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New criteria for the oscillation and asymptotic behavior of second-order neutral differential equations with several delays

Başak KARPUZ[1,*], **Shyam S. SANTRA**[2]
[1]Department of Mathematics, Faculty of Science, Dokuz Eylül University, İzmir, Turkey
[2]Department of Mathematics, JIS College of Engineering, Kalyani, India

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Abstract: In this paper, necessary and sufficient conditions for asymptotic behavior are established of the solutions to second-order neutral delay differential equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-p(t)x(\tau(t))]\right)^{\gamma}\right)+\sum_{i=1}^{m}q_i(t)f_i\big(x(\sigma_i(t))\big)=0 \quad \text{for } t\geq t_0.$$

We consider two cases when $f_i(u)/u^{\beta}$ is nonincreasing for $\gamma>\beta$, and nondecreasing for $\beta>\gamma$, where β and γ are quotients of two positive odd integers. Our main tool is Lebesgue's dominated convergence theorem. Examples illustrating the applicability of the results are also given, and state an open problem.

Key words: Oscillation, nonoscillation, nonlinear, delay argument, second-order neutral differential equations, Lebesgue's dominated convergence theorem

1. Introduction

In this article, we consider the neutral differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-p(t)x(\tau(t))]\right)^{\gamma}\right)+\sum_{i=1}^{m}q_i(t)f_i\big(x(\sigma_i(t))\big)=0 \quad \text{for } t\geq t_0, \tag{1.1}$$

where γ is quotient of two positive odd integers, and the functions $f_i,p,q_i,r,\sigma_i,\tau$ are continuous that satisfy the conditions stated below:

(A1) $\tau,\sigma_i\in\mathrm{C}([t_0,\infty),[0,\infty))$ satisfy $\tau(t)\leq t$ and $\sigma_i(t)\leq t$ for $t\geq t_0$, $\lim_{t\to\infty}\tau(t)=\infty$ and $\lim_{t\to\infty}\sigma_i(t)=\infty$ for $i=1,2,\ldots,m$.

(A2) $r\in\mathrm{C}([t_0,\infty),(0,\infty))$, $q_i\in\mathrm{C}([t_0,\infty),[0,\infty))$ such that $\sum_{i=1}^{m}q_i\not\equiv 0$ on any interval of the form $[T,\infty)$.

(A3) $f_i\in\mathrm{C}(\mathbb{R},\mathbb{R})$ is nondecreasing and $uf_i(u)>0$ for $u\neq 0$, $i=1,2,\ldots,m$.

(A4) $\lim_{t\to\infty}R(t)=\infty$, where $R(t):=\int_{t_0}^{t}r^{-1/\gamma}(\eta)\mathrm{d}\eta$ for $t\geq t_0$.

*Correspondence: bkarpuz@gmail.com

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(A5) $p \in \mathrm{C}([t_0,\infty),[0,\infty))$ satisfies $0 \le p(t) \le p_0 < 1$ for all $t \ge t_0$, where $p_0 \in \mathbb{R}^+$.

The main feature of this article is having conditions that are both necessary and sufficient for the oscillation of all solutions to (1.1).

In 1978, Brands [10] has proved that for bounded delays, the solutions of

$$x''(t) + q(t)x(\sigma(t)) = 0 \quad \text{for } t \ge t_0,$$

where $t - M \le \sigma(t) \le t$ for some M, are oscillatory if and only if the solutions of $x''(t) + q(t)x(t) = 0$ are oscillatory. In [11, 13], Chatzarakis et al. have considered a more general second-order half-linear differential equation of the form

$$\big(r(x')^{\alpha}\big)'(t) + q(t)\big(x(\sigma(t))\big)^{\alpha} = 0 \quad \text{for } t \ge t_0, \tag{1.2}$$

and established new oscillation criteria for (1.2) in both of the cases $\lim_{t\to\infty} R(t) = \infty$ and $\lim_{t\to\infty} R(t) < \infty$.

Wong [35] has obtained the necessary and sufficient conditions for oscillation of solutions of

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}[x(t) - px(t-\tau)] + q(t)f\big(x(t-\sigma)\big) = 0 \quad \text{for } t \ge t_0$$

in which the neutral coefficient satisfies $p \in (0,1)$ and delays are constants. However, we have seen in [6, 14] that the authors Baculíková and Džurina have studied

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}[x(t) + p(t)x(\tau(t))]\right)^{\gamma}\right) + q(t)\big(x(\sigma(t))\big)^{\alpha} = 0 \quad \text{for } t \ge t_0 \tag{1.3}$$

and established sufficient conditions for oscillation of solutions of (1.3) using comparison techniques when $\gamma = \alpha = 1$, $0 \le p(t) < \infty$ and $\lim_{t\to\infty} R(t) = \infty$. By the same technique, Baculíková and Džurina [7] have considered (1.3) and obtained sufficient conditions for oscillation of the solutions of (1.3) by considering the assumptions $0 \le p(t) < \infty$ and $\lim_{t\to\infty} R(t) = \infty$. In [34], Tripathy et al. have studied (1.3) and established several sufficient conditions for oscillations of the solutions of (1.3) by considering the assumptions $\lim_{t\to\infty} R(t) = \infty$ and $\lim_{t\to\infty} R(t) < \infty$ for various ranges of the neutral coefficient p. In [9], Bohner et al. have obtained sufficient conditions for oscillation of solutions of (1.3) when $\gamma = \alpha$, $\lim_{t\to\infty} R(t) < \infty$ and $0 \le p(t) < 1$. Grace et al. [16] have established sufficient conditions for the oscillation of the solutions of (1.3) when $\gamma = \alpha$ and by considering the cases $\lim_{t\to\infty} R(t) < \infty$ and $\lim_{t\to\infty} R(t) = \infty$ when $0 \le p(t) < 1$. In [19], Li et al. have established sufficient conditions for the oscillation of the solutions of (1.3), under the assumptions $\lim_{t\to\infty} R(t) < \infty$ and $p(t) \ge 0$. Karpuz and Santra [18] have obtained several sufficient conditions for the oscillatory and asymptotic behavior of the solutions of

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r(t)\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - p(t)x(\tau(t))]\right) + q(t)f\big(x(\sigma(t))\big) = 0 \quad \text{for } t \ge t_0$$

by considering the cases $\lim_{t\to\infty} R(t) < \infty$ and $\lim_{t\to\infty} R(t) = \infty$ for different ranges of p.

For more information on oscillation of second-order neutral differential equations, we refer []

[1–5, 8, 12, 15, 16, 20–33, 36] to the reader and the references cited therein. Note that most of the works have been considered for sufficient conditions, and merely a few works have been concerned with the necessary and sufficient conditions. Hence, unlike the above methods, the main feature of this article is to establish conditions that are both necessary and sufficient for oscillation of all solutions of (1.1).

Neutral differential equations have several applications in the natural sciences and engineering. For example, they often appear in the study of distributed networks containing lossless transmission lines (see, e.g., [17]). In this paper, we restrict our attention to study oscillation and nonoscillation of (1.1), which includes the class of functional differential equations of neutral type.

By a solution of (1.1), we mean a function $x \in \mathrm{C}([T_x,\infty),\mathbb{R})$, where $T_x \geq t_0$, such that $x - p\cdot x\circ\tau \in \mathrm{C}^1([T_x,\infty),\mathbb{R})$ and $r\big([x-p\cdot x\circ\tau]'\big)^{\gamma} \in \mathrm{C}^1([T_x,\infty),\mathbb{R})$, and satisfies (1.1) on the interval $[T_x,\infty)$. A solution x of (1.1) is said to be proper if it is not identically zero eventually, i.e. $\sup\{|x(\eta)|:\ \eta \geq T\} > 0$ for all $T \geq T_x$. We assume that (1.1) possesses such solutions. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise, it is said to be nonoscillatory. Equation (1.1) itself is said to be oscillatory if all of its solutions are oscillatory.

When a domain is not specified explicitly for mathematical expressions, they are assumed to hold eventually, i.e. they are satisfied for all t large enough.

2. Results

Lemma 2.1 *Assume (A1)–(A5), and that x is an eventually positive solution of (1.1). Then, only one of the following two cases hold.*

(i) $\lim_{t\to\infty} x(t) = 0$.

(ii) There exist $T \geq t_0$ and $\delta > 0$ such that

$$0 < z(t) := x(t) - p(t)x(\tau(t)) \leq \delta R(t) \quad \textit{for all } t \geq T \tag{2.1}$$

and

$$\big(R(t)-R(T)\big)\left(\int_t^{\infty}\sum_{i=1}^{m} q_i(\eta) f_i\big(x(\sigma_i(\eta))\big)\mathrm{d}\eta\right)^{1/\gamma} \leq z(t) \leq x(t) \quad \textit{for all } t \geq T. \tag{2.2}$$

Proof Let x be an eventually positive solution. Then, by (A1) there exists a t_1 such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma_i(t)) > 0$ for all $t \geq t_1$ and $i = 1,2,\ldots,m$. Note that z defined in (2.1) is continuous and satisfies $z(t) \leq x(t)$ for $t \geq t_1$. From (1.1), it follows that

$$\big(r(z')^{\gamma}\big)'(t) = -\sum_{i=1}^{m} q_i(t) f_i\big(x(\sigma_i(t))\big) \leq 0 \quad \text{for } t \geq t_1. \tag{2.3}$$

Therefore, $r(z')^{\gamma}$ is nonincreasing on $[t_1,\infty)$. Next, we show that $r(z')^{\gamma}$ is positive on $[t_1,\infty)$. Assume the contrary that there exists $t_2 \geq t_1$ such that $r(t_2)\big(z'(t_2)\big)^{\gamma} \leq 0$. Using (A2) and (A3), it follows from (2.3) that there exists $t_3 \geq t_2$ such that

$$r(t)\big(z'(t)\big)^{\gamma} \leq r(t_3)\big(z'(t_3)\big)^{\gamma} < 0 \quad \text{for all } t \geq t_3.$$

Then,

$$z'(t) \leq \left(\frac{r(t_3)}{r(t)}\right)^{1/\gamma} z'(t_3) \quad \text{for } t \geq t_3.$$

Integrating from t_3 to t, we have

$$z(t) \leq z(t_3) + \big(r(t_3)\big)^{1/\gamma} z'(t_3)\big(R(t) - R(t_3)\big). \tag{2.4}$$

By (A4), the right-hand side tends to $(-\infty)$, then $\lim_{t\to\infty} z(t) = -\infty$. Since p is bounded and z is unbounded, x cannot be bounded. This allows the existence of an increasing unbounded sequence $\{\xi_k\}$ such that $x(\xi_k) = \sup\{x(\eta) : \eta \leq \xi_k\}$ for $k \in \mathbb{N}$. Then $x(\tau(\xi_k)) \leq x(\xi_k)$ and

$$z(\xi_k) = x(\xi_k) - p(\xi_k)x(\tau(\xi_k)) \geq \big(1 - p(\xi_k)\big)x(\xi_k) \geq (1 - p_0)x(\xi_k) \geq 0,$$

which contradicts $\lim_{k\to\infty} z(\xi_k) = -\infty$. Therefore, $r(z')^\gamma > 0$ on $[t_1, \infty)$.

From $r(z')^\gamma > 0$ and $r > 0$ on $[t_1, \infty)$, it follows that $z' > 0$ on $[t_1, \infty)$. Then, there is $t_2 \geq t_1$ such that only one of the following two cases happens.

Case 1. Let $z(t) < 0$ for all $t \geq t_2$. Note that by (A1), $\limsup_{t\to\infty} x(t) = \limsup_{t\to\infty} x(\tau(t))$. Then, $0 > z(t) \geq x(t) - p_0 x(\tau(t))$ for all $t \geq t_2$, which implies $0 \geq (1 - p_0)\limsup_{t\to\infty} x(t)$. Since $(1 - p_0) > 0$, it follows that $\limsup_{t\to\infty} x(t) = 0$, hence $\lim_{t\to\infty} x(t) = 0$.

Case 2. Let $z(t) > 0$ for all $t \geq t_2$. Note that $x(t) \geq z(t)$ for all $t \geq t_2$, and z is positive and increasing on $[t_2, \infty)$, so x cannot converge to zero. By nonincreasing nature of $r(z')^\gamma$, we have

$$z'(t) \leq \left(\frac{r(t_2)}{r(t)}\right)^{1/\gamma} z'(t_2) \quad \text{for } t \geq t_2.$$

Integrating the above inequality over $[t_2, t)$, we obtain

$$z(t) \leq z(t_2) + \big(r(t_2)\big)^{1/\gamma} z'(t_2)\big(R(t) - R(t_2)\big) \quad \text{for } t \geq t_2.$$

By (A4), there exists a constant $\delta > 0$ such that (2.1) holds.

Since $r(z')^\gamma$ is positive and nonincreasing on $[t_2, \infty)$, $\lim_{t\to\infty} r(t)\big(z'(t)\big)^\gamma$ exists and is nonnegative. Integrating (1.1) over $[t, s)$, we have

$$r(s)\big(z'(s)\big)^\gamma - r(t)\big(z'(t)\big)^\gamma + \int_t^s \sum_{i=1}^m q_i(\eta) f_i\big(x(\sigma_i(\eta))\big)\mathrm{d}\eta = 0 \quad \text{for all } s \geq t \geq t_2.$$

Dropping the positive term $r(s)\big(z'(s)\big)^\gamma$ and then letting $s \to \infty$ yields

$$r(t)\big(z'(t)\big)^\gamma \geq \int_t^\infty \sum_{i=1}^m q_i(\eta) f_i\big(x(\sigma_i(\eta))\big)\mathrm{d}\eta \quad \text{for all } t \geq t_2. \tag{2.5}$$

Then, we get

$$z'(t) \geq \left(\frac{1}{r(t)} \int_t^\infty \sum_{i=1}^m q_i(\eta) f_i\big(x(\sigma_i(\eta))\big)\mathrm{d}\eta\right)^{1/\gamma} \quad \text{for all } t \geq t_2.$$

Since $z(t_2) > 0$, integrating the above inequality over $[t_2, t)$ yields

$$z(t) \geq \int_{t_2}^t \left(\frac{1}{r(\eta)} \int_\eta^\infty \sum_{i=1}^m q_i(\zeta) f_i\big(x(\sigma_i(\zeta))\big)\mathrm{d}\zeta\right)^{1/\gamma} \mathrm{d}\eta \quad \text{for all } t \geq t_2.$$

Taking the inner integration out at its minimum value and using (A4), we arrive at

$$z(t) \geq \big(R(t) - R(t_2)\big)\left(\int_t^\infty \sum_{i=1}^m q_i(\zeta) f_i\big(x(\sigma_i(\zeta))\big)\mathrm{d}\zeta\right)^{1/\gamma} \quad \text{for all } t \geq t_2,$$

which yields (2.2).

This completes the proof. □

Remark 2.2 *Assume (A1)–(A5), and that x is an eventually positive unbounded solution of* (1.1)*. Then, (i) of Lemma 2.1 cannot hold.*

For the next theorem, we introduce a new additional condition.

(C1) There exists a constant $\beta > 0$, which is a quotient of two positive odd integers, with $\gamma > \beta$, such that

$$\frac{f_i(u)}{u^\beta} \text{ is non-increasing on } (-\infty, 0) \text{ and } (0, \infty),\ i = 1, 2, \ldots, m.$$

For example, $f_i(u) := |u|^{\alpha_i} \operatorname{sgn}(u)$, where $\beta > \alpha_i > 0$, satisfies (C1).

Theorem 2.3 *Assume (A1)–(A5) and (C1). Every solution of* (1.1) *is oscillatory or converges to zero, if and only if*

$$\int_{t_0}^\infty \sum_{i=1}^m q_i(\eta) f_i\big(\delta R(\sigma_i(\eta))\big)\mathrm{d}\eta = \infty \quad \textit{for all } \delta > 0. \tag{2.6}$$

Proof We prove sufficiency by contradiction. Initially, we assume that a solution x is eventually positive, which does not converge to zero. Then, Case 1 in Lemma 2.1 leads to $\lim_{t\to\infty} x(t) = 0$, which contradicts the assumption that x does not converge to zero. Next, we show that Case 2 of Lemma 2.1 also leads to a contradiction. In Case 2, there exists t_1 such that

$$x(t) \geq z(t) \geq \big(R(t) - R(t_1)\big) w^{1/\gamma}(t) \geq 0 \quad \text{for all } t \geq t_1,$$

where

$$w(t) := \int_t^\infty \sum_{i=1}^m q_i(\zeta) f_i\big(x(\sigma_i(\zeta))\big)\mathrm{d}\zeta \quad \text{for } t \geq t_1.$$

Since $\lim_{t\to\infty} R(t) = \infty$, there exists $t_2 \geq t_1$ such that $R(t) - R(t_1) \geq \frac{1}{2} R(t)$ for $t \geq t_2$. Then,

$$z(t) \geq \frac{1}{2} R(t) w^{1/\gamma}(t) \quad \text{for all } t \geq t_2. \tag{2.7}$$

Computing the derivative of w, we have

$$w'(t) = -\sum_{i=1}^m q_i(t) f_i\big(x(\sigma_i(t))\big) \quad \text{for all } t \geq t_2.$$

Thus, w is nonnegative and nonincreasing. Since $x > 0$, by (A3), $f_i \circ x \circ \sigma_i > 0$, and by (A2), it follows that $\sum_{i=1}^{m} q_i \cdot f_i \circ x \circ \sigma_i \not\equiv 0$ on any interval of the form $[T, \infty)$, thus w' cannot be identically zero, and w cannot be constant on any interval $[T, \infty)$. Therefore, $w(t) > 0$ for $t \geq t_1$. Computing the derivative, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} w^{1-\beta/\gamma}(t) = \left(1 - \frac{\beta}{\gamma}\right) w^{-\beta/\gamma}(t) w'(t) \quad \text{for } t \geq t_2. \tag{2.8}$$

Integrating (2.8) over $[t_2, t)$, and using positivity of w, we have

$$\begin{aligned} w^{1-\beta/\gamma}(t_2) &\geq \left(1 - \frac{\beta}{\gamma}\right)\left(-\int_{t_2}^{t} w^{-\beta/\gamma}(\eta) w'(\eta)\mathrm{d}\eta\right) \\ &= \left(1 - \frac{\beta}{\gamma}\right)\left(\int_{t_2}^{t} w^{-\beta/\gamma}(\eta) \sum_{i=1}^{m} q_i(\eta) f_i\big(x(\sigma_i(\eta))\big)\mathrm{d}\eta\right) \end{aligned} \tag{2.9}$$

for $t \geq t_2$. Next, we find a lower bound for the right-hand side of (2.9), independent of the solution x. Since $x \geq z$, by (A3), (C1), (2.1), and (2.7), we have

$$\begin{aligned} f_i\big(x(t)\big) &\geq f_i\big(z(t)\big) \frac{\big(z(t)\big)^{\beta}}{\big(z(t)\big)^{\beta}} \geq \frac{f_i\big(\delta R(t)\big)}{\big(\delta R(t)\big)^{\beta}} \big(z(t)\big)^{\beta} \geq \frac{f_i\big(\delta R(t)\big)}{\big(\delta R(t)\big)^{\beta}} \left(\frac{R(t) w^{1/\gamma}(t)}{2}\right)^{\beta} \\ &= \frac{f_i\big(\delta R(t)\big)}{(2\delta)^{\beta}} w^{\beta/\gamma}(t) \end{aligned}$$

for $t \geq t_2$. Since w is nonincreasing, $\beta/\gamma > 0$, and σ_i is a delay, it follows that

$$f_i\big(x(\sigma_i(t))\big) \geq \frac{f_i\big(\delta R(\sigma_i(t))\big)}{(2\delta)^{\beta}} w^{\beta/\gamma}(\sigma_i(t)) \geq \frac{f_i\big(\delta R(\sigma_i(t))\big)}{(2\delta)^{\beta}} w^{\beta/\gamma}(t) \quad \text{for } t \geq t_2. \tag{2.10}$$

Going back to (2.9), we have

$$w^{1-\beta/\gamma}(t_2) \geq \frac{\left(1 - \frac{\beta}{\gamma}\right)}{(2\delta)^{\beta}} \int_{t_2}^{t} \sum_{i=1}^{m} q_i(\eta) f_i\big(\delta R(\sigma_i(\eta))\big)\mathrm{d}\eta \quad \text{for } t \geq t_2. \tag{2.11}$$

Since $(1 - \beta/\gamma) > 0$, by (2.6) the right-hand side tends to ∞ as $t \to \infty$. This contradicts (2.11) and completes the proof of sufficiency for eventually positive solutions.

For an eventually negative solution x, we introduce the variables $y := -x$ and $g_i(u) := -f_i(-u)$. Then, y is an eventually positive solution of (1.1) with g_i instead of f_i. Note that g_i satisfies (A3) and (C1) so can apply the above process for the solution y.

Next, we show the necessity part by a contrapositive argument. When (2.6) does not hold we find an eventually positive solution that does not converge to zero. If (2.6) does not hold for some $\delta > 0$, then for every $\varepsilon > 0$, there exists $t_1 \geq t_0$ such that

$$\int_{t}^{\infty} \sum_{i=1}^{m} q_i(\zeta) f_i\big(\delta R(\sigma_i(\zeta))\big)\mathrm{d}\zeta \leq \varepsilon \quad \text{for all } t \geq t_1. \tag{2.12}$$

We can pick $\alpha > 0$ such that $(1-p_0)\delta > \alpha$, which yields $\delta > \alpha$. Now, let (2.12) hold with $\varepsilon := (1-p_0)^\gamma \delta^\gamma - \alpha^\gamma > 0$. Define the set of continuous functions

$$M := \{x \in \mathrm{C}([t_0, \infty), [0, \infty)) : \ \alpha\psi(t) \le x(t) \le \delta\psi(t) \quad \text{for all } t \ge t_0\},$$

where

$$\psi(t) := \begin{cases} 0, & t_1 \ge t \ge t_0 \\ \displaystyle\int_{t_1}^{t} \frac{1}{\big(r(\eta)\big)^{1/\gamma}} \mathrm{d}\eta, & t \ge t_1. \end{cases}$$

Then, we define an operator Φ on M by

$$(\Phi x)(t) := \begin{cases} 0, & t_1 \ge t \ge t_0 \\ p(t)x(\tau(t)) + \displaystyle\int_{t_1}^{t} \left(\frac{1}{r(\eta)} \left(\alpha^\gamma + \int_{\eta}^{\infty} \sum_{i=1}^{m} q_i(\zeta) f_i\big(x(\sigma_i(\zeta))\big) \mathrm{d}\zeta \right) \right)^{1/\gamma} \mathrm{d}\eta, & t \ge t_1. \end{cases}$$

Note that when x is continuous, Φx is also continuous on $[t_0, \infty)$. If x is a fixed point of Φ, i.e. $\Phi x = x$, then x is a solution of (1.1).

First, we estimate a lower bound for Φx. By (A3), we have $f_i \circ x \circ \sigma_i \ge 0$ and by (A2), we have

$$(\Phi x)(t) \ge 0 + \int_{t_1}^{t} \left(\frac{1}{r(\eta)} (\alpha^\gamma + 0) \right)^{1/\gamma} \mathrm{d}\eta = \alpha\psi(t) \quad \text{for } t \ge t_1.$$

Now, we estimate an upper bound for Φx. For $x \in M$, by (A2) and (A3), we have $f_i \circ x \circ \sigma_i \le f_i \circ (\delta R) \circ \sigma_i$. Then, by (2.12), we get

$$\begin{aligned} (\Phi x)(t) &\le p_0 \delta \psi(t) + \int_{t_1}^{t} \left(\frac{1}{r(\eta)} \left(\alpha^\gamma + \int_{\eta}^{\infty} \sum_{i=1}^{m} q_i(\zeta) f_i\big(\delta R(\sigma_i(\zeta))\big) \mathrm{d}\zeta \right) \right)^{1/\gamma} \mathrm{d}\eta \\ &\le p_0 \delta \psi(t) + (\alpha^\gamma + \varepsilon)^{1/\gamma} \psi(t) = \delta\psi(t) \end{aligned}$$

for $t \ge t_1$. Therefore, Φ maps M into M.

Next, we find a fixed point for Φ in M. Let us define a sequence of functions in M by the recurrence relation

$$u_n(t) := \begin{cases} 0, & n = 0 \\ (\Phi u_{n-1})(t), & n \in \mathbb{N} \end{cases} \quad \text{for } t \ge t_0.$$

Note that we have $u_1(t) \ge u_0(t)$ for $t \ge t_0$. Using that f_i is nondecreasing and mathematical induction, we can show that $u_{n+1}(t) \ge u_n(t)$ for $t \ge t_0$. Therefore, the sequence $\{u_n\}$ converges pointwise to a function u. Using Lebesgue's dominated convergence theorem, we can show that u is a fixed point of Φ in M. This shows that under assumption (2.12), there is a nonoscillatory solution that does not converge to zero. This completes the proof. $\square$

Corollary 2.4 *Under the assumptions of Theorem 2.3, every unbounded solution of* (1.1) *is oscillatory if and only if* (2.6) *holds.*

Proof The proof of the corollary directly follows from Remark 2.2 and Theorem 2.3. Hence, the details are omitted. □

For the next theorem, we introduce two new additional conditions.

(C2) Assume the existence of a differentiable function σ_0 and a positive constant κ such that

$$\sigma_i(t) \geq \sigma_0(t) \text{ and } \sigma_0'(t) \geq \kappa \quad \text{for } t \geq t_0 \text{ and } i = 1, 2, \dots, m.$$

(C3) There exists a constant $\beta > 0$, which is a quotient of two positive odd integers, with $\beta > \gamma$, such that

$$\frac{f_i(u)}{u^\beta} \text{ is non-decreasing on } (-\infty, 0) \text{ and } (0, \infty),\ i = 1, 2, \dots, m.$$

For example, $f_i(u) := |u|^{\alpha_i} \operatorname{sgn}(u)$, where $\alpha_i > \beta > 0$, satisfies (C3).

Theorem 2.5 *Assume (A1)–(A5), (C2), (C3), and let r be differentiable and nondecreasing. Every solution of (1.1) is oscillatory or converges to zero, if and only if*

$$\int_{t_1}^{\infty} \left(\frac{1}{r(\eta)} \int_{\eta}^{\infty} \sum_{i=1}^{m} q_i(\zeta) \mathrm{d}\zeta \right)^{1/\gamma} \mathrm{d}\eta = \infty. \tag{2.13}$$

Proof We prove sufficiency by contradiction. Initially, assume that x is an eventually positive solution that does not converge to zero. Using the same argument as in Lemma 2.1, there exists $t_1 \geq t_0$ such that $x(\sigma_i(t)) > 0$, $x(\tau(t)) > 0$, and $r(z')^\gamma$ is positive and nonincreasing. Case 1 of Lemma 2.1 leads to $\lim_{t\to\infty} x(t) = 0$, which contradicts the assumption that x does not converge to zero.

Case 2 of Lemma 2.1 also leads to a contradiction. In Case 2, $z(t)$ is positive and increasing for $t \geq t_1$. It follows from (A5) and (2.1) that $z(t) \leq x(t)$ for $t \geq t_1$. From (A3), $z(t) \geq z(t_1)$ and (C3), we have

$$f_i(x(t)) \geq \frac{f_i(z(t))}{\big(z(t)\big)^\beta} \big(z(t)\big)^\beta \geq \frac{f_i(z(t_1))}{\big(z(t_1)\big)^\beta} \big(z(t)\big)^\beta \quad \text{for all } t \geq t_1.$$

By (A1), there exists a $t_2 \geq t_1$ such that $\sigma_i(t) \geq t_1$ for $t \geq t_2$. Then,

$$f_i\big(x(\sigma_i(t))\big) \geq \frac{f_i(z(t_1))}{\big(z(t_1)\big)^\beta} \big(z(\sigma_i(t))\big)^\beta \quad \text{for all } t \geq t_2. \tag{2.14}$$

Using (2.14), $\sigma_i \geq \sigma_0$, which is an increasing function, and that z is increasing, it follows from (2.5) that

$$r(t)\big(z'(t)\big)^\gamma \geq \frac{z^\beta(\sigma_0(t))}{\big(z(t_1)\big)^\beta} \int_t^\infty \sum_{i=1}^{m} q_i(\eta) f_i(z(t_1)) \mathrm{d}\eta \quad \text{for all } t \geq t_2.$$

From $r(z')^\gamma$ being nonincreasing and σ_0 being a delay, we have

$$r(\sigma_0(t))\big(z'(\sigma_0(t))\big)^\gamma \geq r(t)\big(z'(t)\big)^\gamma \quad \text{for all } t \geq t_2.$$

We use this in the left-hand side of the above inequality. Then, dividing by $r(\sigma_0(t)) > 0$, raising both sides to the power of $1/\gamma$, and dividing by $z^{\beta/\gamma}(\sigma_0(t)) > 0$, we have

$$\frac{z'(\sigma_0(t))}{\big(z(\sigma_0(t))\big)^{\beta/\gamma}} \geq \left(\frac{1}{r(\sigma_0(t))\big(z(t_1)\big)^{\beta}}\int_t^\infty \sum_{i=1}^m q_i(\eta) f_i(z(t_1))\mathrm{d}\eta\right)^{1/\gamma} \quad \text{for all } t \geq t_2.$$

Multiplying the left-hand side by $\sigma_0'(t)/\kappa \geq 1$, and integrating over $[t_2, t)$, we get

$$\frac{1}{\kappa}\int_{t_2}^t \frac{z'(\sigma_0(\eta))\sigma_0'(\eta)}{z^{\beta/\gamma}(\sigma_0(\eta))}\mathrm{d}\eta \geq \frac{1}{z^{\beta/\gamma}(t_2)}\int_{t_2}^t \left(\frac{1}{r(\sigma_0(\eta))}\int_\eta^\infty \sum_{i=1}^m q_i(\zeta) f_i(z(t_1))\mathrm{d}\zeta\right)^{1/\gamma}\mathrm{d}\eta \quad \text{for all } t \geq t_2. \tag{2.15}$$

On the left-hand side, since $\beta > \gamma$, integrating gives us

$$\frac{1}{\kappa(1-\beta/\gamma)}\big(z(\sigma_0(\eta))\big)^{1-\beta/\gamma}\Big|_{\eta=t_2}^{t} \leq \frac{1}{\kappa(\beta/\gamma-1)}\big(z(\sigma_0(t_2))\big)^{1-\beta/\gamma} \quad \text{for all } t \geq t_2.$$

On the right-hand side of (2.15), we use that $\min_{1\leq i\leq m} f_i(z(t_1)) > 0$ and that $r \circ \sigma_0 \leq r$, to conclude that (2.13) implies the right-hand side approaching ∞, as $t \to \infty$. This contradiction implies that the solution x cannot be eventually positive.

For eventually negative solutions, we use the same change of variables as in the proof of Theorem 2.3, and proceed as above.

To prove the necessity part, we assume that (2.13) does not hold, and obtain an eventually positive solution that does not converge to zero. If (2.13) does not hold, then for each $\varepsilon > 0$ there exists $t_1 \geq t_0$ such that

$$\int_t^\infty \left(\frac{1}{r(\eta)}\int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\mathrm{d}\zeta\right)^{1/\gamma}\mathrm{d}\eta \leq \varepsilon \quad \text{for all } t \geq t_1.$$

Pick $\delta, \alpha > 0$ such that $(1-p_0)\delta > \alpha > 0$. For $\varepsilon := (1-p_0)\delta - \alpha > 0$, we can find t_1 such that

$$\int_t^\infty \left(\frac{1}{r(\eta)}\int_\eta^\infty \sum_{i=1}^m q_i(\zeta) f_i(\delta)\mathrm{d}\zeta\right)^{1/\gamma}\mathrm{d}\eta \leq \varepsilon \quad \text{for all } t \geq t_1. \tag{2.16}$$

Let us consider the set of continuous functions

$$M := \{x \in \mathrm{C}([t_0,\infty),[0,\infty)) : \ \alpha \leq x(t) \leq \delta \text{ for } t \geq t_0\}.$$

Then, we define the operator

$$(\Phi x)(t) := \begin{cases} \alpha + p(t_1)x(\tau(t_1)), & t_1 \geq t \geq t_0, \\ \alpha + p(t)x(\tau(t)) + \displaystyle\int_{t_1}^t \frac{1}{r(\eta)}\left(\int_\eta^\infty \sum_{i=1}^m q_i(\zeta) f_i\big(x(\sigma_i(\zeta))\big)\mathrm{d}\zeta\right)^{1/\gamma}\mathrm{d}\eta, & t \geq t_1. \end{cases}$$

Note that if x is continuous, Φx is also continuous at t_1. This follows by taking the right and left limits in the three possible cases in the definition of Φ. Also note that if $\Phi x = x$, then x is solution of (1.1).

First, we estimate a lower bound for Φx. Let $x \in M$. Then $x \geq \alpha$ and by (A3), we have $(\Phi x)(t) \geq \alpha$ for $t \geq t_1$.

Now, we estimate an upper bound for Φx. Let $x \in M$, then $x \leq \delta$ and

$$(\Phi x)(t) \leq \alpha + p_0\delta + \int_{t_1}^{t}\left(\frac{1}{r(\eta)}\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta) f_i(\delta)\mathrm{d}\zeta\right)^{1/\gamma}\mathrm{d}\eta \quad \text{for } t \geq t_1.$$

Since σ_0 is a delay and r is nondecreasing, we can replace r by $r \circ \sigma_0$ and the above inequality is still valid. By (2.16) and the definition of ε, we have

$$(\Phi x)(t) \leq \alpha + p_0\delta + \varepsilon = \delta \quad \text{for } t \geq t_1.$$

Therefore, Φ maps M to M.

To find a fixed point for Φ in M, we define a sequence of functions by the recurrence relation

$$\begin{aligned} u_0(t) &= 0 \quad \text{for } t \geq t_1, \\ u_1(t) &= (\Phi u_0)(t) = 1 \quad \text{for } t \geq t_1, \\ u_{n+1}(t) &= (\Phi u_n)(t) \quad \text{for } n \geq 1,\ t \geq t_1. \end{aligned}$$

Note that we have $u_1 \geq u_0$ on $[t_1, \infty)$. Using that f is nondecreasing and mathematical induction, we can prove that $u_{n+1} \geq u_n$ on $[t_1, \infty)$. Therefore, $\{u_n\}$ converges pointwise to a function u in M. Then, u is a fixed point of Φ and a positive solution to (1.1) that does not converge to zero. □

Corollary 2.6 *Under the assumptions of Theorem 2.5, every unbounded solution of (1.1) is oscillatory if and only if (2.13) holds.*

Proof The proof of the corollary directly follows from Remark 2.2 and Theorem 2.5. Hence, the details are omitted. □

The next theorem requires neither (C1) nor (C3) but considers only bounded solutions.

Theorem 2.7 *Under assumptions (A1)–(A5), every bounded solution of (1.1) is oscillatory or converges to zero if and only if (2.13) holds.*

Proof We prove sufficiency by contradiction. Assume x is an eventually positive solution that does not converge to zero. Then, we proceed as in Lemma 2.1 up to equation (2.4). Since x and p are bounded so z is bounded. Then, the left-hand side of (2.4) is bounded, while the right-hand side approaches $(-\infty)$ as $t \to \infty$. This contradiction implies that $z'(t) > 0$ for $t \geq t_1$. As in Lemma 2.1, we have two possible cases.

Case 1. $z(t) < 0$ for all $t \geq t_1$. This leads to a contradiction. As in Case 1 of Lemma 2.1, we have $\lim_{t\to\infty} x(t) = 0$, which contradicts the assumption that x does not converge to zero.

Case 2. $z(t) > 0$ for all $t \geq t_1$. This also leads to a contradiction. Since z is positive and increasing, $z(t) \geq z(t_1)$ for $t \geq t_1$. Recall that $x \geq z$ so x cannot converge to zero. By (A2), there is a $t_2 \geq t_1$ such that $\sigma_i(t) \geq t_1$ and $x(\sigma_i(t)) \geq z(t_1)$ for $t \geq t_2$ and $i = 1, 2, \ldots, m$. From (A4), $f \circ x \circ \sigma \geq f\big(z(t_1)\big) > 0$. Then, integrating as we did for (2.5), we have

$$z(t) \geq z(t_2) + \int_{t_2}^{t} \left(\frac{1}{r(\eta)} \int_{\eta}^{\infty} \sum_{i=1}^{m} q_i(\zeta) f_i\big(z(t_1)\big) \mathrm{d}\zeta \right)^{1/\gamma} \mathrm{d}\eta \quad \text{for } t \geq t_2,$$

which shows that $z(t) \to \infty$ as $t \to \infty$. This contradicts the fact that z is bounded.

For eventually negative solutions, we proceed as above to obtain also a contradiction. Therefore, every bounded solution must be oscillatory or converge to zero.

The necessity part of the proof follows from that of Theorem 2.5. Thus, the proof is complete. □

Corollary 2.8 *Under the assumptions of Theorem 2.7, every unbounded solution of* (1.1) *is oscillatory if and only if* (2.13) *holds.*

Proof The proof of the corollary directly follows from Remark 2.2 and Theorem 2.7. Hence, the details are omitted. □

Example 2.9 *Consider the neutral differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-t} \left(\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - \mathrm{e}^{-t} x(\tau(t))] \right)^{11/3} \right) + \frac{1}{t+1}\big(x(t-2)\big)^{1/3} + \frac{1}{t+2}\big(x(t-1)\big)^{5/3} = 0, \quad t \geq 1. \tag{2.17}$$

Here, $\gamma := 11/3$, $r(t) := \mathrm{e}^{-t}$, $0 \leq p(t) := \mathrm{e}^{-t} \leq 1/\mathrm{e} < 1$, $\sigma_1(t) := t-2$, $\sigma_2(t) := t-1$, $R(t) := \int_1^t \mathrm{e}^{3\eta/11} \mathrm{d}\eta = \frac{11}{3}\big(\mathrm{e}^{3t/11} - \mathrm{e}^{3/11}\big)$ *for* $t \geq 1$, $f_1(u) := u^{1/3}$ *and* $f_2(u) := u^{5/3}$ *for* $u \in \mathbb{R}$. *With* $\beta := 7/3$, *we see that (C1) holds, i.e.* $f_1(u)/u^{\beta} = u^{-2}$ *and* $f_2(u)/u^{\beta} = u^{-2/3}$, *both of which are decreasing functions. To check* (2.6), *we compute*

$$\begin{aligned} \int_1^{\infty} \sum_{i=1}^{m} q_i(\eta) f_i\big(\delta R(\sigma_i(\eta))\big) \mathrm{d}\eta &\geq \int_1^{\infty} \sum_{i=1}^{m} q_i(\eta) f_i\big(\delta R(\sigma_i(\eta))\big) \mathrm{d}\eta \\ &\geq \int_1^{\infty} q_1(\eta) f_1\big(\delta R(\sigma_1(\eta))\big) \mathrm{d}\eta \\ &= \int_1^{\infty} \frac{1}{\eta+1} \left(\delta \frac{11}{3} \big(\mathrm{e}^{3(\eta-2)/11} - \mathrm{e}^{3/11}\big) \right)^{1/3} \mathrm{d}\eta = \infty \end{aligned}$$

for all $\delta > 0$. *So, all the conditions of Theorem 2.3 hold, and therefore, each solution of* (2.17) *is oscillatory or converges to zero.*

Example 2.10 *Consider the neutral differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - \mathrm{e}^{-t} x(\tau(t))] \right)^{1/3} \right) + t\big(x(t-2)\big)^{7/3} + (t+1)\big(x(t-1)\big)^{11/3} = 0, \quad t \geq 1. \tag{2.18}$$

Here, $\gamma := 1/3$, $r(t) :\equiv 1$, $\sigma_1(t) := t-2$, $\sigma_2(t) := t-1$ *for* $t \geq 1$, $f_1(u) := u^{7/3}$ *and* $f_2(u) := u^{11/3}$ *for* $u \in \mathbb{R}$. *With* $\beta := 5/3$, *we see that (C3) holds, i.e.* $f_1(u)/u^{\beta} = u^{2/3}$ *and* $f_2(u)/u^{\beta} = u^2$ *both of which are increasing* *functions. To check* (2.13)*, we compute*

$$\int_0^\infty \left(\frac{1}{r(\eta)}\int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\mathrm{d}\zeta\right)^{1/\gamma}\mathrm{d}\eta \geq \int_0^\infty \left(\frac{1}{r(\eta)}\int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\mathrm{d}\zeta\right)^{1/\gamma}\mathrm{d}\eta$$
$$\geq \int_0^\infty \left(\frac{1}{r(\eta)}\int_\eta^\infty q_1(\zeta)\mathrm{d}\zeta\right)^{1/\gamma}\mathrm{d}\eta$$
$$\geq \int_0^\infty \left(\int_\eta^\infty \zeta\mathrm{d}\zeta\right)^{3}\mathrm{d}\eta = \infty.$$

So, all the conditions of Theorem 2.5 *hold. Thus, every solution of* (2.18) *is oscillatory or converges to zero.*

3. Final Comments

Based on this work and [6, 7, 9, 14, 16, 18–20, 31, 34] it would be interesting to fill the gap in establishing necessary and sufficient conditions for the oscillation of the solutions of the second-order nonlinear neutral differential equation (1.1) under the conditions $p \leq 0$ and/or $p \geq 1$.

We would like to mention that the papers [1, 23] are concerned with the oscillation of neutral differential equations in the case where $p \geq 1$ and $p \not\equiv 1$ eventually, whereas [25] is concerned with the oscillation of neutral differential equations in the case where $p \equiv p_0 \geq 0$ and $p_0 \neq 1$, which suggest a possible/useful technique for studying the oscillation problem here. Furthermore, new criteria presented in this paper complement and improve related results obtained in [1, 23, 25].

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