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Introduction to N-soft algebraic structures

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Abstract: This paper is dedicated to two main objectives. The first of these is to develop some new operations on N-soft set, which is the generalization of soft set. The second is to highlight the concepts of N-soft group, N-soft ring, N-soft ideal, completely semiprime N-soft ideal, N-soft field and N-soft lattice. Moreover, in this study, it is attempted to derive certain properties for these concepts and to analyze the relations between them.

Key words: N-soft sets, N-soft operations, N-soft groups, N-soft rings, N-soft fields, N-soft lattices

1. Introduction

The soft set, explored by Molodtsov [19], is an effective parametric tool used to manage uncertain issues. Almost all of the operations on the classical sets have been tried to be adapted into the soft sets. In this direction, Maji et al. [18] published an introductory paper about the operations on soft sets. They discussed the soft operations like complement, union, intersection, And, Or. In [3, 22], the authors extended the soft operations proposed in [18] and thus presented various kinds of them. The researchers in [14, 15, 24] focused on the difference and symmetric difference of two soft sets. Zhu and Wen [30] decided to draw a new direction in the operations of the soft sets. Çağman and Enginoğlu [8] recreated some basic operations and products on the soft sets to make them more helpful in specific cases. In [5, 13, 25], some of the aforementioned operations were studied for a family of soft sets.

The first contributions to the algebraic properties of soft sets started with the concept of soft group [2]. Thereafter, many soft algebraic structures such as soft semiring [13], soft ring [1, 10, 17], soft module [26], soft substructures of ring, field, module [6], soft intersection group [7], soft intersection ring [11] and soft union ring [28] were developed. In [20, 21], the authors investigated the soft p-ideal and implicative ideal based on $\mathcal{N}$-functions in BCK and BCI algebras. Several findings and conclusions regarding the lattice structures of soft sets were discussed in [4, 16]. In those years, the authors focused on the soft topology and its subconcepts such as soft closure, soft open sets, soft closed sets, soft interior points, soft neighborhood of a point, soft compactness and soft separation axioms [9, 27, 29].

In 2018, Fatimah et al. [12] initiated the N-soft set theory. They argue that for $N=2$, the N-soft set is returned to the soft set, so that the N-soft set is the extension of Molodtsov's soft set. In 2019, Riaz et al. [23] described some basic operations on N-soft sets. Also, they introduced the N-soft topology and its properties including N-soft exterior, N-soft interior and N-soft closure. In this study, we aim to describe some

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new operations and products on the N-soft sets. Moreover, we focus on N-soft algebraic structures and thus pioneer the emergence of many algebraic properties of N-soft sets.

The rest of this paper is organized as follows. Section 2 recalls the requisite preliminary concepts. Section 3 contributes to the operations emerged for the N-soft sets. In Section 4, the notions of N-soft group and normal N-group are introduced. In addition, the intersection, union and products of (normal) N-soft groups are proved to be (normal) N-soft group. In Section 5, it is discussed the concepts of N-soft ring, N-soft ideal, completely semiprime N-soft ideal, N-soft field and their related results. Section 6 presents the N-soft lattice and some illustrations to explain this concept. Also, in this section, the relationship between N-soft ideal and N-soft lattice is addressed. The final section is devoted to the conclusions.

2. Preliminaries

In this part, we provide a review of some basic concepts for latter.

Let $\mathcal{G}$ be a set and "$\cdot$" be a binary operation on $\mathcal{G}$. Then, the algebraic structure $(\mathcal{G},\cdot)$ is called a *group* iff the following conditions are satisfied:

(i) $g_k \cdot g_l \in \mathcal{G} \ \ \forall g_k, g_l \in \mathcal{G}$.

(ii) $g_k \cdot (g_l \cdot g_m) = (g_k \cdot g_l) \cdot g_m \ \ \forall g_k, g_l, g_m \in \mathcal{G}$. (associative law)

(iii) There exists an element $e \in \mathcal{G}$ such that $g_k \cdot e = e \cdot g_k = g_k \ \ \forall g_k \in \mathcal{G}$. (identity element)

(iv) For each $g_k \in \mathcal{G}$, there exists an element $g_l \in \mathcal{G}$ such that $g_k \cdot g_l = g_l \cdot g_k = e$.

If the conditions (i) and (ii) are satisfied then $(\mathcal{G},\cdot)$ is called a semigroup. If the conditions (i), (ii) and (iii) are satisfied then $(\mathcal{G},\cdot)$ is called a monoid.
In addition to the conditions (i)-(iv), the following condition is satisfied then $(\mathcal{G},\cdot)$ is termed to be abelian (or commutative) group.

(v) $g_k \cdot g_l = g_l \cdot g_k \ \ \forall g_k, g_l \in \mathcal{G}$. (commutative law)

Let $\mathcal{R}$ be a set and "$+$" and "$\cdot$" be two binary operation on $\mathcal{R}$. Then, the algebraic structure $(\mathcal{R},+,\cdot)$ is called a *ring* iff the following conditions are satisfied:

(i) $(\mathcal{R},+)$ is an abelian group.

(ii) $(\mathcal{R},\cdot)$ is a semigroup.

(iii) $r_k \cdot (r_l + r_m) = r_k \cdot r_l + r_k \cdot r_m$ and $(r_k + r_l) \cdot r_m = r_k \cdot r_m + r_l \cdot r_m \ \ \forall r_k, r_l, r_m \in \mathcal{R}$. (distributive law)

An element $r_k \in \mathcal{R}$ is said to be an idempotent if ${r_k}^2 = r_k$. If every element of $\mathcal{R}$ is idempotent then $\mathcal{R}$ is called a *Boolean ring*.

A ring $(\mathcal{F},+,\cdot)$ is named a *field* if $(\mathcal{F}-\{0_\mathcal{F}\},\cdot)$ is a commutative group, where $O_\mathcal{F}$ indicates the identity element of the group $(\mathcal{F},+)$.

Let $\mathcal{L}$ be a nonempty set, and "$\curlyvee$", "$\curlywedge$" be two binary operations on $\mathcal{L}$. An algebraic structure $(\mathcal{L},\curlyvee,\curlywedge)$ is called a lattice if both $\curlyvee$ and $\curlywedge$ are commutative and associative and they are connected by the absorption law. Here, absorption law means that $\ell_m \curlyvee (\ell_m \curlywedge \ell_n) = \ell_m$ and $\ell_m \curlywedge (\ell_m \curlyvee \ell_n) = \ell_m$ for all $\ell_m, \ell_n \in \mathcal{L}$.

In 1999, Molodtsov [19] initiated the soft set theory. In 2018, Fatimah et al. [12] defined N-soft set, which is a generalization of the soft set, as follows:

Definition 2.1 *([12]) Let $\mathcal{U}$ be a universal set and $\mathfrak{X}$ be a set of parameters (attributes). Also, let $T = \{0,1,...,N-1\}$ be a set of ordered grades, where $N = \{2,3,...\}$. Then, the triplet $(S,\mathfrak{X},N)$ is called an N-soft set over $\mathcal{U}$ if S is a mapping as given $S:\mathfrak{X} \to 2^{\mathcal{U}\times T}$, where for each $x \in \mathfrak{X}$*

$$S(x) = \{(u,\tau_x^S(u)) : \ u \in \mathcal{U} \text{ and } \tau_x^S(u) \in T\}$$

such that each u has a unique grade $\tau_x^S(u)$ for each $x \in X$.

Note that the set of all N-soft sets over $\mathcal{U}$ is denoted by $NSset(\mathcal{U})$.

If $N = 2$ then the 2-soft set can also be considered as Molodtsov's soft set.

Let us consider a questionnaire such that there are two options we can have when evaluating alternatives according to parameters: good and bad. Thus, we create a soft set. It is also a 2-soft set for bad=0 and good=1. If the number of our options increases to be very good, good and bad, then we have to create a 3-soft set for bad=0, good=1 and very good=2.

Now, let us visualize the N-soft set with the following example.

Example 2.2 *Suppose that $\mathcal{U} = \{u_1,u_2,u_3\}$ is a set of hotels in a resort. Let $\mathfrak{X} = \{x_1,x_2,x_3\}$ be a set of parameters to be used to evaluate these hotels, where x_j for $j = 1,2,3,4$ symbolize "location", "room", "pool" and "activities and services" respectively. As a result of evaluating hotels under these parameters, one can create the following 3-soft sets.*

$$(S,\mathfrak{X},N) = \left\{ \begin{array}{l} (x_1,\{(u_1,2),(u_2,0),(u_3,1),(u_4,2)\}),(x_2,\{(u_1,1),(u_2,1),(u_3,2),(u_4,2)\}), \\ (x_3,\{(u_1,1),(u_2,0),(u_3,0),(u_4,2)\}),(x_4,\{(u_1,1),(u_2,0),(u_3,2),(u_4,1)\}) \end{array} \right\}.$$

Definition 2.3 *([23]) Let $(S_1,\mathfrak{X},N)$ and $(S_2,\mathfrak{X},N)$ be two N-soft sets over $\mathcal{U}$. Then, $(S_1,\mathfrak{X},N)$ is an N-soft subset of $(S_2,\mathfrak{X},N)$, denoted by $(S_1,\mathfrak{X},N)\widetilde{\subseteq}(S_2,\mathfrak{X},N)$, if for each $x \in \mathfrak{X}$*

$$\tau_x^{S_1}(u) \leq \tau_x^{S_2}(u) \quad \forall u \in \mathcal{U}$$

where $\tau_x^{S_1}(u), \tau_x^{S_2}(u) \in T$ and represent the grades of $S_1(x)$ and $S_2(x)$, respectively.

Definition 2.4 *([23]) Let $(S_1,\mathfrak{X},N)$ and $(S_2,\mathfrak{X},N)$ be two N-soft sets over $\mathcal{U}$. Then, $(S_1,\mathfrak{X},N)$ and $(S_2,\mathfrak{X},N)$ are the equal N-soft sets, denoted by $(S_1,\mathfrak{X},N) = (S_2,\mathfrak{X},N)$, if for each $x \in \mathfrak{X}$*

$$\tau_x^{S_1}(u) = \tau_x^{S_2}(u) \quad \forall u \in \mathcal{U}$$

where $\tau_x^{S_1}(u), \tau_x^{S_2}(u) \in T$ and represent the grades of $S_1(x)$ and $S_2(x)$, respectively.

3. Contributions to operations on N-soft sets

In this part, we define the intersection, union and some products for N-soft sets and investigate their basic properties.

Definition 3.1 *Let $\mathcal{U}$ be a universal set. For $i \in I = \{1,2,...,w\}$, the* intersection *of the N_i-soft sets $(S_i,\mathfrak{X},N_i)$ over $\mathcal{U}$ is defined by*

$$\bigcap_{i=1}^{w}(S_i,\mathfrak{X},N_i)=(\bigcap_{i=1}^{w}S_i,\mathfrak{X},min_{i\in I}\{N_i\})$$

where for every $x\in\mathfrak{X}$,

$$\bigcap_{i=1}^{w}S_i(x)=\{(u,\tau_x^{\bigcap_{i=1}^{w}S_i}(u)):u\in\mathcal{U}\text{ and }\tau_x^{\bigcap_{i=1}^{w}S_i}(u)\in T\}$$

such that $\tau_x^{\bigcap_{i=1}^{w}S_i}(u)=\bigwedge_{i=1}^{w}\tau_x^{S_i}(u)=min_{i\in I}\{\tau_x^{S_i}(u)\}$ *for each* $u\in\mathcal{U}$.

Definition 3.2 *Let* $\mathcal{U}$ *be a universal set. For* $i\in I=\{1,2,...,w\}$, *the* union *of the* N_i*-soft sets* $(S_i,\mathfrak{X},N_i)$ *over* $\mathcal{U}$ *is defined by*

$$\bigcup_{i=1}^{w}(S_i,\mathfrak{X},N_i)=(\bigcup_{i=1}^{w}S_i,\mathfrak{X},max_{i\in I}\{N_i\})$$

where for every $x\in\mathfrak{X}$,

$$\bigcup_{i=1}^{w}S_i(x)=\{(u,\tau_x^{\bigcup_{i=1}^{w}S_i}(u)):u\in\mathcal{U}\text{ and }\tau_x^{\bigcup_{i=1}^{w}S_i}(u)\in T\}$$

such that $\tau_x^{\bigcup_{i=1}^{w}S_i}(u)=\bigvee_{i=1}^{w}\tau_x^{S_i}(u)=max_{i\in I}\{\tau_x^{S_i}(u)\}$ *for each* $u\in\mathcal{U}$.

Proposition 3.3 *The operations of intersection and union on* N*-soft sets are hold the properties: idempotent law, associative law, commutative law and distributive law.*

Proof The proof is straightforward. □

From now on, for $i\in I=\{1,2,...,w\}$, $(x^i)_{i\in I}$, $(u^i)_{i\in I}$ and $(u_{k_i})_{i\in I}$ stand for $(x^1,x^2,...,x^w)$, $(u^1,u^2,...,u^w)$ and $(u_{k_1},u_{k_2},...,u_{k_w})$ respectively.

Definition 3.4 *Let* $\mathcal{U}$ *be a universal set. For* $i\in I=\{1,2,...,w\}$, *the* And product *of the* N_i*-soft sets* $(S_i,\mathfrak{X}_i,N_i)$ *over* $\mathcal{U}$ *is defined by*

$$\bigwedge_{i=1}^{w}(S_i,\mathfrak{X}_i,N_i)=(\bigwedge_{i=1}^{w}S_i,\prod_{i=1}^{w}\mathfrak{X}_i,min_{i\in I}\{N_i\})$$

where for every $(x^i)_{i\in I}\in\prod_{i=1}^{w}\mathfrak{X}_i$,

$$\bigwedge_{i=1}^{w}S_i((x^i)_{i\in I})=\{(u,\tau_{(x^i)_{i\in I}}^{\bigwedge_{i=1}^{w}S_i}(u)):u\in\mathcal{U}\text{ and }\tau_{(x^i)_{i\in I}}^{\bigwedge_{i=1}^{w}S_i}(u)\in T\}$$

such that $\tau_{(x^i)_{i\in I}}^{\bigwedge_{i=1}^{w}S_i}(u)=\bigwedge_{i=1}^{w}\tau_{x^i}^{S_i}(u)=min_{i\in I}\{\tau_{x^i}^{S_i}(u)\}$ *for each* $u\in\mathcal{U}$.

Definition 3.5 *Let* $\mathcal{U}$ *be a universal set. For* $i\in I=\{1,2,...,w\}$, *the* Or product *of the* N_i*-soft sets* $(S_i,\mathfrak{X}_i,N_i)$ *over* $\mathcal{U}$ *is defined by*

$$\bigvee_{i=1}^{w}(S_i,\mathfrak{X}_i,N_i)=(\bigvee_{i=1}^{w}S_i,\prod_{i=1}^{w}\mathfrak{X}_i,max_{i\in I}\{N_i\})$$

where for every $(x^i)_{i\in I}\in\prod_{i=1}^{w}\mathfrak{X}_i$ *,*

$$\bigvee_{i=1}^{w}S_i((x^i)_{i\in I})=\{(u,\tau_{(x^i)_{i\in I}}^{\bigvee_{i=1}^{w}S_i}(u)):u\in\mathcal{U}\text{ and }\tau_{(x^i)_{i\in I}}^{\bigvee_{i=1}^{w}S_i}(u)\in T\}$$

such that $\tau_{(x^i)_{i\in I}}^{\bigvee_{i=1}^{w}S_i}(u)=\bigvee_{i=1}^{w}\tau_{x^i}^{S_i}(u)=max_{i\in I}\{\tau_{x^i}^{S_i}(u)\}$ *for each* $u\in\mathcal{U}$ *.*

Proposition 3.6 *The And product and Or product on* N *-soft sets are hold the properties: associative law and distributive law.*

Proof The proof is straightforward. □

Definition 3.7 *Let* $\mathcal{U}$ *be a universal set. For* $i\in I=\{1,2,...,w\}$ *, the* internal direct product *of the* N_i *-soft sets* $(S_i,\mathfrak{X}_i,N_i)$ *over* U *is defined by*

$$\prod_{i=1}^{w}(S_i,\mathfrak{X}_i,N_i)=(\prod_{i=1}^{w}S_i,\prod_{i=1}^{w}\mathfrak{X}_i,min_{i\in I}\{N_i\})$$

where for every $(x^i)_{i\in I}\in\prod_{i=1}^{w}\mathfrak{X}_i$ *,*

$$\prod_{i=1}^{w}S_i((x^i)_{i\in I})=\{((u_{k_i})_{i\in I},\tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w}S_i}((u_{k_i})_{i\in I})):(u_{k_i})_{i\in I}\in\mathcal{U}^w\text{ and }\tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w}S_i}((u_{k_i})_{i\in I})\in T\}$$

such that $\tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w}S_i}((u_{k_i})_{i\in I})=\bigwedge_{i=1}^{w}\tau_{x^i}^{S_i}(u_{k_i})=min_{i\in I}\{\tau_{x^i}^{S_i}(u_{k_i})\}$ *for each* $(u_{k_i})_{i\in I}\in\mathcal{U}^w$ *.*

Definition 3.8 *For* $i\in I=\{1,2,...,w\}$ *, let* $\mathcal{U}_i$ *be the universal sets and* $(S_i,\mathfrak{X}_i,N_i)$ *be the* N_i *-soft sets over* $\mathcal{U}_i$ *. The* external direct product *of the* N_i *-soft sets* $(S_i,\mathfrak{X}_i,N_i)$ *over* $\mathcal{U}_i$ *is defined by*

$$\prod_{i=1}^{w}(S_i,\mathfrak{X}_i,N_i)=(\prod_{i=1}^{w}S_i,\prod_{i=1}^{w}\mathfrak{X}_i,min_{i\in I}\{N_i\})$$

where for every $(x^i)_{i\in I}\in\prod_{i=1}^{w}\mathfrak{X}_i$ *,*

$$\prod_{i=1}^{w}S_i((x^i)_{i\in I})=\{((u^i)_{i\in I},\tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w}S_i}((u^i)_{i\in I})):(u^i)_{i\in I}\in\prod_{i=1}^{w}\mathcal{U}_i\text{ and }\tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w}S_i}((u^i)_{i\in I})\in T\}$$

such that $\tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w}S_i}((u^i)_{i\in I})=\bigwedge_{i=1}^{w}\tau_{x^i}^{S_i}(u^i)=min_{i\in I}\{\tau_{x^i}^{S_i}(u^i)\}$ *for each* $(u^i)_{i\in I}\in\prod_{i=1}^{w}\mathcal{U}_i$ *.*

The internal direct product and external direct product of N_i-soft sets $(S_i,\mathfrak{X}_i,N_i)$ are also named cartesian product of $(S_i,\mathfrak{X}_i,N_i)$ over $\mathcal{U}^w$ and $\prod_{i=1}^{w}\mathcal{U}_i$, respectively.

Example 3.9 *Let* $\mathcal{U}=\{u_1,u_2\}$ *be a universal set, and also* $\mathfrak{X}_1=\{x_1^1\}$, $\mathfrak{X}_2=\{x_1^2\}$ *and* $\mathfrak{X}_3=\{x_1^3,x_2^3\}$ *be three different sets of parameters. Let us consider 7-soft set, 5- soft set and 10-soft set as follows:*

$(S_1,\mathfrak{X}_1,7)=\{(x_1^1,\{(u_1,6),(u_2,4)\})\}$,
$(S_2,\mathfrak{X}_2,5)=\{(x_1^2,\{(u_1,2),(u_2,4)\})\}$,
$(S_3,\mathfrak{X}_3,10)=\{(x_1^3,\{(u_1,4),(u_2,8)\}),(x_2^3,\{(u_1,7),(u_2,2)\})\}$.

Then, the Or product of the above 7-soft set, 5-soft set and 10-soft set over $\mathcal{U}$ *is the following 10-soft set:*

$$\bigvee_{i=1}^{3}(S_i,\mathfrak{X}_i,N_i)=(\bigvee_{i=1}^{3}S_i,\prod_{i=1}^{3}\mathfrak{X}_i,10)=\{((x_1^1,x_1^2,x_1^3),\{(u_1,6),(u_2,8)\}),((x_1^1,x_1^2,x_2^3),\{(u_1,7),(u_2,4)\})\}.$$

Then, the internal direct product of the above 7-soft set, 5-soft set and 10-soft set over $\mathcal{U}$ *is the following 5-soft set:*

$$\prod_{i=1}^{3}(S_i,\mathfrak{X}_i,N_i)=(\prod_{i=1}^{3}S_i,\prod_{i=1}^{3}\mathfrak{X}_i,5)$$

$$=\left\{\begin{array}{l}\left((x_1^1,x_1^2,x_1^3),\left\{\begin{array}{l}((u_1,u_1,u_1),2),((u_1,u_1,u_2),2),((u_1,u_2,u_1),4),((u_1,u_2,u_2),4),\\((u_2,u_1,u_1),2),((u_2,u_1,u_2),2),((u_2,u_2,u_1),4),((u_2,u_2,u_2),4)\end{array}\right\}\right),\\ \left((x_1^1,x_1^2,x_2^3),\left\{\begin{array}{l}((u_1,u_1,u_1),2),((u_1,u_1,u_2),2),((u_1,u_2,u_1),4),((u_1,u_2,u_2),2),\\((u_2,u_1,u_1),2),((u_2,u_1,u_2),2),((u_2,u_2,u_1),4),((u_2,u_2,u_2),2)\})\end{array}\right\}\right)\end{array}\right\}.$$

4. N-soft groups

In this part, the concepts of N-soft group and normal N-soft group are defined. Some illustrations to explain these concepts are given.

4.1. N-soft group

Definition 4.1 *Let* $(\mathcal{G},\cdot)$ *be a group and* $(S,\mathfrak{X},N)$ *be an* N*-soft set over* $\mathcal{G}$. *Then,* $(S,\mathfrak{X},N)$ *is said to be an* N-soft group *over* $\mathcal{G}$ *iff for each* $x\in\mathfrak{X}$,

(i) $\tau_x^S(g_k\cdot g_l)\geq\tau_x^S(g_k)\wedge\tau_x^S(g_l)\quad\forall g_k,g_l\in\mathcal{G}$.

(ii) $\tau_x^S(g_k{}^{-1})=\tau_x^S(g_k)\quad\forall g_k\in\mathcal{G}$.

Note that the set of all N-soft groups over $\mathcal{G}$ is denoted by $NSgroup(\mathcal{G})$.

Example 4.2 *Let us consider the classical group* $\mathcal{G}=\{1,-1,i,-i\}$ *with the natural multiplication. Also, we take* $\mathfrak{X}=\{x_1,x_2\}$. *For* $N=6$, *define the 6-soft set* $(S,\mathfrak{X},6)$ *as follows:*

$$(S,\mathfrak{X},6)=\{(x_1,\{(1,5),(-1,5),(i,2),(-i,2)\}),(x_2,\{(1,4),(-1,2),(i,1),(-i,1)\})\}$$

It is obvious that $(S,\mathfrak{X},6)$ *is a 6-soft group over* $\mathcal{G}$.

Proposition 4.3 *Let* $\mathcal{G}$ *be a group and* $(S,\mathfrak{X},N)$ *be an* N*-soft group over* $\mathcal{G}$. *Then, for each* $x\in\mathfrak{X}$

(i) $\tau_x^S(e)\geq\tau_x^S(g_k)$ *for all* $g_k\in\mathcal{G}$.

(ii) $\tau_x^S({g_k}^n) \geq \tau_x^S(g_k)$ *for all* $g_k \in \mathcal{G}$*, where* $n \in \mathbb{N}$*.*

Proof (i) By Definition 4.1, we obtain that for each $x \in \mathfrak{X}$,

$$\tau_x^S(e) = \tau_x^S(g_k{g_k}^{-1}) \geq \tau_x^S(g_k) \wedge \tau_x^S({g_k}^{-1}) = \tau_x^S(g_k) \wedge \tau_x^S(g_k) = \tau_x^S(g_k) \quad \forall g_k \in \mathcal{G}. \tag{4.1}$$

Thus, the proof of (i) is completed.
(ii) It is proved in a similar way. □

Theorem 4.4 *Let* $\mathcal{G}$ *be a group and* $(S, \mathfrak{X}, N)$ *be an* N*-soft set over* $\mathcal{G}$*. Then,* $(S, \mathfrak{X}, N)$ *is an* N*-soft group over* $\mathcal{G}$*.* $\Leftrightarrow$ *For each* $x \in \mathfrak{X}$*,*

$$\tau_x^S(g_k{g_l}^{-1}) \geq \tau_x^S(g_k) \wedge \tau_x^S(g_l) \ \forall g_k, g_l \in \mathcal{G}.$$

Proof Assume that $\mathcal{G}$ is a group and $(S, \mathfrak{X}, N)$ is an N-soft set over $\mathcal{G}$.
$\Rightarrow$: It is clear, thus omitted.
$\Leftarrow$: We have for each $x \in \mathfrak{X}$,

$$\tau_x^S(g_k{g_l}^{-1}) \geq \tau_x^S(g_k) \wedge \tau_x^S(g_l) \quad \forall g_k, g_l \in \mathcal{G}. \tag{4.2}$$

In (4.2), if it is taken $g_k = e$ then it is seen that

$$\tau_x^S({g_l}^{-1}) \geq \tau_x^S(g_l). \tag{4.3}$$

From (4.3), we obtain that

$$\tau_x^S(g_l) = \tau_x^S(({g_l}^{-1})^{-1}) \geq \tau_x^S({g_l}^{-1}). \tag{4.4}$$

Thus, by (4.3) and (4.4), we have for each $x \in \mathfrak{X}$

$$\tau_x^S(g_l) = \tau_x^S({g_l}^{-1}) \quad \forall g_l \in \mathcal{G}. \tag{4.5}$$

Furthermore, we can write from the hypothesis that

$$\tau_x^S(g_k g_l) = \tau_x^S(g_k({g_l}^{-1})^{-1}) \geq \tau_x^S(g_k) \wedge \tau_x^S({g_l}^{-1}) \quad \forall g_k, g_l \in \mathcal{G}. \tag{4.6}$$

By (4.5), we have for each $x \in \mathfrak{X}$

$$\tau_x^S(g_k g_l) \geq \tau_x^S(g_k) \wedge \tau_x^S(g_l) \quad \forall g_k, g_l \in \mathcal{G}. \tag{4.7}$$

This is complete the proof. □

Theorem 4.5 *Let* $\mathcal{G}$ *be a group and* $(S_i, \mathfrak{X}, N_i)$ *be the* N_i*-soft groups over* $\mathcal{G}$ *for* $i \in I = \{1, 2, ..., w\}$*. Then,*

(i) $\bigcap_{i=1}^{w} (S_i, \mathfrak{X}, N_i)$ *is an* N^**-soft group over* $\mathcal{G}$*, where* $N^* = min_{i \in I}\{N_i\}$*.*

(ii) $\bigcup_{i=1}^{w} (S_i, \mathfrak{X}, N_i)$ *is an* N^**-soft group over* $\mathcal{G}$*, where* $N^* = max_{i \in I}\{N_i\}$*.*

Proof Suppose that $\mathcal{G}$ is a group and $(S_i, \mathfrak{X}, N_i)$ for $i \in I = \{1, 2, ..., w\}$ are the N_i-soft groups over $\mathcal{G}$.
(i) To complete the proof, it is enough to demonstrate that for each $x \in \mathfrak{X}$,

$$\tau_x^{\bigcap_{i=1}^{w} S_i}(g_k {g_l}^{-1}) \geq \tau_x^{\bigcap_{i=1}^{w} S_i}(g_k) \wedge \tau_x^{\bigcap_{i=1}^{w} S_i}(g_l) \quad \forall g_k, g_l \in \mathcal{G}. \tag{4.8}$$

$$\begin{aligned}
\tau_x^{\bigcap_{i=1}^{w} S_i}(g_k {g_l}^{-1}) &= \bigwedge_{i=1}^{w} \tau_x^{S_i}(g_k {g_l}^{-1}) \\
&\geq \bigwedge_{i=1}^{w} (\tau_x^{S_i}(g_k) \wedge \tau_x^{S_i}(g_l)) \\
&= (\bigwedge_{i=1}^{w} \tau_x^{S_i}(g_k)) \wedge (\bigwedge_{i=1}^{w} \tau_x^{S_i}(g_l)) \\
&= \tau_x^{\bigcap_{i=1}^{w} S_i}(g_k) \wedge \tau_x^{\bigcap_{i=1}^{w} S_i}(g_l).
\end{aligned} \tag{4.9}$$

By Definition 3.1, it is obvious that $N^* = min_{i \in I}\{N_i\}$. Thus, $\bigcap_{i=1}^{w}(S_i, \mathfrak{X}, N_i)$ is an N^*-soft group over $\mathcal{G}$.

(ii) Similarly, for each $x \in \mathfrak{X}$,

$$\begin{aligned}
\tau_x^{\bigcup_{i=1}^{w} S_i}(g_k {g_l}^{-1}) &= \bigvee_{i=1}^{w} \tau_x^{S_i}(g_k {g_l}^{-1}) \\
&\geq \bigvee_{i=1}^{w} (\tau_x^{S_i}(g_k) \wedge \tau_x^{S_i}(g_l)) \\
&= (\bigvee_{i=1}^{w} \tau_x^{S_i}(g_k)) \wedge (\bigvee_{i=1}^{w} \tau_x^{S_i}(g_l)) \\
&= \tau_x^{\bigcup_{i=1}^{w} S_i}(g_k) \wedge \tau_x^{\bigcup_{i=1}^{w} S_i}(g_l).
\end{aligned} \tag{4.10}$$

By Definition 3.2, it is obvious that $N^* = max_{i \in I}\{N_i\}$. Thus, $\bigcap_{i=1}^{w}(S_i, \mathfrak{X}, N_i)$ is an N^*-soft group over $\mathcal{G}$.

□

Theorem 4.6 *Let $\mathcal{G}$ be a group and $(S_i, \mathfrak{X}_i, N_i)$ be the N_i-soft groups over $\mathcal{G}$ for $i \in I = \{1, 2, ..., w\}$. Then,*

(i) $\bigwedge_{i=1}^{w}(S_i, \mathfrak{X}_i, N_i)$ *is an N^*-soft group over $\mathcal{G}$, where $N^* = min_{i \in I}\{N_i\}$.*

(ii) $\bigvee_{i=1}^{w}(S_i, \mathfrak{X}_i, N_i)$ *is an N^*-soft group over $\mathcal{G}$, where $N^* = max_{i \in I}\{N_i\}$.*

Proof Assume that $\mathcal{G}$ is a group and $(S_i, \mathfrak{X}_i, N_i)$ for $i \in I = \{1, 2, ..., w\}$ are the N_i-soft groups over $\mathcal{G}$.
(i) To complete the proof, we have to show that for each $(x^i)_{i \in I} \in \prod_{i=1}^{w} \mathfrak{X}_i$,

$$\tau_{(x^i)_{i \in I}}^{\bigwedge_{i=1}^{w} S_i}(g_k {g_l}^{-1}) \geq \tau_{(x^i)_{i \in I}}^{\bigwedge_{i=1}^{w} S_i}(g_k) \wedge \tau_{(x^i)_{i \in I}}^{\bigwedge_{i=1}^{w} S_i}(g_l) \quad \forall g_k, g_l \in \mathcal{G}. \tag{4.11}$$

$$\begin{aligned}\tau_{(x^i)_{i\in I}}^{\bigwedge_{i=1}^{w} S_i}(g_k {g_l}^{-1}) &= \bigwedge_{i=1}^{w} \tau_{x^i}^{S_i}(g_k {g_l}^{-1}) \\ &\geq \bigwedge_{i=1}^{w} (\tau_{x^i}^{S_i}(g_k) \wedge \tau_{x^i}^{S_i}(g_l)) \\ &= (\bigwedge_{i=1}^{w} \tau_{x^i}^{S_i}(g_k)) \wedge (\bigwedge_{i=1}^{w} \tau_{x^i}^{S_i}(g_l)) \\ &= \tau_{(x^i)_{i\in I}}^{\bigwedge_{i=1}^{w} S_i}(g_k) \wedge \tau_{(x^i)_{i\in I}}^{\bigwedge_{i=1}^{w} S_i}(g_l). \end{aligned} \tag{4.12}$$

By Definition 3.4, it is obvious that $N^* = min_{i\in I}\{N_i\}$. Thus, $\bigwedge_{i=1}^{w}(S_i, \mathfrak{X}_i, N_i)$ is an N^*-soft group over $\mathcal{G}$.

(ii) It can be shown in a similar way to the proof of (i), therefore omitted. □

Theorem 4.7 *Let $\mathcal{G}_i$ be groups and $(S_i, \mathfrak{X}_i, N_i)$ be the N_i-soft groups over $\mathcal{G}_i$ for $i \in I = \{1, 2, ..., w\}$. Then, $\prod_{i=1}^{w}(S_i, \mathfrak{X}_i, N_i)$ is an N^*-soft group over the group $\prod_{i=1}^{w}\mathcal{G}_i$, where $N^* = min_{i\in I}\{N_i\}$.*

Proof Assume that $(S_i, \mathfrak{X}_i, N_i)$ are the N_i-soft groups over $\mathcal{G}_i$ for $i \in I = \{1, 2, ..., w\}$.
To complete the proof, we have to prove that for each $(x^i)_{i\in I} \in \prod_{i=1}^{w}\mathfrak{X}_i$,

$$\tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w} S_i}((g_k^i)_{i\in I}(g_l^i)_{i\in I}^{-1}) \geq \tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w} S_i}((g_k^i)_{i\in I}) \wedge \tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w} S_i}((g_l^i)_{i\in I}) \quad \forall (g_k^i)_{i\in I}, (g_l^i)_{i\in I} \in \prod_{i=1}^{w}\mathcal{G}_i. \tag{4.13}$$

$$\begin{aligned}\tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w} S_i}((g_k^i)_{i\in I}(g_l^i)_{i\in I}^{-1}) &= \tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w} S_i}((g_k^i {g_l^i}^{-1})_{i\in I}) \\ &= \bigwedge_{i=1}^{w} \tau_{x^i}^{S_i}(g_k^i {g_l^i}^{-1}) \\ &\geq \bigwedge_{i=1}^{w} (\tau_{x^i}^{S_i}(g_k^i) \wedge \tau_{x^i}^{S_i}(g_l^i)) \\ &= (\bigwedge_{i=1}^{w} \tau_{x^i}^{S_i}(g_k^i)) \wedge (\bigwedge_{i=1}^{w} \tau_{x^i}^{S_i}(g_l^i)) \\ &= \tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w} S_i}((g_k^i)_{i\in I}) \wedge \tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w} S_i}((g_l^i)_{i\in I}). \end{aligned} \tag{4.14}$$

By Definition 3.8, we find that $N^* = min_{i\in I}\{N_i\}$. Hence, $\prod_{i=1}^{w}(S_i, \mathfrak{X}_i, N_i)$ is an N^*-soft group over $\prod_{i=1}^{w}\mathcal{G}_i$. □

Example 4.8 *Let us consider the abelian groups $(\mathbb{Z}_2, +)$ and $(\mathbb{Z}_3, +)$.*
For $\mathfrak{X}_1 = \{x_1^1, x_2^1\}$, the following 11-soft set is an 11-soft group over the group $\mathbb{Z}_2$:

$(S_1, \mathfrak{X}_1, N_1) = (S_1, \mathfrak{X}_1, 11) = \{(x_1^1, \{(\bar{0}, 9), (\bar{1}, 7)\}), (x_2^1, \{(\bar{0}, 5), (\bar{1}, 3)\})\}$.

For $\mathfrak{X}_2 = \{x_1^2, x_2^2\}$, the following 7-soft set is an 7-soft group over the group $\mathbb{Z}_3$:

$(S_2, \mathfrak{X}_2, N_2) = (S_2, \mathfrak{X}_2, 7) = \{(x_1^2, \{(\bar{\bar{0}}, 5), (\bar{\bar{1}}, 4), (\bar{\bar{2}}, 4)\}), (x_2^2, \{(\bar{\bar{0}}, 6), (\bar{\bar{1}}, 4), (\bar{\bar{2}}, 4)\})\}$.

Then, we obtain that

$$\prod_{i=1}^{2}(S_i,\mathfrak{X}_i,N_i)=\left\{\begin{array}{l}((x_1^1,x_1^2),\{((\bar{0},\bar{\bar{0}}),5),((\bar{0},\bar{\bar{1}}),4),((\bar{0},\bar{\bar{2}}),4),((\bar{1},\bar{\bar{0}}),5),((\bar{1},\bar{\bar{1}}),4),((\bar{1},\bar{\bar{2}}),4)\}),\\ ((x_1^1,x_2^2),\{((\bar{0},\bar{\bar{0}}),6),((\bar{0},\bar{\bar{1}}),4),((\bar{0},\bar{\bar{2}}),4),((\bar{1},\bar{\bar{0}}),6),((\bar{1},\bar{\bar{1}}),4),((\bar{1},\bar{\bar{2}}),4)\}),\\ ((x_2^1,x_1^2),\{((\bar{0},\bar{\bar{0}}),5),((\bar{0},\bar{\bar{1}}),4),((\bar{0},\bar{\bar{2}}),4),((\bar{1},\bar{\bar{0}}),3),((\bar{1},\bar{\bar{1}}),3),((\bar{1},\bar{\bar{2}}),3)\}),\\ ((x_2^1,x_2^2),\{((\bar{0},\bar{\bar{0}}),5),((\bar{0},\bar{\bar{1}}),4),((\bar{0},\bar{\bar{2}}),4),((\bar{1},\bar{\bar{0}}),3),((\bar{1},\bar{\bar{1}}),3),((\bar{1},\bar{\bar{2}}),3)\})\end{array}\right\}.$$

is a 7-soft group over the group $\mathbb{Z}_2\times\mathbb{Z}_3$*.*

Theorem 4.9 *Let* $\mathcal{G}_i$ *be groups and* $(S_i,\mathfrak{X}_i,N_i)$ *be the* N_i*-soft sets over* $\mathcal{G}_i$ *for* $i\in I=\{1,2,...,w\}$*. If* $\prod_{i=1}^{w}(S_i,\mathfrak{X}_i,N_i)$ *is an* N^**-soft group over the group* $\prod_{i=1}^{w}\mathcal{G}_i$ *(where* $N^*=min_{i\in I}\{N_i\}$*) then for at least one* $i'\in I=\{1,2,...,w\}$*,* $(S_{i'},\mathfrak{X}_{i'},N_{i'})$ *is an* $N_{i'}$*-soft group over* $\mathcal{G}_{i'}$*.*

Proof For each $i\in I=\{1,2,...,w\}$, e^i is the identity element of $\mathcal{G}_i$. Then, we can write that

$$\tau_{x^{i'}}^{S_{i'}}(g^{i'})\le\tau_{(x^1,...,x^{i'-1},x^{i'+1},...,x^w)}^{\prod_{i\in I-\{i'\}}S_i}((e^1,...,e^{i'-1},e^{i'+1},...,e^w))\tag{4.15}$$

and so $(S_{i'},\mathfrak{X}_{i'},N_{i'})$ is an $N_{i'}$-soft group over $\mathcal{G}_{i'}$.

On the contrary, assume that there does not exist an i' such that (4.15) holds, i.e. we can find $x^{i'}\in\mathfrak{X}_{i'}$ such that

$$\tau_{(x^1,...,x^{i'-1},x^{i'+1},...,x^w)}^{\prod_{i\in I-\{i'\}}S_i}((e^1,...,e^{i'-1},e^{i'+1},...,e^w))<\tau_{x^{i'}}^{S_{i'}}(g^{i'})\quad\forall i'\in I\tag{4.16}$$

Then, we obtain that for $(g^i)_{i\in I}\in\prod_{i=1}^{w}\mathcal{G}_i$

$$\begin{aligned}\tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w}S_i}((g^i)_{i\in I}) &= \bigwedge_{i=1}^{w}\tau_{x^i}^{S_i}(g^i)>\tau_{(x^1,...,x^{i'-1},x^{i'+1},...,x^w)}^{\prod_{i\in I-\{i'\}}S_i}((e^1,...,e^{i'-1},e^{i'+1},...,e^w))\\ &= \bigwedge_{i\in I-\{i'\}}\tau_{x^i}^{S_i}(e^i)=\bigwedge_{i=1}^{w}\tau_{x^i}^{S_i}(e^i)=\tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w}S_i}((e^i)_{i\in I})\end{aligned}\tag{4.17}$$

Therefore, by Definition 4.1 and Proposition 4.3, $\prod_{i=1}^{w}(S_i,\mathfrak{X}_i,N_i)$ is not an N^*-soft group over the group $\prod_{i=1}^{w}\mathcal{G}_i$.

So, (4.15) holds.

For a particular i' which satisfies the inequality (4.15), we have the following:

$$\begin{aligned}\tau_{x^{i'}}^{S_{i'}}(g_k^{i'}g_l^{i'}) &= \tau_{x^{i'}}^{S_{i'}}(g_k^{i'}g_l^{i'})\wedge\tau_{(x^1,...,x^{i'-1},x^{i'+1},...,x^w)}^{\prod_{i\in I-\{i'\}}S_i}((e^1,...,e^{i'-1},e^{i'+1},...,e^w)(e^1,...,e^{i'-1},e^{i'+1},...,e^w))\\ &= \tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w}S_i}((e^1,...,e^{i'-1},g_k^{i'},e^{i'+1},...,e^w)(e^1,...,e^{i'-1},g_l^{i'},e^{i'+1},...,e^w))\\ &\ge \tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w}S_i}((e^1,...,e^{i'-1},g_k^{i'},e^{i'+1},...,e^w))\wedge\tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w}S_i}((e^1,...,e^{i'-1},g_l^{i'},e^{i'+1},...,e^w))\\ &= (\tau_{x^{i'}}^{S_{i'}}(g_k^{i'})\wedge\tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w}S_i}((e^1,...,e^{i'-1},g_k^{i'},e^{i'+1},...,e^w)))\wedge(\tau_{x^{i'}}^{S_{i'}}(g_l^{i'})\wedge\tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w}S_i}((e^1,...,e^{i'-1},g_l^{i'},e^{i'+1},...,e^w)))\\ &= \tau_{x^{i'}}^{S_{i'}}(g_k^{i'})\wedge\tau_{x^{i'}}^{S_{i'}}(g_l^{i'})\end{aligned}\tag{4.18}$$

It can be demonstrated in a similar way that $\tau^{S_{i'}}_{x^{i'}}(g^{i'^{-1}}_k) = \tau^{S_{i'}}_{x^{i'}}(g^{i'}_k) \ \ \forall g^{i'}_k \in \mathcal{G}_{i'}$.
Hence, we say that $(S_{i'}, \mathfrak{X}_{i'}, N_{i'})$ is an $N_{i'}$-soft group over $\mathcal{G}_{i'}$. □

Example 4.10 *We consider* $(S_1, \mathfrak{X}_1, 11)$ *and* $(S_2, \mathfrak{X}_2, 7)$ *in Example 4.8. If we take* $(x^1_1, \{(\bar{0}, 9), (\bar{1}, 10)\})$ *instead of* $(x^1_1, \{(\bar{0}, 9), (\bar{1}, 7)\})$ *in the 11-soft set* $(S_1, \mathfrak{X}_1, 11)$ *then it is not an 11-soft group over* $\mathbb{Z}_2$. *However,* $\prod_{i=1}^{2}(S_i, \mathfrak{X}_i, N_i)$ *obtained under these conditions is the same as* $\prod_{i=1}^{2}(S_i, \mathfrak{X}_i, N_i)$ *in Example 4.8 and it is a 7-soft group over* $\mathbb{Z}_2 \times \mathbb{Z}_3$.

4.2. Normal N-soft group

Definition 4.11 *Let* $\mathcal{G}$ *be a group and* $(S, \mathfrak{X}, N)$ *be an* N*-soft group over* $\mathcal{G}$. *Then,* $(S, \mathfrak{X}, N)$ *is said to be a* normal N-soft group *over* $\mathcal{G}$ *if for each* $x \in \mathfrak{X}$,

$$\tau^S_x(g_k g_l) = \tau^S_x(g_l g_k) \ \forall g_k, g_l \in \mathcal{G} \tag{4.19}$$

Note that the set of all normal N-soft groups over $\mathcal{G}$ is denoted by $NS\mathcal{N}group(\mathcal{G})$.

Example 4.12 *Let us consider the symmetric group* S_3. *For* $\mathfrak{X} = \mathbb{Z}_4$, *we create the following 99-soft set.*

$$(S, \mathbb{Z}_4, 99) = \left\{ \begin{array}{l} (\bar{0}, \{(e, 95), ((12), 76), ((13), 76), ((23), 76), ((123), 88), ((132), 88)\}), \\ (\bar{1}, \{(e, 13), ((12), 5), ((13), 5), ((23), 5), ((123), 13), ((132), 13)\}), \\ (\bar{2}, \{(e, 47), ((12), 47), ((13), 47), ((23), 47), ((123), 47), ((132), 47)\}), \\ (\bar{3}, \{(e, 66), ((12), 59), ((13), 59), ((23), 59), ((123), 59), ((132), 59)\}) \end{array} \right\}.$$

Then, it is seen that $(S, \mathbb{Z}_4, 99)$ *is a 99-soft group over* S_3. *Also, it is a normal 99-soft group over* S_3 *since the assertion (4.19) is satisfied.*

By Definition 4.11, it is clear that if $\mathcal{G}$ is an abelian group then every N-soft group over $\mathcal{G}$ is a normal N-soft group.

We consider the 7-soft group (S_2, X_2, N_2) over $\mathbb{Z}_3$ in Example 4.8. Since $\mathbb{Z}_3$ is an abelian group, (S_2, X_2, N_2) is also a normal 7-soft group over $\mathbb{Z}_3$.

Proposition 4.13 *Let* $(S, \mathfrak{X}, N)$ *be a normal* N*-soft group over* $\mathcal{G}$. *Then, the following are equivalent:*
For each $x \in \mathfrak{X}$,

(1) $\tau^S_x(g_k g_l) = \tau^S_x(g_l g_k) \quad \forall g_k, g_l \in \mathcal{G}$.

(2) $\tau^S_x(g_k g_l g_k^{-1}) = \tau^S_x(g_l) \quad \forall g_k, g_l \in \mathcal{G}$.

Proof The proof is straightforward, hence omitted. □

Theorem 4.14 *Let* $\mathcal{G}$ *be a group and* $(S_i, \mathfrak{X}, N_i)$ *be the normal* N_i*-soft groups over* $\mathcal{G}$ *for* $i \in I = \{1, 2, ..., w\}$. *Then,*

(i) $\bigcap_{i=1}^{w}(S_i, \mathfrak{X}, N_i)$ *is a normal* N^**-soft group over* $\mathcal{G}$, *where* $N^* = min_{i \in I}\{N_i\}$.

(ii) $\bigcup_{i=1}^{w}(S_i,\mathfrak{X},N_i)$ *is a normal* N^**-soft group over* $\mathcal{G}$*, where* $N^* = max_{i\in I}\{N_i\}$.

Proof Suppose that $\mathcal{G}$ is a group and $(S_i,\mathfrak{X},N_i)$ for $i \in I = \{1,2,...,w\}$ are the normal N_i-soft groups over $\mathcal{G}$.

(i) To complete the proof, we must achieve that for each $x \in \mathfrak{X}$, $\tau_x^{\bigcap_{i=1}^{w} S_i}(g_k g_l) = \tau_x^{\bigcap_{i=1}^{w} S_i}(g_l g_k)$ $\forall g_k, g_l \in \mathcal{G}$.

$$\tau_x^{\bigcap_{i=1}^{w} S_i}(g_k g_l) = \bigwedge_{i=1}^{w} \tau_x^{S_i}(g_k g_l) = \bigwedge_{i=1}^{w} \tau_x^{S_i}(g_l g_k) = \tau_x^{\bigcap_{i=1}^{w} S_i}(g_l g_k). \tag{4.20}$$

By Definition 3.1, it is obvious that $N^* = min_{i\in I}\{N_i\}$. So, $\bigcap_{i=1}^{w}(S_i,\mathfrak{X},N_i)$ is a normal N^*-soft group over $\mathcal{G}$.

(ii) It can be proved similar to the proof of (i). □

Theorem 4.15 *Let* $\mathcal{G}$ *be a group and* $(S_i,\mathfrak{X}_i,N_i)$ *be the normal* N_i*-soft groups over* $\mathcal{G}$ *for* $i \in I = \{1,2,...,w\}$*. Then,*

(i) $\bigwedge_{i=1}^{w}(S_i,\mathfrak{X}_i,N_i)$ *is a normal* N^**-soft group over* $\mathcal{G}$*, where* $N^* = min_{i\in I}\{N_i\}$.

(ii) $\bigvee_{i=1}^{w}(S_i,\mathfrak{X}_i,N_i)$ *is a normal* N^**-soft group over* $\mathcal{G}$*, where* $N^* = max_{i\in I}\{N_i\}$.

Proof Considering the proofs of Theorems 4.6 and 4.14, it can be easily proved. □

Theorem 4.16 *Let* $\mathcal{G}_i$ *be groups and* $(S_i,\mathfrak{X}_i,N_i)$ *be the normal* N_i*-soft groups over* $\mathcal{G}_i$ *for* $i \in I = \{1,2,...,w\}$*. Then,* $\prod_{i=1}^{w}(S_i,\mathfrak{X}_i,N_i)$ *is a normal* N^**-soft group over the group* $\prod_{i=1}^{w}\mathcal{G}_i$*, where* $N^* = min_{i\in I}\{N_i\}$.

Proof Considering the proofs of Theorems 4.7 and 4.14, it can be easily proved. □

5. N-soft ring and field

This part focuses on the notions of N-soft ring, N-soft ideal, completely prime N-soft ideal, completely semiprime N-soft ideal and N-soft field. Also, their related results and certain properties are presented.

5.1. N-soft ring

Definition 5.1 *Let* $(\mathcal{R},+,\cdot)$ *be a ring and* $(S,\mathfrak{X},N)$ *be an* N*-soft set over* $\mathcal{R}$*. Then,* $(S,\mathfrak{X},N)$ *is termed to be an* N-soft ring *over* $\mathcal{R}$ *iff for each* $x \in \mathfrak{X}$*,*

(i) $\tau_x^S(r_k - r_l) \geq \tau_x^S(r_k) \wedge \tau_x^S(r_l)$ $\forall r_k, r_l \in \mathcal{R}$.

(ii) $\tau_x^S(r_k \cdot r_l) \geq \tau_x^S(r_k) \wedge \tau_x^S(r_l)$ $\forall r_k, r_l \in \mathcal{R}$.

In addition to these, if for each $x \in \mathfrak{X}$ $\tau_x^S(r_k \cdot r_l) = \tau_x^S(r_l \cdot r_k)$ $\forall r_k, r_l \in \mathcal{R}$ then $(S,\mathfrak{X},N)$ is called a commutative N-soft ring over $\mathcal{R}$.

Note that the set of all N-soft rings over $\mathcal{R}$ is denoted by $NSring(\mathcal{R})$.

Example 5.2 *Let us consider the ring* $4\mathbb{Z}$ *and the parameter set* $X = \{x_1, x_2\}$. *For* $N = 15$, *we describe the following 15-soft set:*

$$S(x_1) = \begin{cases} (11, r), & \text{if } r = 8k + 4, k \in \mathbb{Z} \\ (13, r), & \text{if } r = 8k, k \in \mathbb{Z} \end{cases} \text{ and } S(x_2) = \begin{cases} (3, r), & \text{if } r = 8k + 4, k \in \mathbb{Z} \\ (8, r), & \text{if } r = 8k, k \in \mathbb{Z} \end{cases}.$$

Then, we have that $(S, \mathfrak{X}, 15)$ *is a 15-soft ring over* $\mathcal{R}$.

Definition 5.3 *Let* $(\mathcal{R}, +, \cdot)$ *be a ring and* $(S, \mathfrak{X}, N)$ *be an* N*-soft ring over* $\mathcal{R}$. *Then,* $(S, \mathfrak{X}, N)$ *is termed to be an* N-soft left ideal *over* $\mathcal{R}$ *if for each* $x \in \mathfrak{X}$, $\tau_x^S(r_k \cdot r_l) \geq \tau_x^S(r_l)$ $\forall r_k, r_l \in \mathcal{R}$. *It is termed to be an* N-soft right ideal *over* $\mathcal{R}$ *if for each* $x \in \mathfrak{X}$, $\tau_x^S(r_k \cdot r_l) \geq \tau_x^S(r_k)$ $\forall r_k, r_l \in \mathcal{R}$. *If* $(S, \mathfrak{X}, N)$ *is both* N*-soft left ideal and* N*-soft right ideal over* $\mathcal{R}$, *then it is called an* N-soft ideal *over* $\mathcal{R}$.

Theorem 5.4 *Let* $\mathcal{R}$ *be a ring and* $(S, \mathfrak{X}, N)$ *be an* N*-soft set over* $\mathcal{R}$. *Then,* $(S, \mathfrak{X}, N)$ *is an* N-soft ideal *over* $\mathcal{R}$ *iff for each* $x \in \mathfrak{X}$,

(i) $\tau_x^S(r_k - r_l) \geq \tau_x^S(r_k) \wedge \tau_x^S(r_l) \quad \forall r_k, r_l \in \mathcal{R}$.

(ii) $\tau_x^S(r_k \cdot r_l) \geq \tau_x^S(r_k) \vee \tau_x^S(r_l) \quad \forall r_k, r_l \in \mathcal{R}$.

Proof The proof is apparent from Definitions 5.1 and 5.3. □

Example 5.5 *Let us consider the ring* $\mathbb{Z}/8\mathbb{Z}$ *and the parameter set* $\mathfrak{X} = \mathbb{Z}_2$. *For* $N = 30$, *we create the following 30-soft set:*

$$(S, \mathbb{Z}_2, 30) = \left\{ \begin{array}{l} (\bar{0}, \{(25, 8\mathbb{Z}), (5, 1+8\mathbb{Z}), (15, 2+8\mathbb{Z}), (5, 3+8\mathbb{Z}), (20, 4+8\mathbb{Z}), (5, 5+8\mathbb{Z}), (15, 6+8\mathbb{Z}), (5, 7+8\mathbb{Z})\}) \\ (\bar{1}, \{(17, 8\mathbb{Z}), (9, 1+8\mathbb{Z}), (12, 2+8\mathbb{Z}), (9, 3+8\mathbb{Z}), (12, 4+8\mathbb{Z}), (9, 5+8\mathbb{Z}), (12, 6+8\mathbb{Z}), (9, 7+8\mathbb{Z})\}) \end{array} \right\}$$

By Theorem 5.4, we can verify that $(S, \mathbb{Z}_2, 30)$ *is a 30-soft ideal over* $\mathbb{Z}/8\mathbb{Z}$.

Example 5.6 *Let* $\mathcal{R}$ *be a classical ring and let* $\mathfrak{X}$ *be a parameter set. Also, the center of classical ring* $\mathcal{R}$ *is* $C(\mathcal{R}) = \{c \in \mathcal{R} : rc = cr \text{ for every } r \in \mathcal{R}\}$. *For* $N = 5$, *we define the 5-soft set as follows:*
For each $x \in \mathfrak{X}$,

$$S(x) = \begin{cases} (4, r), & \text{if } r \in C(\mathcal{R}) \\ (0, r), & \text{if } r \notin C(\mathcal{R}) \end{cases}.$$

For a 5-soft set $(S, \mathfrak{X}, 5)$ *described above, we can say that this is a 5-soft ring over* $\mathcal{R}$, *but may not be a 5-soft ideal. On the other hand, it is clear from Definitions 5.1 and 5.3 that each* N*-soft ideal over* $\mathcal{R}$ *is an* N*-soft ring over* $\mathcal{R}$.

Proposition 5.7 *If* $(S, \mathfrak{X}, N)$ *is an* N*-soft ring/ideal over the ring* $\mathcal{R}$ *then for each* $x \in \mathfrak{X}$, $\tau_x^S(0_{\mathcal{R}}) \geq \tau_x^S(r_k)$ $\forall r_k \in \mathcal{R}$.

Proof For each $x \in \mathfrak{X}$, $\tau_x^S(0_{\mathcal{R}}) = \tau_x^S(r_k - r_k) \geq \tau_x^S(r_k) \wedge \tau_x^S(r_k) = \tau_x^S(r_k)$ $\forall r_k \in \mathcal{R}$. So, the proof is over. □

Proposition 5.8 *Let $\mathcal{R}$ be a ring with identity. If $(S,\mathfrak{X},N)$ is an N-soft ideal over $\mathcal{R}$ then for each $x\in\mathfrak{X}$, $\tau_x^S(r_k)\geq\tau_x^S(1_{\mathcal{R}})$ $\forall r_k\in\mathcal{R}$.*

Proof For each $x\in\mathfrak{X}$, $\tau_x^S(r_k)=\tau_x^S(r_k 1_{\mathcal{R}})\geq\tau_x^S(1_{\mathcal{R}})$ $\forall r_k\in\mathcal{R}$. So, the proof is over. □

Example 5.9 *Consider the 30-soft ideal $(S,\mathbb{Z}_2,30)$ over $\mathbb{Z}/8\mathbb{Z}$. It is known that $\mathbb{Z}/8\mathbb{Z}$ is a ring with identity (where $0_{\mathbb{Z}/8\mathbb{Z}}=8\mathbb{Z}$ and $1_{\mathbb{Z}/8\mathbb{Z}}=1+8\mathbb{Z}$). It is obvious that $\tau_{\bar{0}}^S(r_k)\leq\tau_{\bar{0}}^S(0_{\mathbb{Z}/8\mathbb{Z}})$, $\tau_{\bar{1}}^S(r_k)\leq\tau_{\bar{1}}^S(0_{\mathbb{Z}/8\mathbb{Z}})$, $\tau_{\bar{0}}^S(r_k)\geq\tau_{\bar{0}}^S(1_{\mathbb{Z}/8\mathbb{Z}})$, and $\tau_{\bar{1}}^S(r_k)\geq\tau_{\bar{1}}^S(1_{\mathbb{Z}/8\mathbb{Z}})$ for all $r_k\in\mathbb{Z}/8\mathbb{Z}$.*

Theorem 5.10 *Let $\mathcal{R}$ be a ring and $(S_i,\mathfrak{X},N_i)$ be the N_i-soft rings (ideals) over $\mathcal{R}$ for $i\in I=\{1,2,...,w\}$. Then,*

(i) $\bigcap\limits_{i=1}^{w}(S_i,\mathfrak{X},N_i)$ *is an N^*-soft ring (ideal) over $\mathcal{R}$, where $N^*=min_{i\in I}\{N_i\}$.*

(ii) $\bigcup\limits_{i=1}^{w}(S_i,\mathfrak{X},N_i)$ *is an N^*-soft ring (ideal) over $\mathcal{R}$, where $N^*=max_{i\in I}\{N_i\}$.*

Proof Assume that $(S_i,\mathfrak{X},N_i)$ is the N_i-soft rings (ideals) over the ring $\mathcal{R}$ for $i\in I=\{1,2,...,w\}$.
(i) If we prove that for each $x\in\mathfrak{X}$,

$$\tau_x^{\bigcap_{i=1}^{w}S_i}(r_k-r_l)\geq\tau_x^{\bigcap_{i=1}^{w}S_i}(r_k)\wedge\tau_x^{\bigcap_{i=1}^{w}S_i}(r_l)\text{ and }\tau_x^{\bigcap_{i=1}^{w}S_i}(r_kr_l)\geq\tau_x^{\bigcap_{i=1}^{w}S_i}(r_k)\wedge\tau_x^{\bigcap_{i=1}^{w}S_i}(r_l)\quad\forall r_k,r_l\in\mathcal{R}\tag{5.1}$$

then the proof is completed for N^*-soft ring.

$$\begin{aligned}\tau_x^{\bigcap_{i=1}^{w}S_i}(r_k-r_l) &= \bigwedge_{i=1}^{w}\tau_x^{S_i}(r_k-r_l)\\ &\geq \bigwedge_{i=1}^{w}(\tau_x^{S_i}(r_k)\wedge\tau_x^{S_i}(r_l))\\ &= (\bigwedge_{i=1}^{w}\tau_x^{S_i}(r_k))\wedge(\bigwedge_{i=1}^{w}\tau_x^{S_i}(r_l))\\ &= \tau_x^{\bigcap_{i=1}^{w}S_i}(r_k)\wedge\tau_x^{\bigcap_{i=1}^{w}S_i}(r_l)\end{aligned}\tag{5.2}$$

and

$$\begin{aligned}\tau_x^{\bigcap_{i=1}^{w}S_i}(r_kr_l) &= \bigwedge_{i=1}^{w}\tau_x^{S_i}(r_kr_l)\\ &\geq \bigwedge_{i=1}^{w}(\tau_x^{S_i}(r_k)\wedge\tau_x^{S_i}(r_l))\\ &= (\bigwedge_{i=1}^{w}\tau_x^{S_i}(r_k))\wedge(\bigwedge_{i=1}^{w}\tau_x^{S_i}(r_l))\\ &= \tau_x^{\bigcap_{i=1}^{w}S_i}(r_k)\wedge\tau_x^{\bigcap_{i=1}^{w}S_i}(r_l).\end{aligned}\tag{5.3}$$

Moreover, from Definition 3.1, we have that $N^* = min_{i \in I}\{N_i\}$. Thus, $\bigcap_{i=1}^{w}(S_i, \mathfrak{X}, N_i)$ is an N^*-soft ring over $\mathcal{R}$. Also, for each $x \in \mathfrak{X}$ and $\forall r_k, r_l \in \mathcal{R}$

$$\tau_x^{\bigcap_{i=1}^{w} S_i}(r_k r_l) = \bigwedge_{i=1}^{w} \tau_x^{S_i}(r_k r_l) \geq \bigwedge_{i=1}^{w} \tau_x^{S_i}(r_l) = \tau_x^{\bigcap_{i=1}^{w} S_i}(r_l) \tag{5.4}$$

$$\tau_x^{\bigcap_{i=1}^{w} S_i}(r_k r_l) = \bigwedge_{i=1}^{w} \tau_x^{S_i}(r_k r_l) \geq \bigwedge_{i=1}^{w} \tau_x^{S_i}(r_k) = \tau_x^{\bigcap_{i=1}^{w} S_i}(r_k) \tag{5.5}$$

Thus, $\bigcap_{i=1}^{w}(S_i, \mathfrak{X}, N_i)$ is an N^*-soft ideal over $\mathcal{R}$ where $N^* = min_{i \in I}\{N_i\}$.

(ii) It can be demonstrated similar to the proof of (i). □

Theorem 5.11 *Let $\mathcal{R}$ be a ring and $(S_i, \mathfrak{X}_i, N_i)$ be the N_i-soft rings (ideals) over $\mathcal{R}$ for $i \in I = \{1, 2, ..., w\}$. Then,*

(i) $\bigwedge_{i=1}^{w}(S_i, \mathfrak{X}_i, N_i)$ *is an N^*-soft ring (ideal) over $\mathcal{R}$, where $N^* = min_{i \in I}\{N_i\}$.*

(ii) $\bigvee_{i=1}^{w}(S_i, \mathfrak{X}_i, N_i)$ *is an N^*-soft ring (ideal) over $\mathcal{R}$, where $N^* = max_{i \in I}\{N_i\}$.*

Proof It is similar to the proof of Theorem 5.10. □

Theorem 5.12 *Let $\mathcal{R}_i$ be rings and $(S_i, \mathfrak{X}_i, N_i)$ be the N_i-soft rings (ideals) over $\mathcal{R}_i$ for $i \in I = \{1, 2, ..., w\}$. Then, $\prod_{i=1}^{w}(S_i, \mathfrak{X}_i, N_i)$ is an N^*-soft ring (ideal) over $\prod_{i=1}^{w}\mathcal{R}_i$, where $N^* = min_{i \in I}\{N_i\}$.*

Proof It is similar to the proof of Theorem 5.10. □

5.2. Completely semiprime N-soft ideal

Definition 5.13 *Let $\mathcal{R}$ be a ring and $(S, \mathfrak{X}, N)$ be an N-soft ideal over $\mathcal{R}$. If for each for each $x \in \mathfrak{X}$,*

$$\tau_x^S(r_k r_l) = \tau_x^S(r_k) \quad \text{or} \quad \tau_x^S(r_k r_l) = \tau_x^S(r_l) \quad \forall r_k, r_l \in \mathcal{R} \tag{5.6}$$

then $(S, \mathfrak{X}, N)$ is called a completely prime N-soft ideal *over $\mathcal{R}$.*

Definition 5.14 *Let $\mathcal{R}$ be a ring and $(S, \mathfrak{X}, N)$ be an N-soft ideal over $\mathcal{R}$. If for each for each $x \in \mathfrak{X}$,*

$$\tau_x^S({r_k}^2) = \tau_x^S(r_k) \ \forall r_k \in \mathcal{R} \tag{5.7}$$

then $(S, \mathfrak{X}, N)$ is called a completely semiprime N-soft ideal *over $\mathcal{R}$.*

Lemma 5.15 *Let $\mathcal{R}$ be a ring and $(S, \mathfrak{X}, N)$ be an N-soft ideal over $\mathcal{R}$. If $(S, \mathfrak{X}, N)$ is a completely prime N-soft ideal then it is a completely semiprime N-soft ideal.*

Proof If it is taken $r_l = r_k$ in Definition 5.13 then the proof is easily seen. □

In general, the opposite claim does not apply. Let us illustrate this with the example below.

Example 5.16 *Let* $\mathfrak{X} = \{x_1, x_2\}$. *We consider the following 25-soft set* $(S, \mathfrak{X}, 25)$ *over the ring* $\mathbb{Z}_6$.

$$(S,\mathfrak{X},25)=\{(x_1,\{(\bar{0},21),(\bar{1},18),(\bar{2},18),(\bar{3},18),(\bar{4},18),(\bar{5},18)\}),(x_2,\{(\bar{0},9),(\bar{1},9),(\bar{2},9),(\bar{3},9),(\bar{4},9),(\bar{5},9)\})\}.$$

Then $(S, \mathfrak{X}, 25)$ *is a 25-soft ideal over* $\mathbb{Z}_6$. *Moreover, it is a completely semiprime 25-soft ideal over* $\mathbb{Z}_6$. *However it is not a completely prime 25-soft ideal since* $\tau^S_{x_1}(\bar{2}\cdot\bar{3}) \neq \tau^S_{x_1}(\bar{2})$ *and* $\tau^S_{x_1}(\bar{2}\cdot\bar{3}) \neq \tau^S_{x_1}(\bar{3})$.

Proposition 5.17 *If* $(S, \mathfrak{X}, N)$ *is a completely semiprime* N*-soft ideal over* $\mathcal{R}$ *then it is a commutative* N*-soft ring over* $\mathcal{R}$.

Proof Suppose that $(S, \mathfrak{X}, N)$ is a completely semiprime N-soft ideal over $\mathcal{R}$. Then, $(S, \mathfrak{X}, N)$ is an N-soft ideal and so it is an N-soft ring. Now we have to show that this N-soft ring $(S, \mathfrak{X}, N)$ is commutative, i.e. for each $x \in \mathfrak{X}$ $\tau^S_x(r_k r_l) = \tau^S_x(r_l r_k)$ $\forall r_k, r_l \in \mathcal{R}$.

$$\tau^S_x(r_kr_l) = \tau^S_x((r_kr_l)^2) = \tau^S_x(r_kr_lr_kr_l) \geq \tau^S_x(r_kr_lr_k) \vee \tau^S_x(r_l) \geq \tau^S_x(r_k) \vee \tau^S_x(r_lr_k) \vee \tau^S_x(r_l) \geq \tau^S_x(r_lr_k). \tag{5.8}$$

Similarly, we obtain that for each $x \in \mathfrak{X}$ $\tau^S_x(r_l r_k) \geq \tau^S_x(r_k r_l)$ $\forall r_k, r_l \in \mathcal{R}$. It follows that $\tau^S_x(r_k r_l) = \tau^S_x(r_l r_k)$ $\forall r_k, r_l \in \mathcal{R}$. Thus, the proof is completed. □

Theorem 5.18 *Let* $\mathcal{R}$ *be a ring and* $(S, \mathfrak{X}, N)$ *be an* N*-soft ideal over* $\mathcal{R}$. $(S, \mathfrak{X}, N)$ *is a completely semiprime* N*-soft ideal over* $\mathcal{R}$. $\Leftrightarrow$ *For all* $n \in \mathbb{Z}^+$ $(n \geq 2)$, $\tau^S_x({r_k}^n) = \tau^S_x(r_k)$ *for each* $x \in \mathfrak{X}$ *and* $\forall r_k \in \mathcal{R}$.

Proof Let $\mathcal{R}$ be a ring and $(S, \mathfrak{X}, N)$ be an N-soft ideal over $\mathcal{R}$.
$\Rightarrow$: Suppose that $(S, \mathfrak{X}, N)$ is a completely semiprime N-soft ideal over $\mathcal{R}$. We prove this result by induction. Clearly, the result holds for $n = 2$. Let $2 \leq q$ be any positive integer. Let's assume that for each $x \in \mathfrak{X}$ $\tau^S_x({r_k}^q) = \tau^S_x(r_k)$ holds. Then, we assert that $\tau^S_x({r_k}^{q+1}) = \tau^S_x(r_k)$ for each $x \in X$ and $\forall r_k \in \mathcal{R}$. Indeed:
Case 1. If q is odd, i.e. $q = 2v + 1$ for $v \in \mathbb{Z}^+$. Then, for each $x \in X$

$$\tau^S_x({r_k}^{q+1}) = \tau^S_x(({r_k}^{v+1})^2) = \tau^S_x({r_k}^{v+1}). \tag{5.9}$$

Since $v + 1 < q$, by the induction hypothesis, $\tau^S_x({r_k}^{v+1}) = \tau^S_x(r_k)$. So we have for each $x \in X$

$$\tau^S_x({r_k}^{q+1}) = \tau^S_x(r_k) \ \forall r_k \in \mathcal{R}, \tag{5.10}$$

where q is odd.
Case 2. If q is even, i.e. $q = 2v$ for $v \in \mathbb{Z}^+$. From Definition 5.3 (ii), we can write that for each $x \in X$

$$\tau^S_x(r_k) \leq \tau^S_x({r_k}^{q+1}) \ \forall r_k \in \mathcal{R}. \tag{5.11}$$

Moreover, again by the induction hypothesis, we have for each $x \in X$

$$\tau^S_x({r_k}^{q+1}) = \tau^S_x({r_k}^{2v+1}) \leq \tau^S_x({r_k}^{2v+2}) = \tau^S_x(({r_k}^{v+1})^2) = \tau^S_x({r_k}^{v+1}). \tag{5.12}$$

Since $v+1<q$, by the induction hypothesis, $\tau_x^S({r_k}^{v+1}) \leq \tau_x^S(r_k)$. Thus we have for each $x \in X$

$$\tau_x^S({r_k}^{q+1}) \leq \tau_x^S(r_k) \ \forall r_k \in \mathcal{R}, \tag{5.13}$$

By (5.11) and (5.13), we obtain that for each $x \in X$

$$\tau_x^S({r_k}^{q+1}) = \tau_x^S(r_k) \ \forall r_k \in \mathcal{R}, \tag{5.14}$$

where q is even. It is completed with (5.10) and (5.14).
$\Leftarrow$: It is obvious. □

Theorem 5.19 *Let $\mathcal{R}$ be a ring and $(S_i,\mathfrak{X},N_i)$ be the completely semiprime N_i-soft ideals over $\mathcal{R}$ for $i \in I=\{1,2,...,w\}$. Then,*

(i) $\bigcap\limits_{i=1}^{w}(S_i,\mathfrak{X},N_i)$ *is a completely semiprime N^*-soft ideal over $\mathcal{R}$, where $N^* = min_{i\in I}\{N_i\}$.*

(ii) $\bigcup\limits_{i=1}^{w}(S_i,\mathfrak{X},N_i)$ *is a completely semiprime N^*-soft ideal over $\mathcal{R}$, where $N^* = max_{i\in I}\{N_i\}$.*

Proof It is straightforward. □

Theorem 5.20 *Let $\mathcal{R}$ be a ring and $(S_i,\mathfrak{X}_i,N_i)$ be the completely semiprime N_i-soft ideals over $\mathcal{R}$ for $i \in I=\{1,2,...,w\}$. Then,*

(i) $\bigwedge\limits_{i=1}^{w}(S_i,\mathfrak{X}_i,N_i)$ *is a completely semiprime N^*-soft ideal over $\mathcal{R}$, where $N^* = min_{i\in I}\{N_i\}$.*

(ii) $\bigvee\limits_{i=1}^{w}(S_i,\mathfrak{X}_i,N_i)$ *is a completely semiprime N^*-soft ideal over $\mathcal{R}$, where $N^* = max_{i\in I}\{N_i\}$.*

Proof It is straightforward. □

Theorem 5.21 *Let $\mathcal{R}_i$ be rings and $(S_i,\mathfrak{X}_i,N_i)$ be the completely semiprime N_i-soft ideals over $\mathcal{R}_i$ for $i \in I=\{1,2,...,w\}$. Then, $\prod\limits_{i=1}^{w}(S_i,\mathfrak{X}_i,N_i)$ is a completely semiprime N^*-soft ideal over $\prod\limits_{i=1}^{w}\mathcal{R}_i$, where $N^* = min_{i\in I}\{N_i\}$.*

Proof It is obvious since for each $(x^i)_{i\in I} \in \prod\limits_{i=1}^{w}\mathfrak{X}_i$ and $\forall (r_k^i)_{i\in I} \in \prod\limits_{i=1}^{w}\mathcal{R}_i$,

$$\tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w} S_i}(((r_k^i)_{i\in I})^2) = \tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w} S_i}(((r_k^i)^2)_{i\in I}) = \bigwedge_{i=1}^{w}\tau_{x^i}^{S_i}((r_k^i)^2) = \bigwedge_{i=1}^{w}\tau_{x^i}^{S_i}(r_k^i) = \tau_{(x^i)_{i\in I}}^{\prod_{i=1}^{w} S_i}((r_k^i)_{i\in I}). \tag{5.15}$$

□

Example 5.22 *Let us consider the rings $(\mathbb{Z}_2,+,.)$, $(\mathbb{Z}_3,+,.)$ and $(\mathbb{Z}_5,+,.)$.*
For $\mathfrak{X}_1=\{x_1^1\}$, the following 6-soft set is a completely semiprime 6-soft ideal over $\mathbb{Z}_2$:

$(S_1,\mathfrak{X}_1,N_1) = (S_1,X_1,6) = \{(x_1^1,\{(\bar{0},4),(\bar{1},3)\})\}$.

For $\mathfrak{X}_2 = \{x_1^2, x_2^2\}$*, the following* 4*-soft set is a completely semiprime 4-soft ideal over* $\mathbb{Z}_3$*:*

$$(S_2, \mathfrak{X}_2, N_2) = (S_2, X_2, 4) = \{(x_1^2, \{(\bar{\bar{0}}, 3), (\bar{\bar{1}}, 1), (\bar{\bar{2}}, 1)\}), (x_2^2, \{(\bar{\bar{0}}, 3), (\bar{\bar{1}}, 2), (\bar{\bar{2}}, 2)\})\}.$$

For $\mathfrak{X}_3 = \{x_1^3\}$*, the following* 7*-soft set is a completely semiprime 7-soft ideal over* $\mathbb{Z}_5$*:*

$$(S_3, \mathfrak{X}_3, N_3) = (S_3, X_3, 7) = \{(x_1^3, \{(\bar{\bar{\bar{0}}}, 5), (\bar{\bar{\bar{1}}}, 2), (\bar{\bar{\bar{2}}}, 2), (\bar{\bar{\bar{3}}}, 2), (\bar{\bar{\bar{4}}}, 2)\}.$$

Then, we obtain that $\prod_{i=1}^{3}(S_i, \mathfrak{X}_i, N_i)$ *is a completely semiprime 4-soft ideal over* $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$*.*

Lemma 5.23 *If* $\mathcal{R}$ *is a Booelan ring, then every* N*-soft ideal over* $\mathcal{R}$ *is a completely semiprime* N*-soft ideal.*

Proof It is clear. □

5.3. N-soft field

Definition 5.24 *Let* $(\mathcal{F}, +, \cdot)$ *be a field and* $(S, \mathfrak{X}, N)$ *be an* N*-soft set over* $\mathcal{F}$*. Then,* $(S, \mathfrak{X}, N)$ *is termed to be an* N-soft field *over* $\mathcal{F}$ *iff for each* $x \in \mathfrak{X}$*,*

(i) $\tau_x^S(f_k - f_l) \geq \tau_x^S(f_k) \wedge \tau_x^S(f_l) \quad \forall f_k, f_l \in \mathcal{F}$.

(ii) $\tau_x^S(f_k \cdot {f_l}^{-1}) \geq \tau_x^S(f_k) \wedge \tau_x^S(f_l) \quad \forall f_k, f_l (\neq 0_{\mathcal{F}}) \in \mathcal{F}$.

Note that the set of all N-soft fields over $\mathcal{F}$ is denoted by $NSfield(\mathcal{F})$.

Example 5.25 *Let For* $X = \{x_1, x_2\}$*. Also, we consider the field* $(\mathbb{Z}_3, +, .)$*. If we take 34-soft set over* $\mathbb{Z}_3$ *as follows:*

$$(S, X, 34) = \{(x_1, \{(\bar{0}, 5), (\bar{1}, 2), (\bar{2}, 2)\}), (x_2, \{(\bar{0}, 32), (\bar{1}, 10), (\bar{2}, 14)\})\},$$

then $(S, X, 34)$ *is not a 34-soft field over* $\mathbb{Z}_3$*. Because* $\tau_{x_2}^S(\bar{2} \cdot \bar{2}^{-1}) = \tau_{x_2}^S(\bar{2} \cdot \bar{2}) \ngeq \tau_{x_2}^S(\bar{2})$*.*
If we take 34-soft set over $\mathbb{Z}_3$ *as follows:*

$$(S, X, 34) = \{(x_1, \{(\bar{0}, 5), (\bar{1}, 2), (\bar{2}, 2)\}), (x_2, \{(\bar{0}, 32), (\bar{1}, 14), (\bar{2}, 14)\})\},$$

then $(S, X, 34)$ *is a 34-soft field over* $\mathbb{Z}_3$*.*

Theorem 5.26 *Let* $\mathcal{F}$ *be a field and* $(S_i, \mathfrak{X}, N_i)$ *be the* N_i*-soft fields over* $\mathcal{F}$ *for* $i \in I = \{1, 2, ..., w\}$*. Then,*

(i) $\bigcap_{i=1}^{w}(S_i, \mathfrak{X}, N_i)$ *is an* N^**-soft field over* $\mathcal{F}$*, where* $N^* = min_{i \in I}\{N_i\}$*.*

(ii) $\bigcup_{i=1}^{w}(S_i, \mathfrak{X}, N_i)$ *is an* N^**-soft field over* $\mathcal{F}$*, where* $N^* = max_{i \in I}\{N_i\}$*.*

Proof It is straightforward (see Theorem 5.10). □

Theorem 5.27 *Let* $\mathcal{F}$ *be a field and* $(S_i, \mathfrak{X}_i, N_i)$ *be the* N_i*-soft fields over* $\mathcal{F}$ *for* $i \in I = \{1, 2, ..., w\}$*. Then,*

(i) $\bigwedge_{i=1}^{w}(S_i, \mathfrak{X}_i, N_i)$ *is an* N^**-soft field over* $\mathcal{F}$*, where* $N^* = min_{i\in I}\{N_i\}$*.*

(ii) $\bigvee_{i=1}^{w}(S_i, \mathfrak{X}_i, N_i)$ *is an* N^**-soft field over* $\mathcal{F}$*, where* $N^* = max_{i\in I}\{N_i\}$*.*

Proof Assume that $(S_i, \mathfrak{X}_i, N_i)$ is the N_i-soft fields over the field $\mathcal{F}$ for $i \in I = \{1, 2, ..., w\}$.
(i) We have to prove that for each $(x^i)_{i\in I} \in \prod_{i=1}^{w} \mathfrak{X}_i$,

$$\tau_x^{\bigwedge_{i=1}^{w} S_i}(f_k - f_l) \geq \tau_x^{\bigwedge_{i=1}^{w} S_i}(f_k) \wedge \tau_x^{\bigwedge_{i=1}^{w} S_i}(f_l) \text{ and } \tau_x^{\bigwedge_{i=1}^{w} S_i}(f_k {f_l}^{-1}) \geq \tau_x^{\bigwedge_{i=1}^{w} S_i}(f_k) \wedge \tau_x^{\bigwedge_{i=1}^{w} S_i}(f_l) \quad \forall f_k, f_l \in \mathcal{R}. \tag{5.16}$$

$$\begin{aligned} \tau_{(x^i)_{i\in I}}^{\bigwedge_{i=1}^{w} S_i}(f_k - f_l) &= \bigwedge_{i=1}^{w} \tau_{x^i}^{S_i}(f_k - f_l) \\ &\geq \bigwedge_{i=1}^{w} (\tau_{x^i}^{S_i}(f_k) \wedge \tau_{x^i}^{S_i}(f_l)) \\ &= (\bigwedge_{i=1}^{w} \tau_{x^i}^{S_i}(f_k)) \wedge (\bigwedge_{i=1}^{w} \tau_{x^i}^{S_i}(f_l)) \\ &= \tau_{(x^i)_{i\in I}}^{\bigwedge_{i=1}^{w} S_i}(f_k) \wedge \tau_{(x^i)_{i\in I}}^{\bigcap_{i=1}^{w} S_i}(f_l). \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} \tau_{(x^i)_{i\in I}}^{\bigwedge_{i=1}^{w} S_i}(f_k {f_l}^{-1}) &= \bigwedge_{i=1}^{w} \tau_{x^i}^{S_i}(f_k {f_l}^{-1}) \\ &\geq \bigwedge_{i=1}^{w} (\tau_{x^i}^{S_i}(f_k) \wedge \tau_{x^i}^{S_i}(f_l)) \\ &= (\bigwedge_{i=1}^{w} \tau_{x^i}^{S_i}(f_k)) \wedge (\bigwedge_{i=1}^{w} \tau_{x^i}^{S_i}(f_l)) \\ &= \tau_{(x^i)_{i\in I}}^{\bigwedge_{i=1}^{w} S_i}(f_k) \wedge \tau_{(x^i)_{i\in I}}^{\bigwedge_{i=1}^{w} S_i}(f_l). \end{aligned} \tag{5.18}$$

Also, from Definition 3.4, we have that $N^* = min_{i\in I}\{N_i\}$. Hence, $\bigwedge_{i=1}^{w}(S_i, \mathfrak{X}_i, N_i)$ is an N^*-soft field over $\mathcal{F}$.
(ii) It can be demonstrated similar to the proof of (i). □

Note: Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two fields. $\mathcal{F}_1 \times \mathcal{F}_2$ may not be a field. For instance, $\mathbb{Z}_3$ is a field but $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not a field.

6. N-soft lattice

In this section, we study the N-soft lattice and its relationship to the N-soft ideal.

Definition 6.1 *Let* $(\mathcal{L}, \curlyvee, \curlywedge)$ *be a lattice and* $(S, \mathfrak{X}, N)$ *be an* N*-soft set over* $\mathcal{L}$*. Then,* $(S, \mathfrak{X}, N)$ *is called an* N-soft lattice *over* $\mathcal{L}$ *iff for each* $x \in \mathfrak{X}$*,*

(i) $\tau_x^S(\ell_m \curlyvee \ell_n) \geq \tau_x^S(\ell_m) \wedge \tau_x^S(\ell_n) \quad \forall \ell_m, \ell_n \in \mathcal{L}$.

(ii) $\tau_x^S(\ell_m \curlywedge \ell_n) \geq \tau_x^S(\ell_m) \wedge \tau_x^S(\ell_n) \quad \forall \ell_m, \ell_n \in \mathcal{L}$.

Theorem 6.2 *Let $\mathcal{L}$ be a ring and $(S, \mathfrak{X}, N)$ be an N-soft set over $\mathcal{L}$. Then, $(S, \mathfrak{X}, N)$ is an N*-soft lattice *over $\mathcal{R}$ if for each $x \in \mathfrak{X}$,*

$$\tau_x^S(\ell_m \curlyvee \ell_n) \wedge \tau_x^S(\ell_m \curlywedge \ell_n) \geq \tau_x^S(\ell_m) \wedge \tau_x^S(\ell_m) \quad \forall \ell_m, \ell_n \in \mathcal{L}. \tag{6.1}$$

Proof It is obvious from Definitions 6.1. □

Example 6.3 *Let $\mathcal{L} = \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6\}$. Also, we consider the following tables for $\curlyvee$ and $\curlywedge$, respectively.*

$\curlyvee$	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6
ℓ_1	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6
ℓ_2	ℓ_2	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6
ℓ_3	ℓ_3	ℓ_3	ℓ_3	ℓ_5	ℓ_5	ℓ_6
ℓ_4	ℓ_4	ℓ_4	ℓ_5	ℓ_4	ℓ_5	ℓ_6
ℓ_5	ℓ_5	ℓ_5	ℓ_5	ℓ_5	ℓ_5	ℓ_6
ℓ_6	ℓ_6	ℓ_6	ℓ_6	ℓ_6	ℓ_6	ℓ_6

$\curlywedge$	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6
ℓ_1	ℓ_1	ℓ_1	ℓ_1	ℓ_1	ℓ_1	ℓ_1
ℓ_2	ℓ_1	ℓ_2	ℓ_2	ℓ_2	ℓ_2	ℓ_2
ℓ_3	ℓ_1	ℓ_2	ℓ_3	ℓ_2	ℓ_3	ℓ_3
ℓ_4	ℓ_1	ℓ_2	ℓ_2	ℓ_4	ℓ_4	ℓ_4
ℓ_5	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_5
ℓ_6	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6

It is clear that $(\mathcal{L}, \curlyvee, \curlywedge)$ is a lattice. For $\mathfrak{X}_1 = \{x_1^1, x_2^1\}$, we take the following 7-soft set over $\mathcal{L}$.

$$(S_1, \mathfrak{X}_1, N_1) = (S_1, \mathfrak{X}_1, 7) = \left\{ \begin{array}{l} (x_1^1, \{(\ell_1, 2), (\ell_2, 6), (\ell_3, 4), (\ell_4, 3), (\ell_5, 3), (\ell_6, 0)\}), \\ (x_2^1, \{(\ell_1, 1), (\ell_2, 5), (\ell_3, 4), (\ell_4, 2), (\ell_5, 2), (\ell_6, 1)\}) \end{array} \right\}.$$

Then, we say that $(S_1, \mathfrak{X}_1, 7)$ is a 7-soft lattice over $\mathcal{L}$.

Theorem 6.4 *Let $\mathcal{L}$ be a lattice and $(S_i, \mathfrak{X}, N_i)$ be the N_i-soft lattices over $\mathcal{L}$ for $i \in I = \{1, 2, ..., w\}$. Then,*

(i) $\bigcap_{i=1}^{w} (S_i, \mathfrak{X}, N_i)$ *is an N^*-soft lattice over $\mathcal{F}$, where $N^* = min_{i \in I}\{N_i\}$.*

(ii) $\bigcup_{i=1}^{w} (S_i, \mathfrak{X}, N_i)$ *is an N^*-soft lattice over $\mathcal{L}$, where $N^* = max_{i \in I}\{N_i\}$.*

Proof Assume that $(S_i, \mathfrak{X}, N_i)$ is the N_i-soft lattices over the lattice $\mathcal{L}$ for $i \in I = \{1, 2, ..., w\}$.
(i) To complete proof, we must demonstrate that for each $x \in \mathfrak{X}$,

$$\tau_x^{\bigcap_{i=1}^{w} S_i}(\ell_m \curlyvee \ell_n) \geq \tau_x^{\bigcap_{i=1}^{w} S_i}(\ell_m) \wedge \tau_x^{\bigcap_{i=1}^{w} S_i}(\ell_n) \text{ and } \tau_x^{\bigcap_{i=1}^{w} S_i}(\ell_m \curlywedge \ell_n) \geq \tau_x^{\bigcap_{i=1}^{w} S_i}(\ell_m) \wedge \tau_x^{\bigcap_{i=1}^{w} S_i}(\ell_n) \quad \forall \ell_m, \ell_n \in \mathcal{L}. \tag{6.2}$$

$$\begin{aligned} \tau_x^{\bigcap_{i=1}^{w} S_i}(\ell_m \curlyvee \ell_n) &= \bigwedge_{i=1}^{w} \tau_x^{S_i}(\ell_m \curlyvee \ell_n) \\ &\geq \bigwedge_{i=1}^{w} (\tau_x^{S_i}(\ell_m) \wedge \tau_x^{S_i}(\ell_n)) \\ &= (\bigwedge_{i=1}^{w} \tau_x^{S_i}(\ell_m)) \wedge (\bigwedge_{i=1}^{w} \tau_x^{S_i}(\ell_n)) \\ &= \tau_x^{\bigcap_{i=1}^{w} S_i}(\ell_m) \wedge \tau_x^{\bigcap_{i=1}^{w} S_i}(\ell_n) \end{aligned} \tag{6.3}$$

and

$$\tau_x^{\overset{w}{\underset{i=1}{\cap}} S_i}(\ell_m \curlywedge \ell_n) = \bigwedge_{i=1}^{w} \tau_x^{S_i}(\ell_m \curlywedge \ell_n)$$
$$\geq \bigwedge_{i=1}^{w}(\tau_x^{S_i}(\ell_m) \wedge \tau_x^{S_i}(\ell_n))$$
$$= (\bigwedge_{i=1}^{w} \tau_x^{S_i}(\ell_m)) \wedge (\bigwedge_{i=1}^{w} \tau_x^{S_i}(\ell_n))$$
$$= \tau_x^{\overset{w}{\underset{i=1}{\cap}} S_i}(\ell_m) \wedge \tau_x^{\overset{w}{\underset{i=1}{\cap}} S_i}(\ell_n) \quad (6.4)$$

Hence, $\overset{w}{\underset{i=1}{\cap}}(S_i, \mathfrak{X}, N_i)$ is an N^*-soft lattice over $\mathcal{L}$, where $N^* = min_{i\in I}\{N_i\}$.

(ii) It can be proved similar to the proof of (i). □

Theorem 6.5 *Let $\mathcal{L}$ be a ring and $(S_i, \mathfrak{X}_i, N_i)$ be the N_i-soft lattices over $\mathcal{L}$ for $i \in I = \{1, 2, ..., w\}$. Then,*

(i) $\bigwedge_{i=1}^{w}(S_i, \mathfrak{X}_i, N_i)$ *is an N^*-soft lattice over $\mathcal{L}$, where $N^* = min_{i\in I}\{N_i\}$.*

(ii) $\bigvee_{i=1}^{w}(S_i, \mathfrak{X}_i, N_i)$ *is an N^*-soft lattice over $\mathcal{L}$, where $N^* = max_{i\in I}\{N_i\}$.*

Proof It is similar to the proof of Theorem 6.4. □

Example 6.6 *Let us consider the 7-soft lattice $(S_1, \mathfrak{X}_1, N_1) = (S_1, \mathfrak{X}_1, 7)$ in Example 6.3. For $\mathfrak{X}_2 = \{x_1^2\}$, we consider the following 9-soft lattice over $\mathcal{L}$.*

$(S_2, \mathfrak{X}_2, N_2) = (S_2, \mathfrak{X}_2, 9) = \{(x_1^2, \{(\ell_1, 2), (\ell_2, 4), (\ell_3, 4), (\ell_4, 4), (\ell_5, 4), (\ell_6, 1)\})$.

Then, we obtain that $\bigwedge_{i=1}^{2}(S_i, \mathfrak{X}_i, N_i)$ is a 7-soft lattice over $\mathcal{L}$ and $\bigvee_{i=1}^{2}(S_i, \mathfrak{X}_i, N_i)$ is a 9-soft lattice over $\mathcal{L}$.

Definition 6.7 *Let $\mathcal{L}$ be a lattice and $(S, \mathfrak{X}, N)$ be an N-soft lattice over $\mathcal{L}$. Then, $(S, \mathfrak{X}, N)$ is an* N-soft ideal *over $\mathcal{L}$ if $\ell_m \leq \ell_n$ implies $\tau_x^S(\ell_m) \geq \tau_x^S(\ell_n)$ for each $x \in \mathfrak{X}$.*

Proposition 6.8 *Let $(S, \mathfrak{X}, N)$ be an N-soft set over the lattice $\mathcal{L}$. Then, the following are equivalent:*
For each $x \in \mathfrak{X}$ and $\ell_m, \ell_n \in \mathcal{L}$,

(1) $\tau_x^S(\ell_m) \geq \tau_x^S(\ell_n)$ *whenever* $\ell_m \leq \ell_n$.

(2) $\tau_x^S(\ell_m \curlyvee \ell_n) \leq \tau_x^S(\ell_m) \wedge \tau_x^S(\ell_n)$.

(3) $\tau_x^S(\ell_m \curlywedge \ell_n) \geq \tau_x^S(\ell_m) \vee \tau_x^S(\ell_n)$.

Proof Let $(S, \mathfrak{X}, N)$ be an N-soft set over the lattice $\mathcal{L}$.
(1) ⇔ (2) From assumption (1), $\ell_m \leq \ell_m \curlyvee \ell_n$ and $\ell_n \leq \ell_m \curlyvee \ell_n$ for $\ell_m, \ell_n \in \mathcal{L}$. Also, we can write

$$\tau_x^S(\ell_m) \geq \tau_x^S(\ell_m \curlyvee \ell_n) \text{ and } \tau_x^S(\ell_n) \geq \tau_x^S(\ell_m \curlyvee \ell_n) \tag{6.5}$$

and thus

$$\tau_x^S(\ell_m \curlyvee \ell_n) \leq \tau_x^S(\ell_m) \wedge \tau_x^S(\ell_n). \tag{6.6}$$

On the other side, if $\ell_m \leq \ell_n$ then we have $\ell_m \curlyvee \ell_n = \ell_n$ and so

$$\tau_x^S(\ell_n) = \tau_x^S(\ell_m \curlyvee \ell_n) \leq \tau_x^S(\ell_m) \wedge \tau_x^S(\ell_n). \tag{6.7}$$

Hence, $\tau_x^S(\ell_m) \geq \tau_x^S(\ell_n)$.
(1) ⇔ (3) From assumption (1), $\ell_m \geq \ell_m \curlywedge \ell_n$ and $\ell_n \geq \ell_m \curlywedge \ell_n$ for $\ell_m, \ell_n \in \mathcal{L}$. Also, we can write

$$\tau_x^S(\ell_m) \leq \tau_x^S(\ell_m \curlywedge \ell_n) \text{ and } \tau_x^S(\ell_n) \leq \tau_x^S(\ell_m \curlywedge \ell_n) \tag{6.8}$$

and thus

$$\tau_x^S(\ell_m \curlywedge \ell_n) \geq \tau_x^S(\ell_m) \vee \tau_x^S(\ell_n). \tag{6.9}$$

On the other side, if $\ell_m \leq \ell_n$ then we have $\ell_m \curlywedge \ell_n = \ell_m$ and so

$$\tau_x^S(\ell_m) = \tau_x^S(\ell_m \curlywedge \ell_n) \geq \tau_x^S(\ell_m) \vee \tau_x^S(\ell_n). \tag{6.10}$$

Hence, $\tau_x^S(\ell_m) \geq \tau_x^S(\ell_n)$. □

Corollary 6.9 *Let $\mathcal{L}$ be a lattice and $(S, \mathfrak{X}, N)$ be an N-soft lattice over $\mathcal{L}$. $(S, \mathfrak{X}, N)$ is an N-soft ideal over $\mathcal{L}$ if for each $x \in \mathfrak{X}$, $\tau_x^S(\ell_m \curlyvee \ell_n) = \tau_x^S(\ell_m) \wedge \tau_x^S(\ell_n)$ $\forall \ell_m, \ell_n \in \mathcal{L}$.*

7. Conclusion

In this study, we discussed the algebraic properties of N-soft sets in some algebraic structures. By generating new operations on the N-soft sets, we investigated several findings and results for these N-soft algebraic structures.
The operations and products presented in this study may be proposed for the application of N-soft sets in many areas containing uncertain data in the real world scene. Inspired by this work, the properties of N-soft sets in other algebraic structures can be developed. Therefore, it will allow new perspectives for future work on the algebraic properties of N-soft sets.

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