genocchi polynomials for $|\theta|<\pi$. Besides, using the function $g(\theta,x)$, moment generating functions and the other generating functions for special numbers and polynomials may be obtained.

By considering the infinite series comprising of a quotient of the numbers $V_n(\lambda)$ and the numbers C_n, we get the following result:

Theorem 6.5 *Assuming*

$$\left|\frac{2\lambda^2}{(\lambda-1)^2}\right|<1,$$

we have

$$h(\lambda):=\frac{2(\lambda-1)}{3\lambda^2-2\lambda+1}=\sum_{n=0}^{\infty}\frac{V_n(\lambda)}{C_n}.$$

Remark 6.6 *Note that the function $h(\lambda)$ satisfy the following relation:*

$$h(\lambda)=2(\lambda-1)H(\lambda;2,-3;1,1,1)$$

where

$$H(\lambda;x,y;a,b,c)=\frac{1}{1-x^a\lambda-y^b\lambda^{b+c}}\qquad (a,b,c\in\mathbb{N}_0)$$

which is a generating function for the Fibonacci-type polynomials in two variables (for details, see [24]). *In the same manner of the above calculations, by investigating infinite series representations for the Catalan-type numbers and polynomials under the special conditions, one may encounter with the generating functions for the Lucas numbers and poynomials, the Pell numbers and polynomials derived from the Pell equations, and Chebyshev polynomials.*

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