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Existence results for ψ–Caputo fractional neutral functional integro-differential equations with finite delay

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Abstract: This research article deals with novel two species of initial value problems, one of them, the fractional neutral functional integrodifferential equations, and the other one, the coupled system of fractional neutral functional integrodifferential equations, with finite delay and involving a ψ–Caputo fractional operator. The existence and uniqueness results are studied through Banach's contraction principle and Krasnoselskii's fixed point theorem. We also establish two various kinds of Ulam stability results for the proposed problems. Further, two pertinent examples are presented to demonstrate the reported results.

Key words: Delay fractional differential equations, ψ–Caputo fractional derivative, existence and stability theory, coupled system, neutral, fixed point theorem

1. Introduction

Fractional order derivatives (FODs) [34] grant catching the memory impacts because of its nonlocal nature, therefore, modeling of genuine dynamics having memory impact is more reasonable with FDOs than the traditional integer order derivatives (IODs). Numerous FODs can be found in the literature with the point that various types of real-world phenomena can be demonstrated appropriately (see [18, 19, 27, 37, 40, 42, 43] and the references therein.

Besides, the stated FOD must maintain the classical properties of IODs. Among the current classical FODs, most generally utilized FODs, are Riemann–Liouville and Caputo derivatives. However, as a rule, FODs with the singular kernel cannot characterize the nonlocality of many real-world dynamics. Along these lines, new FODs with nonsingular kernel have been introduced, namely, fractional Caputo–Fabrizio exponential derivative [24] and fractional Atangana–Baleanu–Caputo Mittag–Leffler derivative [17].

Neutral functional differential equations (NFDEs) emerge in the mathematical modelling of biological, physical and engineering problems, see, for instance, the texts [26, 28, 32, 46] and the references referred to in that. Initial value problems (IVPs) for fractional NFDEs with finite delay is a subject of high interest. Various investigators have elaborated various techniques and methods for examining some qualitative results to such equations involving classical FODs, see [1, 3–5, 8–10, 20, 38, 41, 46].

The Ulam–Hyers (HU) stability notion has been taken into consideration in the several pieces of literature.

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The announced stability analysis is a simple manner in this consideration. This specie of stability was done formulated for the primary time by Ulam [44], then it was evolved with the aid of Hyers [29] and Urs [45]. A series of papers have been devoted to the investigation of UH-type stability of solutions of the FDEs with various species of FODs.

On the other side, ψ-fractional operators were introduced by Kilbas [34] as a generalization of Riemann–Liouville operators. These fractional operators are different from the other classical fractional operator due to the kernel appears in terms of another function ψ. Several generalized FODs and their applications were introduced by Agarwal [7].

Recently, Almeida in [12] introduced a version generalized of Caputo FOD with some interesting properties. Several properties of this operator could be found in [12–15]. For some particular cases of ψ, one can realize that ψ-Caputo FOD can be reduced to the (Caputo [34], Caputo–Hadamard [30], Caputo–Katugampola [31], Caputo–Erde'lyi–Kober [35]) FODs.

Motivated by novel developments in ψ-fractional calculus, in the present research, we investigate the existence and uniqueness of the solutions and UH-type stability of the fractional neutral functional integrodifferential equations (FNFIEs) with finite delay described by

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\nu;\psi}\left[z(\tau)-\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k(\tau,z_\tau)\right]=\mathbb{H}(\tau,z_\tau), \tau\in \mathrm{J}:=[a,b],\\ z(\tau)=\alpha(\tau), \tau\in[a-\delta,a], \end{cases} \tag{1.1}$$

and the coupled system of FNFIEs formulated by

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\nu;\psi}\left[\omega(\tau)-\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k^*(\tau,\omega_\tau,\varpi_\tau)\right]=\mathbb{H}_1(\tau,\omega_\tau,\varpi_\tau),\\ {}^{c}\mathbb{D}_{a^+}^{\varsigma;\psi}\left[\varpi(\tau)-\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\xi_k;\psi}\mathbb{G}_k(\tau,\omega_\tau,\varpi_\tau)\right]=\mathbb{H}_2(\tau,\omega_\tau,\varpi_\tau), \end{cases} \quad \tau\in\mathrm{J}, \tag{1.2}$$

with the coupled finite delay

$$\begin{cases} \omega(\tau)=\varphi_1(\tau),\\ \varpi(\tau)=\varphi_2(\tau). \end{cases} \quad \tau\in[a-\delta,a], \tag{1.3}$$

where ${}^{c}\mathbb{D}_{a^+}^{\beta;\psi}$ is the ψ-Caputo FOD of order $\beta\in\{\nu,\varsigma\}\in(0,1]$, $\mathbb{I}_{a^+}^{\theta;\psi}$ is the ψ-RL fractional integral of order $\theta>0$, $\theta\in\{\sigma_1,\sigma_2,\dots,\sigma_m,\xi_1,\xi_2,...,\xi_m\}$, $\sigma_k,\xi_k>0, k=1,2,\dots,m$, $\mathbb{F}_k,\mathbb{H}:\mathrm{J}\times\mathcal{C}_\delta\longrightarrow\mathbb{R}$ $(k=1,2,\dots,m)$ are given continuous functions such that $\mathbb{F}_k(a,z_a)=0$, where $\mathcal{C}_\delta=C([-\delta,0],\mathbb{R})$, $\mathbb{F}_k^*,\mathbb{G}_k,\mathbb{H}_1,\mathbb{H}_2:\mathrm{J}\times\mathcal{C}_\delta\times\mathcal{C}_\delta\longrightarrow\mathbb{R}$ are given continuous functions, such that $\mathbb{F}_k^*(a,\omega_a,\varpi_a)=0, \mathbb{G}(a,\omega_a,\varpi_a)=0$, and $\alpha,\varphi_1,\varphi_2:[a-\delta,a]\longrightarrow\mathbb{R}$ is a continuous functions with $\alpha(a)=0$, $\varphi_1(a)=0$ and $\varphi_2(a)=0$. For any function u defined on $[a-\delta,a]$ and any $\tau\in\mathrm{J}$, we denote by u_τ the element of $\mathcal{C}_\delta$ defined by

$$u_\tau(\rho)=u(\tau+\rho), \quad \rho\in[-\delta,0].$$

Remark 1.1

- *Problem (1.1) and coupled system (1.2)–(1.3) are formulated in an overall form that combines both integrodifferential problems and neutral functional equations associated with finite delay involving generalized FOD. In fact, the choices of $\psi(t)\equiv t$, $\log t$, and t^ρ $(\rho>0)$, reduce problem (1.1) and coupled system (1.2)–(1.3) into corresponding problems involving standard Caputo type, Caputo–Hadamard type, and Caputo–Katugumpola type, respectively.*

- *On problem (1.1), $\sigma_k = 0$, for $k = 1, ..., m$, and $\psi(t) \equiv t$ incorporates the results of Agarwal et al. [8] for fractional neutral FDE.*

- *On problem (1.1), $a = 1$, $\psi(t) \equiv \log t$ incorporates the results of Abbas [1] for fractional neutral functional integrodifferential equations.*

- *On problem (1.1), $a = t_0$, $\mathbb{F}_k(\tau, \omega_\tau, \varpi_\tau) = 0$, for $k = 1, ..., m$, and $\mathbb{H}_1(\tau, \omega_\tau, \varpi_\tau) = f(\tau, \omega, \omega(\tau - \delta)$ incorporates the results of Ameen et al. [16] for fractional delay FDE.*

According to the above remark, our proposed problems cover many of the corresponding problems in the literature, which are considered special cases.

Here is a brief outline of the work. Section 2 provides the definitions and preliminary results that we will require to prove our essential results. Moreover, we give an auxiliary lemma that supplies solution representation for the solutions of the problem (1.1) and system (1.2)–(1.3). In Section 3, we establish existence, uniqueness and stability in the sense of Ulam for the proposed problems. In Section 4, we provide two examples to demonstrate the acquired results.

2. Auxiliary results

In this section, we recall some notations, definitions, and preliminary results of advanced fractional calculus and nonlinear analysis needed in the proofs later.

Consider the space of real and continuous functions $\mathcal{C} = C(J, \mathbb{R})$ space with the norm

$$\|\omega\|_{\mathcal{C}} = \sup_{\tau \in \mathrm{J}} |\omega(\tau)|.$$

Then the product space $\mathcal{U} := \mathcal{C} \times \mathcal{C}$ defined by $\mathcal{U} = \{(\omega, \varpi) : \omega \in \mathcal{C}, \varpi \in \mathcal{C}\}$ is Banach space under the norm

$$\|(\omega, \varpi)\|_{\mathcal{U}} = \|\omega\|_{\mathcal{C}} + \|\varpi\|_{\mathcal{C}}.$$

Also $\mathcal{C}_\delta = C([-\delta, 0], \mathbb{R})$ is endowed with norm

$$\|\varphi\|_{\mathcal{C}_\delta} = \sup_{\tau \in [-\delta, 0]} |\varphi(\tau)|, \text{ and } \|\omega_\tau\|_{\mathcal{C}_\delta} = \sup_{\rho \in [-\delta, 0]} |\omega(\tau + \rho)|.$$

Consider $\mathcal{C}_b = C([a - \delta, b], \mathbb{R})$ the Banach space defined on $[a - \delta, b]$ with the norm

$$\|\omega\|_{\mathcal{C}_b} = \|\omega\|_{\mathcal{C}_\delta} + \|\omega\|_{\mathcal{C}} = \sup_{\tau \in [a - \delta, b]} |\omega(\tau)|.$$

Then the product space $\mathcal{C}_b \times \mathcal{C}_b$ is Banach space under the norm

$$\|(\omega, \varpi)\|_{\mathcal{C}_b} = \|\omega\|_{\mathcal{C}_b} + \|\varpi\|_{\mathcal{C}_b}.$$

Let $\psi \in \mathcal{C}^1 = C^1(J, \mathbb{R})$ be an increasing differentiable function such that $\psi'(\tau) \neq 0$, for all $\tau \in \mathrm{J}$.

Now, we start by defining ψ-FODs as follows:

Definition 2.1 *[34] The ψ–Riemann–Liouville fractional integral of order $\alpha > 0$ for an integrable function $\omega\colon \mathrm{J} \longrightarrow \mathbb{R}$ is given by*

$$\mathbb{I}_{a^+}^{\alpha;\psi}\omega(\tau) = \frac{1}{\Gamma(\alpha)}\int_a^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1}\omega(s)\mathrm{ds}, \tag{2.1}$$

where Γ is the Gamma function.

Definition 2.2 *[34] Let $n-1<\alpha<n$ $(n\in\mathbb{N})$, $\omega : J \to \mathbb{R}$ is an integrable function, and $\psi \in C^n(\mathrm{J},\mathbb{R})$, the ψ–Riemann–Liouville FOD of a function ω of order α is given by*

$$\mathbb{D}_{a^+}^{\alpha;\psi}\omega(\tau) = \left(\frac{D_t}{\psi'(\tau)}\right)^n \ \mathbb{I}_{a^+}^{n-\alpha;\psi}\omega(\tau),$$

where $n=[\alpha]+1$ and $D_t = \frac{d}{dt}$.

Definition 2.3 *[12] For $n-1<\alpha<n$ $(n\in\mathbb{N})$ and $\omega,\psi \in C^n(\mathrm{J},\mathbb{R})$, the ψ–Caputo FOD of a function ω of order α is given by*

$${}^c\mathbb{D}_{a^+}^{\alpha;\psi}\omega(\tau) = \ \mathbb{I}_{a^+}^{n-\alpha;\psi}\omega_\psi^{[n]}(\tau),$$

where $n=[\alpha]+1$ for $\alpha\notin\mathbb{N}$, $n=\alpha$ for $\alpha\in\mathbb{N}$, and $\omega_\psi^{[n]}(\tau) = \left(\frac{D_t}{\psi'(\tau)}\right)^n\omega(\tau)$.

From the above definition, we can express ψ–Caputo FOD by formula

$${}^c\mathbb{D}_{a^+}^{\alpha;\psi}\omega(\tau) = \begin{cases} \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)}\omega_\psi^{[n]}(s)\mathrm{ds} & , \quad \text{if}\,\alpha\notin\mathbb{N}, \\ \omega_\psi^{[n]}(\tau) & , \quad \text{if}\,\alpha\in\mathbb{N}. \end{cases} \tag{2.2}$$

Also, the ψ–Caputo FOD of order α of ω is defined as

$${}^c\mathbb{D}_{a^+}^{\alpha;\psi}\omega(\tau) = \mathbb{D}_{a^+}^{\alpha;\psi}\left[\omega(\tau) - \sum_{k=0}^{n-1}\frac{\omega_\psi^{[k]}(a)}{k!}(\psi(\tau)-\psi(a))^k\right].$$

For more details see [12, Theorem 3].

Lemma 2.4 *[34] For $\alpha,\beta>0$, and $\omega\in C(\mathrm{J},\mathbb{R})$, we have*

$$\mathbb{I}_{a^+}^{\alpha;\psi}\mathbb{I}_{a^+}^{\beta;\psi}\omega(\tau) = \mathbb{I}_{a^+}^{\alpha+\beta;\psi}\omega(\tau), \ \textit{a.e. } \tau\in\mathrm{J}.$$

Lemma 2.5 *[13] Let $\alpha>0$.*
If $\omega\in C(\mathrm{J},\mathbb{R})$, then

$${}^c\mathbb{D}_{a^+}^{\alpha;\psi}\mathbb{I}_{a^+}^{\alpha;\psi}\omega(\tau) = \omega(\tau), \ \tau\in\mathrm{J},$$

and if $\omega\in C^{n-1}(\mathrm{J},\mathbb{R})$, then

$$\mathbb{I}_{a^+}^{\alpha;\psi} \ {}^c\mathbb{D}_{a^+}^{\alpha;\psi}\omega(\tau) = \omega(\tau) - \sum_{k=0}^{n-1}\frac{\omega_\psi^{[k]}(a)}{k!}\left[\psi(\tau)-\psi(a)\right]^k, \quad \tau\in\mathrm{J}.$$

Lemma 2.6 *[12, 34] For $\tau > a,\ \alpha \geq 0,\ \beta > 0,$ and let $\chi(\tau) = \psi(\tau) - \psi(a)$. Then*

- $\mathbb{I}_{a^+}^{\alpha;\psi}(\chi(\tau))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\chi(\tau))^{\beta+\alpha-1}$,
- ${}^c\mathbb{D}_{a^+}^{\alpha;\psi}(\chi(\tau))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\chi(\tau))^{\beta-\alpha-1}$,
- ${}^c\mathbb{D}_{a^+}^{\alpha;\psi}(\chi(\tau))^{k} = 0,$ *for all* $k \in \{0, \dots, n-1\}, n \in \mathbb{N}$.

3. Main result

In this section, we consider a general type of FFDEs (1.1) and (1.2)–(1.3) involving the arbitrary function ψ. For the sake of convenience, we setting the following symbols.

$$\begin{aligned}
\Delta &= \Lambda_1 + \Lambda_2 = \frac{(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\|\mu\| + \sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\sigma_k}}{\Gamma(\sigma_k+1)}\|\upsilon_k\|, \qquad (3.1)\\
\Theta_1 &= \left(\frac{(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\|L_1\| + \sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\sigma_k}}{\Gamma(\sigma_k+1)}\|\omega_{k,1}\|\right),\\
\Theta_2 &= \left(\frac{(\psi(b)-\psi(a))^{\varsigma}}{\Gamma(\varsigma+1)}\|L_2\| + \sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\xi_k}}{\Gamma(\xi_k+1)}\|\omega_{k,2}\|\right),\\
\overline{\Theta}_1 &= \left(\frac{(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\|K_1\| + \sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\sigma_k}}{\Gamma(\sigma_k+1)}\|\varpi_{k,1}\|\right),\\
\overline{\Theta}_2 &= \left(\frac{(\psi(b)-\psi(a))^{\varsigma}}{\Gamma(\varsigma+1)}\|K_2\| + \sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\xi_k}}{\Gamma(\xi_k+1)}\|\varpi_{k,2}\|\right),
\end{aligned}$$

3.1. ψ-Caputo FNFIE (1.1)

Let us define exactly what we mean by a solution of problem (1.1).

Definition 3.1 *A function $z \in \mathcal{C}_b$ is said to be a solution of (1.1) if z fulfills the equation ${}^c\mathbb{D}_{a^+}^{\nu}\left[z(\tau) - \sum_{k=1}^{m}\mathbb{I}^{\sigma_k;\psi}\mathbb{F}_k(\tau, z_\tau)\right] = \mathbb{H}(\tau, z_\tau)$ on J, and the condition $z(\tau) = \varphi(\tau)$ on $[a-\delta, a]$.*

To demonstrate the existence of solutions to (1.1), we need the following Lemma.

Lemma 3.2 *Let $0 < \nu \leq 1,\ \alpha(a) = 0,$ and $g, h : \mathrm{J} \to \mathbb{R}$ are continuous functions with $h(a) = 0$. The linear problem*

$$ {}^c\mathbb{D}_{a^+}^{\nu;\psi}[z(\tau) - h(\tau)] = g(\tau), \quad \tau \in \mathrm{J}, z(\tau) = \alpha(\tau), \quad \tau \in [a-\delta, a],$$

has a unique solution $z(t)$ defined by:

$$z(t) = \begin{cases} h(\tau) + \mathbb{I}_{a^+}^{\nu;\psi}g(\tau), & \text{if } \tau \in \mathrm{J},\\ \alpha(\tau), & \text{if } \tau \in [a-\delta, a]. \end{cases}$$

For the proof of Lemma 3.2, it is useful to refer to [11, 15, 25, 34].

Now, we give the following hypotheses.

(G1) The functions $\mathbb{H}, \mathbb{F}_k : \mathrm{J} \times \mathcal{C}_\delta \to \mathbb{R}$ are continuous.

(G2) There exist positive functions μ, υ_k, $k = 1, 2, \cdots, m$, with bounds $\|\mu\|$ and $\|\upsilon_k\|$, $k = 1, 2, \cdots, m$, respectively such that

$$|\mathbb{H}(\tau, z) - \mathbb{H}(\tau, \overline{z})| \leq \mu(\tau)\|z - \overline{z}\|_{\mathcal{C}_\delta},$$

and

$$|\mathbb{F}_k(\tau, z) - \mathbb{F}_k(\tau, \overline{z})| \leq \upsilon_k(\tau)\|z - \overline{z}\|_{\mathcal{C}_\delta}.$$

for $\tau \in \mathrm{J}$ and $z, \overline{z} \in \mathcal{C}_\delta$.

(G3) There exist two constants $\varphi, \varphi_k \geq 0$, for $k = 1, 2, \cdots, m$ such that

$$|\mathbb{H}(\tau, z)| \leq \varphi\|z\|_{\mathcal{C}_\delta}, \quad |\mathbb{F}_k(\tau, z)| \leq \varphi_k, \quad \forall(\tau, z) \in \mathrm{J} \times \mathcal{C}_\delta.$$

3.1.1. Uniqueness result via Banach FPT

Theorem 3.3 *Suppose that assumptions (G1)–(G2) holds. If*

$$\Delta < 1 \tag{3.2}$$

where Δ is given by (3.1), then there exists a unique solution for (1.1) on the interval $[a - \delta, b]$.

Proof

Define the set

$$U := \left\{ z \in \mathcal{C}_b : z|_{[a-\delta,a]} \in \mathcal{C}_\delta, \ z|_{\mathrm{J}} \in \mathcal{C};\ {}^c\mathbb{D}^{\nu}_{a^+} z \in \mathcal{C} \right\},$$

and the operator $\mathcal{K} : U \to U$

$$\mathcal{K}(z)(t) := \begin{cases} \mathbb{I}^{\nu,\psi}_{a^+}\mathbb{H}(\tau, z_\tau) + \mathbb{F}(\tau, z_\tau), & \text{if } \tau \in \mathrm{J}, \\ \alpha(\tau), & \text{if } \tau \in [a - \delta, a]. \end{cases} \tag{3.3}$$

Notice that $\mathcal{K}$ is well defined. Indeed, for $z \in U$, the map $\tau \mapsto \mathcal{K}(z)(\tau)$ is continuous, for all $\tau \in [a-\delta, b]$. Also, for all $\tau \in \mathrm{J}$, ${}^c\mathbb{D}^{a,\psi}_{a^+}\mathcal{K}[z(\tau) - \mathbb{I}^{\sigma_k,\psi}_{a^+}\mathbb{F}_k(\tau, z_\tau)] = \mathbb{H}(\tau, z_\tau)$ exists and is continuous too due to continuity of $\mathbb{H}$ and Lemma 2.5. Now we need to show that $\mathcal{K}$ is a contraction map. Let $z, \overline{z} \in U$ and $\tau \in [a - \delta, a]$. Then, $|\mathcal{K}(z)(\tau) - \mathcal{K}(\overline{z})(\tau)| = 0$. On the other side, for $\tau \in \mathrm{J}$, and use (G2), it follows that

$$\begin{aligned} |\mathcal{K}(z)(\tau) - \mathcal{K}(\overline{z})(\tau)| &\leq \mathbb{I}^{\nu,\psi}_{a^+}|\mathbb{H}(\tau, z_\tau) - \mathbb{H}(\tau, \overline{z}_\tau)| + \sum_{k=1}^{m}\mathbb{I}^{\sigma_k,\psi}_{a^+}|\mathbb{F}_k(\tau, z_\tau) - \mathbb{F}_k(\tau, \overline{z}_\tau)| \\ &\leq \|\mu\|\|z_\tau - \overline{z}_\tau\|_{\mathcal{C}_\delta}\mathbb{I}^{\nu,\psi}_{a^+}(1)(\tau) + \sum_{k=1}^{m}\|\upsilon_k\|\,\|z_\tau - \overline{z}_\tau\|_{\mathcal{C}_\delta}\mathbb{I}^{\sigma_k,\psi}_{a^+}(1)(\tau) \\ &\leq \left(\frac{(\psi(b) - \psi(a))^{\nu}}{\Gamma(\nu + 1)}\|\mu\| + \sum_{k=1}^{m}\frac{(\psi(b) - \psi(a))^{\sigma_k}}{\Gamma(\sigma_k + 1)}\|\upsilon_k\|\right)\|z_\tau - \overline{z}_\tau\|_{\mathcal{C}_\delta} \\ &\leq (\Lambda_1 + \Lambda_2)\|z - \overline{z}\|_{\mathcal{C}_b}, \end{aligned}$$

which implies

$$\|\mathcal{K}(z)-\mathcal{K}(\overline{z})\|_{\mathcal{C}_b} \leq \Delta \|z-\overline{z}\|_{\mathcal{C}_b}.$$

Since $\Delta < 1$, the operator $\mathcal{K}$ is a contraction. Hence the theorem of Banach fixed point shows that $\mathcal{K}$ admits a unique fixed point. □

3.1.2. Existence result via Kransnoselskii's FPT

Here, we apply the fixed point theorem of Krasnoselskii [33] to obtain the existence result.

Theorem 3.4 *Assume that (G1)–(G3) hold. Then (1.1) has at least one solution on* $[a-\delta, b]$*, provided*

$$\frac{(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\varphi < 1. \tag{3.4}$$

Proof By the assumption (G3), we can fix

$$\rho \geq \frac{\sum_{k=1}^{m} \frac{(\psi(b)-\psi(a))^{\sigma_k}}{\Gamma(\sigma_k+1)}\varphi_k}{1-\frac{(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\varphi},$$

where $B_\rho = \{z \in \mathcal{C}_b : \|z\|_{\mathcal{C}_b} \leq \rho\}$. Let us split the operator $\mathcal{K} : \mathcal{C}_b \to \mathcal{C}_b$ defined by Equation (3.3) as $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$, where $\mathcal{K}_1$ and $\mathcal{K}_2$ are given by

$$\mathcal{K}_1(z)(t) := \begin{cases} \mathbb{I}_{a^+}^{\nu,\psi}\mathbb{H}(\tau, z_\tau), & \text{if } \tau \in \mathrm{J}, \\ 0, & \text{if } \tau \in [a-\delta, a]. \end{cases}$$

$$\mathcal{K}_2(z)(t) := \begin{cases} \sum_{k=1}^{m} \mathbb{I}_{a^+}^{\sigma_k,\psi}\mathbb{F}_k(\tau, z_\tau), & \text{if } \tau \in \mathrm{J}, \\ \alpha(\tau), & \text{if } \tau \in [a-\delta, a]. \end{cases}$$

The proof will be split into numerous steps:

Step 1: $\mathcal{K}_1(z) + \mathcal{K}_2(z) \in B_\rho$.

Case 1. For any $z, \overline{z} \in B_\rho$, and $\tau \in [a-\delta, a]$ we have

$$|\mathcal{K}_1(z)(\tau) + \mathcal{K}_2(\overline{z})(\tau)| \leq |\alpha(\tau)| \leq \|\alpha\|_{\mathcal{C}_\delta} \leq \|\alpha\|_{\mathcal{C}_b} \leq \rho.$$

Case 2. Let $z, \overline{z} \in B_\rho$ and $\tau \in \mathrm{J}$. Then

$$\begin{aligned} |\mathcal{K}_1(z)(\tau) + \mathcal{K}_2(\overline{z})(\tau)| &\leq \mathbb{I}_{a^+}^{\nu,\psi}|\mathbb{H}(\tau, z_\tau)| + \sum_{k=1}^{m} \mathbb{I}_{a^+}^{\sigma_k,\psi}|\mathbb{F}_k(\tau, z_\tau)|\, ds \\ &\leq \varphi\|z_\tau\|_{\mathcal{C}_\delta}\mathbb{I}_{a^+}^{\nu,\psi}(1)(\tau) + \sum_{k=1}^{m}\varphi_k \mathbb{I}_{a^+}^{\sigma_k,\psi}(1)(\tau) \\ &\leq \frac{(\psi(\tau)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\varphi\|z\|_{\mathcal{C}_b} + \sum_{k=1}^{m}\frac{(\psi(\tau)-\psi(a))^{\sigma_k}}{\Gamma(\sigma_k+1)}\varphi_k \\ &\leq \frac{(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\varphi\rho + \sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\sigma_k}}{\Gamma(\sigma_k+1)}\varphi_k \\ &\leq \rho. \end{aligned}$$

Hence

$$\|\mathcal{K}_1(z)+\mathcal{K}_2(\overline{z})\|_{\mathcal{C}_b}\leq \rho,$$

which shows that $\mathcal{K}_1 z+\mathcal{K}_2\overline{z}\in B_\rho$.

Step 2: $\mathcal{K}_1$ is a contration map on B_ρ.

Due to the contractility of $\mathcal{K}$ as in Theorem 3.3, then $\mathcal{K}_1$ is a contraction map too.

Step 3: $\mathcal{K}_2$ is completely continuous on B_ρ.

From the continuity of $\mathbb{F}_k\left(\cdot, z_{(\cdot)}\right)$ and $\alpha(\cdot)$, it follows that $\mathcal{K}_2$ is continuous.

Since

$$\begin{aligned}\|\mathcal{K}_2 z\|_{\mathcal{C}_\delta} &= \sup_{\tau\in[-\delta,0]}|\mathcal{K}_2 z(\tau)| = \sup_{\tau\in[-\delta,0]}|\alpha(\tau)|\\ &= \|\alpha\|_{\mathcal{C}_\delta}=\|\alpha\|_{\mathcal{C}_b}-\|\alpha\|_{\mathcal{C}}\leq\|\alpha\|_{\mathcal{C}_b}\leq\rho,\ z\in B_\rho,\end{aligned}$$

and

$$\|\mathcal{K}_2 z\|_{\mathcal{C}}=\sup_{\tau\in \mathrm{J}}|\mathcal{K}_2 z(\tau)|\leq\sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\sigma_k}}{\Gamma(\sigma_k+1)}\varphi_k:=L,\ z\in B_\rho.$$

we get $\|\mathcal{K}_2 z\|_{\mathcal{C}_b}\leq\rho+L$ which emphasize that $\mathcal{K}_2$ uniformly bounded on B_ρ.

Finally, we prove the compactness of $\mathcal{K}_2$.

For $z\in B_\rho$ and $\tau\in\mathrm{J}$, we can estimate the operator derivative as follows:

$$\begin{aligned}\left|(\mathcal{K}_2 z)_\psi^{(1)}(\tau)\right| &\leq\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k-1,\psi}\left|\mathbb{F}_k\left(\tau,z_\tau\right)\right|\leq\sum_{k=1}^{m}\varphi_k\mathbb{I}_{a^+}^{\sigma_k-1,\psi}(1)(\tau)\\ &\leq\sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\sigma_k-1}}{\Gamma(\sigma_k)}\varphi_k:=\ell,\end{aligned}$$

where we used the fact

$$D_\psi^k\,\mathbb{I}_{a^+}^{\alpha,\psi}=\mathbb{I}_{a^+}^{\alpha-k,\psi},\quad \omega_\psi^{(k)}(\tau)=\left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)^k\omega(\tau)\text{ for }k=0,1,...,n-1.$$

Hence, for each $\tau_1,\tau_2\in\mathrm{J}$ with $a<\tau_1<\tau_2<b$ and for $z\in B_\rho$, we get

$$|(\mathcal{K}_2 z)(t_2)-(\mathcal{K}_2 z)(t_1)|=\int_{\tau_1}^{\tau_2}|(\mathcal{K}_2 z)'(s)|\,ds\leq\ell(\tau_2-\tau_1).$$

which as $(\tau_2-\tau_1)\to 0$ tends to zero independent of z. So, $\mathcal{K}_2$ is equicontinuous. The equicontinuity for the case $\tau_1,\tau_2\in[a-\delta,a]$ is obvious. In view of the foregoing arguments along with Arzela–Ascoli theorem, we infer that $\mathcal{K}_2$ is compact on B_ρ. Thus, the hypotheses of Krasnoselskii fixed point theorem [33] holds, so there exists at least one solution of (1.1) on $[a-\delta,b]$. □

3.1.3. UH stability of the solutions of problem (1.1)

Here, we discuss the UH and generalized UH stability types of (1.1).

Definition 3.5 *Problem (1.1) is UH stable if there exists a $c \in \mathbb{R}^+$ such that, for each $\epsilon \in \mathbb{R}^+$ and for each $\overline{z} \in \mathcal{C}_b$ satisfying*

$$\begin{cases} \left| {}^c\mathbb{D}_{a^+}^{\nu;\psi}\left[\overline{z}(\tau) - \sum_{k=1}^{m} \mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k(\tau, \overline{z}_\tau)\right] = \mathbb{H}(\tau, \overline{z}_\tau)\right| \leq \epsilon, \ \ \tau \in \mathrm{J}, \\ |\overline{z}(\tau) - \alpha(\tau)| \leq \epsilon, \quad \tau \in [a-\delta, a], \end{cases} \tag{3.5}$$

there exists a unique solution $z \in \mathcal{C}_b$ of (1.1) with

$$\|\overline{z} - z\| \leq c\epsilon.$$

Definition 3.6 *Problem (1.1) is generalized UH if there exists $\sigma \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\sigma(0) = 0$ such that for each $\epsilon \in \mathbb{R}^+$ and for each $\overline{z} \in \mathcal{C}_b$ satisfying (3.5), there exists a unique solution $z \in \mathcal{C}_b$ of (1.1) with*

$$\|z - \overline{z}\| \leq \sigma(\epsilon).$$

Remark 3.7 *A function $\overline{z} \in \mathcal{C}_b$ is a solution of the inequality (3.5) if and only if there exists a function $g \in \mathcal{C}$ such that (i) $|g(\tau)| \leq \epsilon, \ \tau \in \mathrm{J}$,*
(ii) ${}^c\mathbb{D}_{a^+}^{\nu;\psi}\left[\overline{z}(\tau) - \sum_{k=1}^{m} \mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k(\tau, \overline{z}_\tau)\right] = \mathbb{H}(\tau, \overline{z}_\tau) + g(\tau), \ \tau \in \mathrm{J}.$

Theorem 3.8 *Suppose that the conditions (G2) and (3.2) are fulfilled. Then, the solution of (1.1) is UH and GUH stable.*

Proof Let $\epsilon \in \mathbb{R}^+$ and $\overline{z} \in \mathcal{C}_b$ be any solution of the inequality (3.5). Then, there exists $g \in \mathcal{C}$ such that $|g(\tau)| \leq \epsilon, \tau \in [a, b]$, and satisfying

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\nu;\psi}\left[\overline{z}(\tau) - \sum_{k=1}^{m} \mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k(\tau, \overline{z}_\tau)\right] = \mathbb{H}(\tau, \overline{z}_\tau) + g(\tau), \tau \in \mathrm{J}, \\ \overline{z}(\tau) = \alpha(\tau), \quad \tau \in [a-\delta, a]. \end{cases} \tag{3.6}$$

Using Lemma 3.2, problem (3.6) has a solution given as

$$\overline{z}(t) := \begin{cases} \mathbb{I}_{a^+}^{\nu,\psi}\left[\mathbb{H}(\tau, \overline{z}_\tau) + g(t)\right] + \sum_{k=1}^{m} \mathbb{I}_{a^+}^{\sigma_k,\psi}\mathbb{F}_k(\tau, \overline{z}_\tau), & \text{if } \tau \in \mathrm{J}, \\ \alpha(\tau), & \text{if } \tau \in [a-\delta, a]. \end{cases}$$

Theorem 3.3 ensures the existence of a unique solution $z \in \mathcal{C}_b$ of equation (1.1) and satisfies the integral equation

$$z(t) := \begin{cases} \mathbb{I}_{a^+}^{\nu,\psi}\mathbb{H}(\tau, z_\tau) + \sum_{k=1}^{m} \mathbb{I}_{a^+}^{\sigma_k,\psi}\mathbb{F}_k(\tau, z_\tau), & \text{if } \tau \in \mathrm{J}, \\ \alpha(\tau), & \text{if } \tau \in [a-\delta, a]. \end{cases}$$

Therefore, for any $\tau \in \mathrm{J}$, we obtain

$$\begin{aligned}
|\overline{z}(\tau)-z(\tau)| &\leq \mathbb{I}_{a^+}^{\nu;\psi}\left|\mathbb{H}\left(\tau,\overline{z}_\tau\right)-\mathbb{H}\left(\tau,z_\tau\right)\right|+\mathbb{I}_{a^+}^{\nu;\psi}|g(\tau)| \\
&\quad+\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k,\psi}\left|\mathbb{F}_k\left(\tau,\overline{z}_\tau\right)-\mathbb{F}_k\left(\tau,z_\tau\right)\right| \\
&\leq \frac{(\psi(\tau)-\psi(a))^\nu}{\Gamma(\nu+1)}\|\mu\|\|\overline{z}_\tau-z_\tau\|_{\mathcal{C}_\delta}+\frac{(\psi(\tau)-\psi(a))^\nu}{\Gamma(\nu+1)}\epsilon \\
&\quad+\sum_{k=1}^{m}\frac{(\psi(\tau)-\psi(a))^{\sigma_k}}{\Gamma\left(\sigma_k+1\right)}\|\upsilon_k\|\,\|\overline{z}_\tau-z_\tau\|_{\mathcal{C}_\delta} \\
&\leq \left(\frac{(\psi(b)-\psi(a))^\nu}{\Gamma(\nu+1)}\|\mu\|+\sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\sigma_k}}{\Gamma\left(\sigma_k+1\right)}\|\upsilon_k\|\right)\|\overline{z}-z\|_{\mathcal{C}_b}+\kappa\epsilon.
\end{aligned}$$

where $\kappa=\frac{(\psi(b)-\psi(a))^\nu}{\Gamma(\nu+1)}$. Therefore, we have proved that

$$\|\overline{z}-z\|_{\mathcal{C}_b}\leq\Delta\|\overline{z}-z\|_{\mathcal{C}_b}+\kappa\epsilon.$$

By the condition in Theorem (3.3), one can deduce that

$$\|\overline{z}-z\|_{\mathcal{C}_b}\leq\frac{\kappa}{1-\Delta}\epsilon.$$

For $c=\frac{\kappa}{1-\Delta}>0$, we infer that the solution of (1.1) is UH stable. Similarly, it shows the existence of a function $\sigma\in\mathcal{C}\left(\mathbb{R}^+,\mathbb{R}^+\right)$ such that $\sigma(\epsilon)=\frac{\kappa}{1-\Delta}\epsilon$ with $\sigma(0)=0$. Hence, the solution of (1.1) is GUH stable. □

3.2. Coupled systems of ψ-Caputo FNFIEs

3.2.1. Uniqueness result via Banach FPT

Let us defining what we mean by a solution of problem (1.2)–(1.3).

A pair of coupled functions $(\omega,\varpi)\in\mathcal{C}_b\times\mathcal{C}_b$ is said to be a solution of the system (1.2)–(1.3) if it satisfies

$$\omega(t)=\begin{cases}\mathbb{I}_{a^+}^{\nu,\psi}\mathbb{H}_1\left(\tau,\omega_\tau,\varpi_\tau\right)+\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k,\psi}\mathbb{F}_k^*\left(\tau,\omega_\tau,\varpi_\tau\right), & \text{if } \tau\in\mathrm{J},\\ \varphi_1(\tau), & \text{if } \tau\in[a-\delta,a],\end{cases}$$

$$\varpi(t)=\begin{cases}\mathbb{I}_{a^+}^{\varsigma,\psi}\mathbb{H}_2\left(\tau,\omega_\tau,\varpi_\tau\right)+\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\xi_k,\psi}\mathbb{G}_k\left(\tau,\omega_\tau,\varpi_\tau\right), & \text{if } \tau\in\mathrm{J},\\ \varphi_2(\tau), & \text{if } \tau\in[a-\delta,a].\end{cases}$$

In the sequel, the following assumptions will be considered fulfilled:

(H1) The functions $\mathbb{H}_i,\mathbb{F}_k^*,\mathbb{G}_k:\mathrm{J}\times\mathcal{C}_\delta\times\mathcal{C}_\delta\to\mathbb{R}$ are continuous, for $i=1,2$ and $k=1,..,m$.

(H2) There exist positive functions $L_i,K_i,\omega_{k,i}$, and $\varpi_{k,i}$, $i=1,2$, $K=1,...,m$, with bounds $\|L_i\|,\|K_i\|,\|\omega_{k,i}\|$ and $\|\varpi_{k,i}\|$, $i=1,2$, $k=1,2,\cdots,m$, respectively such that:

$$\left|\mathbb{H}_1\left(\tau,\omega,\varpi\right)-\mathbb{H}_1\left(\tau,\overline{\omega},\overline{\varpi}\right)\right|\leq L_1(\tau)\|\omega-\overline{\omega}\|_{\mathcal{C}_\delta}+K_1(\tau)\|\varpi-\overline{\varpi}\|_{\mathcal{C}_\delta},$$

$$|\mathbb{H}_2\left(t,\omega,\varpi\right)-\mathbb{H}_2\left(t,\overline{\omega},\overline{\varpi}\right)|\leq L_2(\tau)\|\omega-\overline{\omega}\|_{\mathcal{C}_\delta}+K_2(\tau)\|\varpi-\overline{\varpi}\|_{\mathcal{C}_\delta},$$

and

$$|\mathbb{F}_k^*\left(t,\omega,\varpi\right)-\mathbb{F}_k^*\left(t,\overline{\omega},\overline{\varpi}\right)|\leq \omega_{k,1}(\tau)\|\omega-\overline{\omega}\|_{\mathcal{C}_\delta}+\varpi_{k,1}(\tau)\|\varpi-\overline{\varpi}\|_{\mathcal{C}_\delta},$$

$$|\mathbb{G}_k\left(t,\omega,\varpi\right)-\mathbb{G}_k\left(t,\overline{\omega},\overline{\varpi}\right)|\leq \omega_{k,2}(\tau)\|\omega-\overline{\omega}\|_{\mathcal{C}_\delta}+\varpi_{k,2}(\tau)\|\varpi-\overline{\varpi}\|_{\mathcal{C}_\delta},$$

for $\tau\in \mathrm{J}$ and $\omega,\varpi,\overline{\omega},\overline{\varpi}\in\mathcal{C}_\delta$.

(H3) There exist positive functions $\varphi_i,\overline{\varphi}_{k,i}$, $i=1,2$, $k=1,...,m$, with bounds $\|\varphi_i\|$ and $\left\|\overline{\varphi}_{k,i}\right\|$, $i=1,2$, $k=1,2,\cdots,m$, respectively such that:

$$|\mathbb{H}_1(\tau,\omega,\varpi)|\leq\varphi_1(\tau)\left(\|\omega\|_{\mathcal{C}_\delta}+\|\varpi\|_{\mathcal{C}_\delta}\right),\quad |\mathbb{F}_k^*(\tau,\omega,\varpi)|\leq\overline{\varphi}_{k,1}(\tau),\quad \forall(\tau,\omega,\varpi)\in \mathrm{J}\times\mathcal{C}_\delta\times\mathcal{C}_\delta,$$

$$|\mathbb{H}_2(\tau,\omega,\varpi)|\leq\varphi_2(\tau)\left(\|\omega\|_{\mathcal{C}_\delta}+\|\varpi\|_{\mathcal{C}_\delta}\right),\quad |\mathbb{G}_k(\tau,\omega,\varpi)|\leq\overline{\varphi}_{k,2}(\tau),\quad \forall(\tau,\omega,\varpi)\in \mathrm{J}\times\mathcal{C}_\delta\times\mathcal{C}_\delta.$$

The following result shows the uniqueness of solution for system (1.2)–(1.3) relying on Banach FPT.

Theorem 3.9 *Assume that assumptions (H1)–(H2) holds. If*

$$\Omega:=\sum_{i=1}^{2}\Omega_i=\max\left(\sum_{i=1}^{2}\Theta_i,\sum_{i=1}^{2}\overline{\Theta}_i\right)<1, \tag{3.7}$$

then there exists a unique solution for (1.2)–(1.3) on the interval $[a-\delta,b]$.

Proof Define the set

$$\overline{U}:=\left\{(\omega,\varpi)\in\mathcal{C}_b\times\mathcal{C}_b:\ (\omega,\varpi)|_{[a-\delta,a]}\in\mathcal{C}_\delta,\ \ (\omega,\varpi)|_{\mathrm{J}}\in\mathcal{C};\ {}^c\mathbb{D}_{a^+}^{\nu}(\omega,\varpi)\in\mathcal{C}\times\mathcal{C}\right\},$$

and the operators $\mathcal{K}_1^*:\overline{U}\to\overline{U}$ and $\mathcal{K}_2^*:\overline{U}\to\overline{U}$ defined by

$$\begin{cases}\mathcal{K}_1^*\omega(\tau)=\varpi(\tau),\\ \mathcal{K}_2^*\varpi(\tau)=\omega(\tau),\end{cases}$$

That is,

$$\begin{aligned}\mathcal{K}_1^*(\omega)(\tau)\quad &:=\begin{cases}\mathbb{I}_{a^+}^{\nu,\psi}\mathbb{H}_1\left(\tau,\omega_\tau,\varpi_\tau\right)+\sum_{k=1}^m\mathbb{I}_{a^+}^{\sigma_k,\psi}\mathbb{F}_k^*\left(\tau,\omega_\tau,\varpi_\tau\right), & \text{if } \tau\in \mathrm{J},\\ \varphi_1(\tau), & \text{if } \tau\in[a-\delta,a],\end{cases}\\ \mathcal{K}_2^*(\varpi)(\tau)\quad &:=\begin{cases}\mathbb{I}_{a^+}^{\varsigma,\psi}\mathbb{H}_2\left(\tau,\omega_\tau,\varpi_\tau\right)+\sum_{k=1}^m\mathbb{I}_{a^+}^{\xi_k,\psi}\mathbb{G}_k\left(\tau,\omega_\tau,\varpi_\tau\right), & \text{if } \tau\in \mathrm{J},\\ \varphi_2(\tau), & \text{if } \tau\in[a-\delta,a].\end{cases}\end{aligned} \tag{3.8}$$

Therefore, we define the operator $\mathcal{K}^*:\overline{U}\to\overline{U}$ by

$$\mathcal{K}^*(\omega,\varpi)(\tau)=\mathcal{K}_1^*(\omega)(\tau)+\mathcal{K}_2^*(\varpi)(\tau).$$

Thus, $\mathcal{K}^*$ is well defined. Indeed, for $(\omega,\varpi)\in\overline{U}$, the map $\tau\mapsto\mathcal{K}^*(\omega,\varpi)(\tau)$ is continuous, for all $\tau\in[a-\delta,b]$. Also, for all $\tau\in \mathrm{J}$,

$${}^c\mathbb{D}_{a^+}^{\nu,\psi}\mathcal{K}^*[\omega(\tau)-\mathbb{I}_{a^+}^{\sigma_k,\psi}\mathbb{F}_k^*\left(\tau,\omega_\tau,\varpi_\tau\right)]=\mathbb{H}_1\left(\tau,\omega_\tau,\varpi_\tau\right),$$

and

$$^{c}\mathbb{D}_{a^{+}}^{\varsigma,\psi}\mathcal{K}^{*}[\varpi(\tau)-\mathbb{I}_{a^{+}}^{\xi_{k},\psi}\mathbb{G}_{k}\left(\tau,\omega_{\tau},\varpi_{\tau}\right)]=\mathbb{H}_{2}\left(\tau,\omega_{\tau},\varpi_{\tau}\right).$$

exist and are continuous due to continuity of $\mathbb{H}_1$, $\mathbb{H}_2$ and Lemma 2.5. It rest for us to prove that $\mathcal{K}^*$ is contraction. Let $(\omega,\varpi),(\overline{\omega},\overline{\varpi})\in\overline{U}$ and $\tau\in[a-\delta,a]$. Then,

$$|\mathcal{K}^{*}(\omega,\varpi)(\tau)-\mathcal{K}^{*}(\overline{\omega},\overline{\varpi})(\tau)|=0.$$

On the other side, for $\tau\in$ J, and using (H2), we have that

$$\begin{aligned}
|\mathcal{K}_{1}^{*}(\omega)(\tau)-\mathcal{K}_{1}^{*}(\overline{\omega})(\tau)| &\leq \mathbb{I}_{a^{+}}^{\nu,\psi}\left|\mathbb{H}_{1}\left(\tau,\omega_{\tau},\varpi_{\tau}\right)-\mathbb{H}_{1}\left(\tau,\overline{\omega}_{\tau},\overline{\varpi}_{\tau}\right)\right| \\
&\quad+\sum_{k=1}^{m}\mathbb{I}_{a^{+}}^{\sigma_{k},\psi}\left|\mathbb{F}_{k}^{*}\left(\tau,\omega_{\tau},\varpi_{\tau}\right)-\mathbb{F}_{k}^{*}\left(\tau,\overline{\omega}_{\tau},\overline{\varpi}_{\tau}\right)\right| \\
&\leq \mathbb{I}_{a^{+}}^{\nu,\psi}\left(L_{1}(\tau)\left\|\omega_{\tau}-\overline{\omega}_{\tau}\right\|_{\mathcal{C}_{\delta}}+K_{1}(\tau)\left\|\varpi_{\tau}-\overline{\varpi}_{\tau}\right\|_{\mathcal{C}_{\delta}}\right)(\tau) \\
&\quad+\sum_{k=1}^{m}\mathbb{I}_{a^{+}}^{\sigma_{k},\psi}\left(\omega_{k,1}(\tau)\left\|\omega_{\tau}-\overline{\omega}_{\tau}\right\|_{\mathcal{C}_{\delta}}+\varpi_{k,1}(\tau)\left\|\varpi_{\tau}-\overline{\varpi}_{\tau}\right\|_{\mathcal{C}_{\delta}}\right)(\tau) \\
&\leq \left(\|L_{1}\|\left\|\omega_{\tau}-\overline{\omega}_{\tau}\right\|_{\mathcal{C}_{\delta}}+\|K_{1}\|\left\|\varpi_{\tau}-\overline{\varpi}_{\tau}\right\|_{\mathcal{C}_{\delta}}\right)\mathbb{I}_{a^{+}}^{\nu,\psi}(1)(\tau) \\
&\quad+\sum_{k=1}^{m}\left(\|\omega_{k,1}\|\left\|\omega_{\tau}-\overline{\omega}_{\tau}\right\|_{\mathcal{C}_{\delta}}+\|\varpi_{k,1}\|\left\|\varpi_{\tau}-\overline{\varpi}_{\tau}\right\|_{\mathcal{C}_{\delta}}\right)\mathbb{I}_{a^{+}}^{\sigma_{k,1},\psi}(1)(\tau) \\
&\leq \frac{(\psi(\tau)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\left(\|L_{1}\|\left\|\omega-\overline{\omega}\right\|_{\mathcal{C}_{b}}+\|K_{1}\|\left\|\varpi-\overline{\varpi}\right\|_{\mathcal{C}_{b}}\right) \\
&\quad+\sum_{k=1}^{m}\frac{(\psi(\tau)-\psi(a))^{\sigma_{k}}}{\Gamma\left(\sigma_{k}+1\right)}\left(\|\omega_{k,1}\|\left\|\omega-\overline{\omega}\right\|_{\mathcal{C}_{b}}+\|\varpi_{k,1}\|\left\|\varpi-\overline{\varpi}\right\|_{\mathcal{C}_{b}}\right),
\end{aligned}$$

which gives

$$\begin{aligned}
&\left\|\mathcal{K}_{1}(\omega)-\mathcal{K}_{1}(\overline{\omega})\right\|_{\mathcal{C}_{b}} \\
\leq\ &\left(\frac{(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\|L_{1}\|+\sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\sigma_{k}}}{\Gamma\left(\sigma_{k}+1\right)}\|\omega_{k,1}\|\right)\|\omega-\overline{\omega}\|_{\mathcal{C}_{b}} \\
&+\left(\frac{(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\|K_{1}\|+\sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\sigma_{k}}}{\Gamma\left(\sigma_{k}+1\right)}\|\varpi_{k,1}\|\right)\|\varpi-\overline{\varpi}\|_{\mathcal{C}_{b}} \\
=\ &\Theta_{1}\|\omega-\overline{\omega}\|_{\mathcal{C}_{b}}+\overline{\Theta}_{1}\|\varpi-\overline{\varpi}\|_{\mathcal{C}_{b}}. \qquad (3.9)
\end{aligned}$$

In a similar way, we get

$$\begin{aligned}
&\left\|\mathcal{K}_2^*(\varpi)-\mathcal{K}_2^*(\overline{\varpi})\right\|_{\mathcal{C}_b}\\
\leq\ &\left(\frac{(\psi(b)-\psi(a))^{\varsigma}}{\Gamma(\varsigma+1)}\|L_2\|+\sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\xi_k}}{\Gamma\left(\xi_k+1\right)}\left\|\omega_{k,2}\right\|\right)\|\omega-\overline{\omega}\|_{\mathcal{C}_b}\\
&+\left(\frac{(\psi(b)-\psi(a))^{\varsigma}}{\Gamma(\varsigma+1)}\|K_2\|+\sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\xi_k}}{\Gamma\left(\xi_k+1\right)}\left\|\varpi_{k,2}\right\|\right)\|\varpi-\overline{\varpi}\|_{\mathcal{C}_b}\\
=\ &\Theta_2\|\omega-\overline{\omega}\|_{\mathcal{C}_b}+\overline{\Theta}_2\|\varpi-\overline{\varpi}\|_{\mathcal{C}_b}. \qquad (3.10)
\end{aligned}$$

From (3.9) and (3.10), we obtain

$$\begin{aligned}
\left\|\mathcal{K}^*(\omega,\varpi)-\mathcal{K}^*(\overline{\omega},\overline{\varpi})\right\|_{\overline{U}} &\leq \sum_{i=1}^{2}\Theta_i\|\omega-\overline{\omega}\|_{\mathcal{C}_b}+\sum_{i=1}^{2}\overline{\Theta}_i\|\varpi-\overline{\varpi}\|_{\mathcal{C}_b}\\
&\leq \Omega\left\|(\omega,\varpi)-(\overline{\omega},\overline{\varpi})\right\|_{\overline{U}}.
\end{aligned}$$

As $\Omega<1$, $\mathcal{K}^*$ is a contraction, and hence $\mathcal{K}^*$ has a unique fixed point by Banach FPT. □

3.2.2. Existence result via Kransnoselskii's FPT

The following existence theorem based on Krasnoselskii's FPT.

Theorem 3.10 *Suppose (H1)-(H3) hold. Then system (1.2)–(1.3) has at least one solution on* $[a-\delta,b]$, *provided that* $\Lambda_{1,1}+\Lambda_{2,2}<1$, *where*

$$\Lambda_{1,1}:=\frac{(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\|\varphi_1\|,\quad \Lambda_{2,2}:=\frac{(\psi(b)-\psi(a))^{\varsigma}}{\Gamma(\varsigma+1)}\|\varphi_2\|. \qquad (3.11)$$

Proof Setting

$$B_R=\{(\omega,\varpi)\in\mathcal{C}_b\times\mathcal{C}_b:\|(\omega,\varpi)\|_{\mathcal{C}_b}\leq R\}.$$

By Assumption (H3) we can fix

$$R\geq\left\{\frac{\Lambda_{1,2}+\Lambda_{2,1}}{1-(\Lambda_{1,1}+\Lambda_{2,2})}\right\}, \qquad (3.12)$$

where

$$\Lambda_{1,2}=\sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\sigma_k}}{\Gamma\left(\sigma_k+1\right)}\left\|\overline{\varphi}_{k,1}\right\|, \qquad (3.13)$$

$$\Lambda_{2,1}=\sum_{k=1}^{m}\frac{(\psi(b)-\psi(a))^{\xi_k}}{\Gamma\left(\xi_k+1\right)}\left\|\overline{\varphi}_{k,2}\right\|, \qquad (3.14)$$

Let us split the operator $\mathcal{K}^*:\mathcal{C}_b\times\mathcal{C}_b\to\mathcal{C}_b\times\mathcal{C}_b$ defined by (3.8) as

$$\mathcal{K}_1^*(\omega,\varpi)(\tau)=\mathcal{K}_{1,1}(\omega,\varpi)(\tau)+\mathcal{K}_{1,2}(\omega,\varpi)(\tau),$$

and

$$\mathcal{K}_2^*(\omega,\varpi)(\tau)=\mathcal{K}_{2,1}(\omega,\varpi)(\tau)+\mathcal{K}_{2,2}(\omega,\varpi)(\tau),$$

where $\mathcal{K}_{1,1},\mathcal{K}_{1,2},\mathcal{K}_{2,1}$ and $\mathcal{K}_{2,2}$ are given by

$$\mathcal{K}_{1,1}(\omega,\varpi)(\tau):=\begin{cases}\mathbb{I}_{a^+}^{\nu,\psi}\mathbb{H}_1\left(\tau,\omega_\tau,\varpi_\tau\right), & \tau\in \mathrm{J},\\ \varphi_1(\tau), & \tau\in[a-\delta,a],\end{cases},$$

$$\mathcal{K}_{1,2}(\omega,\varpi)(\tau):=\begin{cases}\sum_{k=1}^m\mathbb{I}_{a^+}^{\sigma_k,\psi}\mathbb{F}_k^*\left(\tau,\omega_\tau,\varpi_\tau\right), & \tau\in \mathrm{J},\\ 0, & \tau\in[a-\delta,a],\end{cases},$$

$$\mathcal{K}_{2,1}(\omega,\varpi)(\tau):=\begin{cases}\mathbb{I}_{a^+}^{\varsigma,\psi}\mathbb{H}_2\left(\tau,\omega_\tau,\varpi_\tau\right), & \tau\in \mathrm{J},\\ \varphi_2(\tau), & \tau\in[a-\delta,a],\end{cases},$$

$$\mathcal{K}_{2,2}(\omega,\varpi)(\tau):=\begin{cases}\sum_{k=1}^m\mathbb{I}_{a^+}^{\xi_k,\psi}\mathbb{G}_k\left(\tau,\omega_\tau,\varpi_\tau\right), & \tau\in \mathrm{J},\\ 0, & \tau\in[a-\delta,a],\end{cases},$$

The proof of the case $\tau\in[a-\delta,a]$ will be omitted. For any $(\omega,\varpi),(\overline{\omega},\overline{\varpi})\in B_R$, and $\tau\in\mathrm{J}$, we have

$$\begin{aligned}
&|\mathcal{K}_{1,1}(\omega,\varpi)(\tau)+\mathcal{K}_{1,2}(\overline{\omega},\overline{\varpi})(\tau)|\\
\leq\ & \mathbb{I}_{a^+}^{\nu,\psi}\left|\mathbb{H}_1\left(\tau,\omega_\tau,\varpi_\tau\right)\right|+\sum_{k=1}^m\mathbb{I}_{a^+}^{\sigma_k,\psi}\left|\mathbb{F}_k^*\left(\tau,\omega_\tau,\varpi_\tau\right)\right|\\
\leq\ & \|\varphi_1\|\left(\|\omega\|_{\mathcal{C}_\delta}+\|\varpi\|_{\mathcal{C}_\delta}\right)\mathbb{I}_{a^+}^{\nu,\psi}(1)(\tau)+\sum_{k=1}^m\left\|\overline{\varphi}_{k,1}\right\|\mathbb{I}_{a^+}^{\sigma_k,\psi}(1)(\tau)\\
\leq\ & \frac{(\psi(\tau)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\|\varphi_1\|\left(\|\omega\|_{\mathcal{C}_b}+\|\varpi\|_{\mathcal{C}_b}\right)+\sum_{k=1}^m\frac{(\psi(\tau)-\psi(a))^{\sigma_k}}{\Gamma\left(\sigma_k+1\right)}\left\|\overline{\varphi}_{k,1}\right\|\\
\leq\ & \frac{(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\|\varphi_1\|\|\left(\omega,\varpi\right)\|_{\mathcal{C}_b}+\sum_{k=1}^m\frac{(\psi(b)-\psi(a))^{\sigma_k}}{\Gamma\left(\sigma_k+1\right)}\left\|\overline{\varphi}_{k,1}\right\|\\
\leq\ & \frac{(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\|\varphi_1\|R+\sum_{k=1}^m\frac{(\psi(b)-\psi(a))^{\sigma_k}}{\Gamma\left(\sigma_k+1\right)}\left\|\overline{\varphi}_{k,1}\right\|.
\end{aligned}$$

Therefore, we proved that

$$\|\mathcal{K}_{1,1}(\omega,\varpi)+\mathcal{K}_{1,2}(\overline{\omega},\overline{\varpi})\|_{\mathcal{C}_b}\leq\Lambda_{1,1}R+\Lambda_{1,2}.\tag{3.15}$$

Similarly, we obtian

$$\|\mathcal{K}_{2,1}(\omega,\varpi)+\mathcal{K}_{2,2}(\overline{\omega},\overline{\varpi})\|_{\mathcal{C}_b}\leq\Lambda_{2,2}R+\Lambda_{2,1}.\tag{3.16}$$

From (3.12), (3.15) and (3.16), we get

$$\|\mathcal{K}_1^*(\omega,\varpi)+\mathcal{K}_2^*(\overline{\omega},\overline{\varpi})\|_{\mathcal{C}_b}\leq\left(\Lambda_{1,1}+\Lambda_{2,2}\right)R+\left(\Lambda_{1,2}+\Lambda_{2,1}\right)\leq R,$$

which shows that $\mathcal{K}_1^*(\omega,\varpi)+\mathcal{K}_2^*(\overline{\omega},\overline{\varpi})\in B_R$. The condition $\Lambda_{1,1}+\Lambda_{2,2}<1$ implies that $\mathcal{K}_1^*=(\mathcal{K}_{1,1}+\mathcal{K}_{1,2})$ is contraction. From the continuity of $\mathbb{H}_2$ and $\mathbb{G}_k$, it follows that $\mathcal{K}_2^*$ is continuous. By (H3) with the help of (3.14) and (3.16), we attain

$$\left\|\mathcal{K}_2^*(\omega,\varpi)\right\|_{\mathcal{C}_b}=\left\|\left(\mathcal{K}_{2,1}+\mathcal{K}_{2,2}\right)(\omega,\varpi)\right\|_{\mathcal{C}_b}\leq\Lambda_{2,2}R+\Lambda_{2,1}.$$

This means that $\mathcal{K}_2^*$ is uniformly bounded on B_R.

Finally, we show the compactness of $\mathcal{K}_2^*$. Take τ_1, $\tau_2 \in \mathrm{J}$, with $\tau_1 < \tau_2$, and $(\omega, \varpi) \in B_R$. Then,

$$\begin{aligned}
& |\mathcal{K}_2^*(\omega,\varpi)(\tau_2) - \mathcal{K}_2^*(\omega,\varpi)(\tau_1)| \\
\leq & \left|\mathbb{I}_{a^+}^{\varsigma,\psi}\mathbb{H}_2\left(s,\omega_s,\varpi_s\right)(\tau_2) - \mathbb{I}_{a^+}^{\varsigma,\psi}\mathbb{H}_2\left(s,\omega_s,\varpi_s\right)(\tau_1)\right| \\
& + \left|\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\xi_k,\psi}\mathbb{G}_k\left(s,\omega_s,\varpi_s\right)(\tau_2) - \sum_{k=1}^{m}\mathbb{I}_{a^+}^{\xi_k,\psi}\mathbb{G}_k\left(s,\omega_s,\varpi_s\right)(\tau_1)\right| \\
\leq & \frac{1}{\Gamma(\varsigma)}\int_a^{\tau_1}\psi'(s)\left|(\psi(\tau_1)-\psi(s))^{\varsigma-1} - (\psi(\tau_2)-\psi(s))^{\varsigma-1}\right|\left|\mathbb{H}_2\left(s,\omega_s,\varpi_s\right)\right| ds \\
& + \frac{1}{\Gamma(\varsigma)}\int_{\tau_1}^{\tau_2}\psi'(s)(\psi(\tau_2)-\psi(s))^{\varsigma-1}\left|\mathbb{H}_2\left(s,\omega_s,\varpi_s\right)\right| ds \\
+ & \sum_{k=1}^{m}\frac{1}{\Gamma(\xi_k)}\int_a^{\tau_1}\psi'(s)\left|(\psi(\tau_1)-\psi(s))^{\xi_k-1} - (\psi(\tau_2)-\psi(s))^{\xi_k-1}\right|\left|\mathbb{G}_k\left(s,\omega_s,\varpi_s\right)\right| ds \\
& + \sum_{k=1}^{m}\frac{1}{\Gamma(\xi_k)}\int_{\tau_1}^{\tau_2}\psi'(s)(\psi(\tau_2)-\psi(s))^{\xi_k-1}\left|\mathbb{G}_k\left(s,\omega_s,\varpi_s\right)\right| ds \\
\leq & \frac{2\|\varphi_2\|R}{\Gamma(\varsigma+1)}\left(\psi(\tau_2)-\psi(\tau_1)\right)^{\varsigma} + \sum_{k=1}^{m}\frac{2\left\|\overline{\varphi}_{k,2}\right\|}{\Gamma(\xi_k+1)}\left(\psi(\tau_2)-\psi(\tau_1)\right)^{\xi_k}.
\end{aligned}$$

which tends to zero as $\tau_1 \to \tau_2$. For $\tau_1, \tau_2 \in [a-\delta, a]$, then

$$|\mathcal{K}_2^*(\omega,\varpi)(\tau_2) - \mathcal{K}_2^*(\omega,\varpi)(\tau_1)| = |\varphi_2(\tau_2) - \varphi_2(\tau_1)| \to 0, \text{ as } \tau_1 \to \tau_2.$$

The equicontinuity for the case $\tau_1 \in [a-\delta, a]$ and $, \tau_2 \in \mathrm{J}$ is obvious.Thus, the set $\{\mathcal{K}_2^*(\omega,\varpi) : (\omega,\varpi) \in B_R\}$ is equicontinuous on B_R. In view of the foregoing arguments along with Arzela–Ascoli theorem, we infer that $\mathcal{K}_2^*$ is compact on B_R.
Thus all the assumptions of Krasnoselskii FPT are satisfied. So, Theorem 3.10 shows that (1.2)–(1.3) has at least one solution on $[a-\delta, b]$. □

3.2.3. UH stability on system (1.2)–(1.3)

In the current subsection, we are interesting to study UH and generalized UH stability types of system (1.2)–(1.3).

Definition 3.11 *System (1.2)–(1.3) is UH stable if there exists a real number $c = \max(c_1, c_2) > 0$ such that for each $\epsilon = \max(\epsilon_1, \epsilon_2) > 0$ and for each $(\overline{\omega}, \overline{\varpi}) \in \mathcal{C}_b \times \mathcal{C}_b$ satisfying*

$$\begin{cases} \left|{}^{c}\mathbb{D}_{a^+}^{\nu;\psi}\left[\overline{\omega}(\tau) - \sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k^*(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)\right] - \mathbb{H}_1(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)\right| \leq \epsilon_1, \ \tau \in \mathrm{J}, \\ |\overline{\omega}(\tau) - \varphi_1(\tau)| \leq \epsilon_1, \quad \tau \in [a-\delta, a], \\ \\ \left|{}^{c}\mathbb{D}_{a^+}^{\varsigma;\psi}\left[\overline{\varpi}(\tau) - \sum_{k=1}^{m}\mathbb{I}_{a^+}^{\xi_k;\psi}\mathbb{G}_k(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)\right] - \mathbb{H}_2(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)\right| \leq \epsilon_1, \ \tau \in \mathrm{J}, \\ |\overline{\varpi}(\tau) - \varphi_2(\tau)| \leq \epsilon_2, \quad \tau \in [a-\delta, a], \end{cases} \tag{3.17}$$

there exists a unique solution $(\omega,\varpi)\in\mathcal{C}_b\times\mathcal{C}_b$ *of* (1.2)–(1.3) *with*

$$\|(\overline{\omega},\overline{\varpi})-(\omega,\varpi)\|_{\mathcal{C}_b\times\mathcal{C}_b}\le c\epsilon.$$

Definition 3.12 *System* (1.2)–(1.3) *is generalized UH stable if there exists* $\sigma=\max(\sigma_1,\sigma_2)\in C(\mathbb{R}^+,\mathbb{R}^+)$ *with* $\sigma(0)=0$ *such that for each* $\epsilon=\max(\epsilon_1,\epsilon_2)>0$ *and for each* $(\overline{\omega},\overline{\varpi})\in\mathcal{C}_b\times\mathcal{C}_b$ *satisfying* (4.3), *there exists a unique solution* $(\omega,\varpi)\in\mathcal{C}_b\times\mathcal{C}_b$ *of* (1.2)–(1.3) *with*

$$\|(\overline{\omega},\overline{\varpi})-(\omega,\varpi)\|_{\mathcal{C}_b\times\mathcal{C}_b}\le\sigma(\epsilon).$$

Remark 3.13 *A solution* $(\overline{\omega},\overline{\varpi})\in\mathcal{C}_b\times\mathcal{C}_b$ *satisfying the system* (4.3) *if and only if there exists a function* $(\eta_1,\eta_2)\in\mathcal{C}_b\times\mathcal{C}_b$ *(which depend on* $(\overline{\omega},\overline{\varpi})$*) such that*

(i) $|\eta_1(\tau)|\le\epsilon_1,$ *and* $|\eta_2(\tau)|\le\epsilon_2,$ *for* $\tau\in\mathrm{J}$.

(ii) *For* $\tau\in\mathrm{J}$,

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\nu;\psi}\left[\overline{\omega}(\tau)-\sum_{k=1}^m\mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k^*(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)\right]=\mathbb{H}_1(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)+\eta_1(\tau),\\ {}^c\mathbb{D}_{a^+}^{\varsigma;\psi}\left[\overline{\varpi}(\tau)-\sum_{k=1}^m\mathbb{I}_{a^+}^{\xi_k;\psi}\mathbb{G}_k(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)\right]=\mathbb{H}_2(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)+\eta_1(\tau).\end{cases}$$

Theorem 3.14 *Suppose that* $(H2)$ *and* (4.3) *are fulfilled. Then, the solution of system* (1.2)–(1.3) *is UH and generalized UH stable, provided that* $(1-\Theta_1)\left(1-\overline{\Theta}_2\right)-\Theta_2\overline{\Theta}_1\neq0$.

Proof For $\epsilon_1,\epsilon_2>0$ and let $(\overline{\omega},\overline{\varpi})\in\mathcal{C}_b\times\mathcal{C}_b$ be any solution of (4.3). By Remark 3.13 and Lemma 3.2, we have

$$\begin{cases}\overline{\omega}(\tau)=\mathbb{I}_{a^+}^{\nu;\psi}\mathbb{H}_1(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)+\sum_{k=1}^m\mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k^*(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)+\mathbb{I}_{a^+}^{\nu;\psi}\eta_1(\tau),\\ \overline{\varpi}(\tau)=\mathbb{I}_{a^+}^{\varsigma;\psi}\mathbb{H}_2(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)+\sum_{k=1}^m\mathbb{I}_{a^+}^{\xi_k;\psi}\mathbb{G}_k(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)+\mathbb{I}_{a^+}^{\varsigma;\psi}\eta_2(\tau).\end{cases}\quad\tau\in\mathrm{J},\tag{3.18}$$

and

$$\begin{cases}\overline{\omega}(\tau)=\varphi_1(\tau),\\ \overline{\varpi}(\tau)=\varphi_2(\tau),\end{cases}\qquad\tau\in[a-\delta,a],$$

From (3.18), for $\tau\in\mathrm{J}$,

$$\begin{cases}\left|\overline{\omega}(\tau)-\mathbb{I}_{a^+}^{\nu;\psi}\mathbb{H}_1(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)-\sum_{k=1}^m\mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k^*(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)\right|\le\mathbb{I}_{a^+}^{\nu;\psi}|\eta_1(\tau)|\le\frac{(\psi(\tau)-\psi(a))^\nu}{\Gamma(\nu+1)}\epsilon_1,\\ \left|\overline{\varpi}(\tau)-\mathbb{I}_{a^+}^{\nu;\psi}\mathbb{H}_2(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)-\sum_{k=1}^m\mathbb{I}_{a^+}^{\xi_k;\psi}\mathbb{G}_k(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)\right|\le\mathbb{I}_{a^+}^{\nu;\psi}|\eta_2(\tau)|\le\frac{(\psi(\tau)-\psi(a))^\varsigma}{\Gamma(\varsigma+1)}\epsilon_2.\end{cases}\tag{3.19}$$

and for $\tau\in[a-\delta,a]$,

$$\begin{cases}|\overline{\omega}(\tau)-\varphi_1(\tau)|\le0,\\ |\overline{\varpi}(\tau)-\varphi_2(\tau)|\le0,\end{cases}$$

Let $(\omega,\varpi)\in\mathcal{C}_b\times\mathcal{C}_b$ be the solution of the system

$$\begin{cases}{}^c\mathbb{D}_{a^+}^{\nu;\psi}\left[\omega(\tau)-\sum_{k=1}^m\mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k^*(\tau,\omega_\tau,\varpi_\tau)\right]=\mathbb{H}_1(\tau,\omega_\tau,\varpi_\tau),\\ {}^c\mathbb{D}_{a^+}^{\varsigma;\psi}\left[\varpi(\tau)-\sum_{k=1}^m\mathbb{I}_{a^+}^{\xi_k;\psi}\mathbb{G}_k(\tau,\omega_\tau,\varpi_\tau)\right]=\mathbb{H}_2(\tau,\omega_\tau,\varpi_\tau),\end{cases}\quad\tau\in\mathrm{J},\tag{3.20}$$

with finite delay

$$\begin{cases}\omega(\tau)=\overline{\omega}(\tau)=\varphi_1(\tau), \\ \varpi(\tau)=\overline{\varpi}(\tau)=\varphi_2(\tau),\end{cases} \qquad \tau\in[a-\delta,a], \tag{3.21}$$

Thanks to Lemma 3.2, the equivalent fractional integral system to problem (3.20)–(3.21) is

$$\omega(\tau) \quad : \quad = \begin{cases} \mathbb{I}_{a^+}^{\nu,\psi}\mathbb{H}_1(\tau,\omega_\tau,\varpi_\tau)+\sum_{k=1}^m \mathbb{I}_{a^+}^{\sigma_k,\psi}\mathbb{F}_k^*(\tau,\omega_\tau,\varpi_\tau), & \text{if } \tau\in \mathrm{J}, \\ \varphi_1(\tau), & \text{if } \tau\in[a-\delta,a], \end{cases} \tag{3.22}$$

$$\varpi(\tau) \quad : \quad = \begin{cases} \mathbb{I}_{a^+}^{\varsigma,\psi}\mathbb{H}_2(\tau,\omega_\tau,\varpi_\tau)+\sum_{k=1}^m \mathbb{I}_{a^+}^{\xi_k,\psi}\mathbb{G}_k(\tau,\omega_\tau,\varpi_\tau), & \text{if } \tau\in \mathrm{J}, \\ \varphi_2(\tau), & \text{if } \tau\in[a-\delta,a]. \end{cases}$$

Since $\omega(\tau)=\overline{\omega}(\tau)$ and $\varpi(\tau)=\overline{\varpi}(\tau), |\overline{\omega}(\tau)-\omega(\tau)|=0$, and $|\overline{\varpi}(\tau)-\varpi(\tau)|=0$, for $\tau\in[a-\delta,a]$. On the other hand, for $\tau\in\mathrm{J}$, then by same arrguments in Theorem 3.9 with (3.19) and (3.22) we get

$$\begin{aligned}
|\overline{\omega}(\tau)-\omega(\tau)| &= \left|\overline{\omega}(\tau)-\mathbb{I}_{a^+}^{\nu,\psi}\mathbb{H}_1(\tau,\omega_\tau,\varpi_\tau)-\sum_{k=1}^m \mathbb{I}_{a^+}^{\sigma_k,\psi}\mathbb{F}_k^*(\tau,\omega_\tau,\varpi_\tau)\right| \\
&\le \left|\overline{\omega}(\tau)-\mathbb{I}_{a^+}^{\nu;\psi}\mathbb{H}_1(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)-\sum_{k=1}^m \mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k^*(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)\right| \\
&+ \left|\mathbb{I}_{a^+}^{\nu,\psi}\left|\mathbb{H}_1(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)-\mathbb{H}(\tau,\omega_\tau,\varpi_\tau)\right|\right. \\
&+ \left.\sum_{k=1}^m \mathbb{I}_{a^+}^{\sigma_k,\psi}\left|\mathbb{F}_k^*(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)-\mathbb{F}_k^*(\tau,\omega_\tau,\varpi_\tau)\right|\right| \\
&\le \frac{(\psi(\tau)-\psi(a))^\nu}{\Gamma(\nu+1)}\epsilon_1+\Theta_1\|\overline{\omega}-\omega\|_{\mathcal{C}_b}+\overline{\Theta}_1\|\overline{\varpi}-\varpi\|_{\mathcal{C}_b},
\end{aligned}$$

which implies

$$(1-\Theta_1)\|\overline{\omega}-\omega\|_{\mathcal{C}_b}-\overline{\Theta}_1\|\overline{\varpi}-\varpi\|_{\mathcal{C}_b}\le \mathcal{A}_1\epsilon_1, \tag{3.23}$$

where $\mathcal{A}_1:=\frac{(\psi(b)-\psi(a))^\nu}{\Gamma(\nu+1)}$. Similarly, we have

$$|\overline{\varpi}(\tau)-\varpi(\tau)|\le \frac{(\psi(\tau)-\psi(a))^\varsigma}{\Gamma(\varsigma+1)}\epsilon_2+\Theta_2\|\overline{\omega}-\omega\|_{\mathcal{C}_b}+\overline{\Theta}_2\|\overline{\varpi}-\varpi\|_{\mathcal{C}_b},$$

which gives

$$(1-\overline{\Theta}_2)\|\overline{\varpi}-\varpi\|_{\mathcal{C}_b}-\Theta_2\|\overline{\omega}-\omega\|_{\mathcal{C}_b}\le \mathcal{A}_2\epsilon_2, \tag{3.24}$$

where $\mathcal{A}_2:=\frac{(\psi(b)-\psi(a))^\varsigma}{\Gamma(\varsigma+1)}$.

Representing (3.23) and (3.24) as matrices as follows:

$$\begin{pmatrix} 1-\Theta_1 & -\overline{\Theta}_1 \\ -\Theta_2 & 1-\overline{\Theta}_2 \end{pmatrix}\begin{pmatrix} \|\overline{\omega}-\omega\|_{\mathcal{C}_b} \\ \|\overline{\varpi}-\varpi\|_{\mathcal{C}_b} \end{pmatrix}\le\begin{pmatrix} \mathcal{A}_1\epsilon_1 \\ \mathcal{A}_2\epsilon_2 \end{pmatrix}. \tag{3.25}$$

After straightforward calculations of (3.25), we find that

$$\|\overline{\omega}-\omega\|_{\mathcal{C}_b}\le \frac{1-\Theta_1}{\Delta}\mathcal{A}_1\epsilon_1+\frac{\overline{\Theta}_1}{\Delta}\mathcal{A}_2\epsilon_2, \tag{3.26}$$

$$\|\overline{\varpi}-\varpi\|_{\mathcal{C}_b} \leq \frac{\Theta_2}{\Delta}\mathcal{A}_1\epsilon_1 + \frac{1-\overline{\Theta}_2}{\Delta}\mathcal{A}_2\epsilon_2, \tag{3.27}$$

where $\Delta = (1-\Theta_1)\left(1-\overline{\Theta}_2\right) - \Theta_2\overline{\Theta}_1 \neq 0$. By collecting (3.26) and (3.27), we obtain

$$\|\overline{\varpi}-\omega\|_{\mathcal{C}_b} + \|\overline{\varpi}-\varpi\|_{\mathcal{C}_b} \leq \left(\frac{1-\Theta_1}{\Delta}+\frac{\Theta_2}{\Delta}\right)\mathcal{A}_1\epsilon_1 + \left(\frac{\overline{\Theta}_1}{\Delta}+\frac{1-\overline{\Theta}_2}{\Delta}\right)\mathcal{A}_2\epsilon_2.$$

For $\epsilon = \max(\epsilon_1, \epsilon_2)$ and $c = \left(\frac{(1-\Theta_1+\Theta_2)\mathcal{A}_1 + \left(\overline{\Theta}_1+1-\overline{\Theta}_2\right)\mathcal{A}_2}{\Delta}\right)$, we get

$$\|(\overline{\omega},\overline{\varpi})-(\omega,\varpi)\|_{\mathcal{C}_b\times\mathcal{C}_b} = \|\overline{\varpi}-\omega\|_{\mathcal{C}_b} + \|\overline{\varpi}-\varpi\|_{\mathcal{C}_b} \leq c\epsilon.$$

Therefore, according to Definition 3.11, the solution of problem (1.2)–(1.3) is UH stable. Similarly, it shows the existence of a function $\sigma \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ such that $\sigma(\epsilon) = c\epsilon$ with $\sigma(0) = 0$. Therefore, the solution of system (1.2)–(1.3) is generalized UH stable. □

4. Examples

In this section, in order to justify our results, we consider two examples.

Example 4.1 *Let us consider problem (1.1) with specific data:*

$$\psi(\tau) = \ln(\tau), \quad a = 1, \quad b = e, \quad \nu = 1/2, \quad \sigma_1 = 1/4, \quad \xi_1 = 5/2.$$

and $\mathbb{H}(\tau,u)$, $\mathbb{F}_k(\tau,u)$ are fixed below.

Using the given data, we find that $\Lambda_1 = 0.1128, \Lambda_2 = 0.1103$, where Λ_1 and Λ_2 are, respectively, given by (3.1). Let us take

$$\mathbb{F}_k(\tau,u) = \frac{\ln(\tau)\cos u}{\sqrt{100+\ln(\tau)}}, \quad \mathbb{H}(\tau,u) = \frac{1}{10\left(1+\ln(\tau)\right)}\sin u + \frac{\ln(\tau)}{25\tau}, \tag{4.1}$$

and note that $\mathbb{H}(\tau,u)$ and $\mathbb{F}_k(\tau,u)$ satisfy the hypothesis of Theorem 3.3 with $\mathbb{F}_k(0,u_0) = 0$, and $\upsilon_1(\tau) = \frac{1}{10(1+\ln(\tau))} + \frac{1}{25}, \mu(\tau) = \frac{\ln(\tau)}{\sqrt{100+\ln(\tau)}}$ and $\varphi = 1/10$. In addition, $\frac{(\psi(b)-\psi(a))^\nu}{\Gamma(\nu+1)}\varphi \approx 0.1128 < 1$. Thus, the conclusion of Theorem 3.4 applies to the problem in (1.1) with $\mathbb{H}(\tau,u)$ and $\mathbb{F}_k(\tau,u)$ given by (4.1).

To explain Theorem 3.3, let us take $\mathbb{H}(\tau,u)$ and $\mathbb{F}_k(\tau,u)$ given by (4.1). Clearly, the conditions (G1) and (G2) hold with $\|\upsilon_1\| = 1/10$ and $\|\mu\| = 1/10$. In addition, $\Delta \approx 0.2231 < 1$. Hence, all hypotheses of Theorem 3.3 are satisfied. So, the problem (1.1) has an existence of a unique solution on $[1,e]$ with $\mathbb{H}(\tau,u)$ and $\mathbb{F}_k(\tau,u)$ given by (4.1).

Example 4.2 *Consider the following coupled system:*

$$\begin{cases} \mathbb{D}^{\frac{1}{2},e^\tau}\left[u(\tau) - \mathbb{I}^{\frac{1}{3},e^\tau} e^{(\tau-1)}\left(\frac{|u_\tau|}{2(4+e^\tau)(1+|u_\tau|)} + \frac{|v_\tau|}{9+\sqrt{e^\tau}}\right)\right] = \frac{\sqrt{3}|u_\tau|\cos^2(2\pi\tau)}{3(27-e^\tau)} + \frac{\sqrt{2}\pi|v_\tau|}{e^\tau(7\pi-\tau)^2}\left(\frac{|v_\tau|}{|v_\tau|+3}+1\right), \\ \tau \in \mathrm{J} = [0,1], \\ \mathbb{D}^{\frac{1}{4},e^\tau}\left[u(\tau) - \mathbb{I}^{\frac{2}{3},e^\tau}\frac{1}{1+e^{\tau^2}}\left(\frac{|u_\tau|}{8(1+|u_\tau|)} + \frac{|u_\tau^2|}{10(1+|u_\tau|)}\right)\right] = \frac{\sqrt{2}\pi|u_\tau|}{164e^{\tau^2}}\left(\frac{|u_\tau|}{|u_\tau|+3}+1\right) + \frac{|v_\tau|\sin^2(2\pi\tau)}{(9+e^\tau)}, \\ \tau \in \mathrm{J} = [0,1], \\ u(\tau) = \varphi_1(\tau), \quad v(\tau) = \varphi_2(\tau), \quad \tau \in \mathrm{J} = [-\delta, 0]. \end{cases} \tag{4.2}$$

Here

$$\psi(\tau)=e^{\tau}\,,\quad a=0\,,\quad b=1\,,\quad \nu=1/2\,,\quad \varsigma=1/4\,,\quad \sigma_1=1/3\,,\quad \xi_1=2/3\,,\quad \delta=1/5\,.$$

Using the given data, we find that $\mathbb{F}_k^*(0,u_0,v_0)=\mathbb{G}_k(0,u_0,v_0)=0$, *and*

$$\begin{aligned}
|\mathbb{H}_1(\tau,u,v)-\mathbb{H}_1(\tau,\overline{u},\overline{v})| &\le \tfrac{\sqrt{3}\cos^2(2\pi\tau)}{3(27-e^{\tau})}\|u-\overline{u}\|_{\mathcal{C}_\delta}+\tfrac{\sqrt{2}\pi}{e^{\tau}(7\pi-\tau)^2}\|v-\overline{v}\|_{\mathcal{C}_\delta}\,,\\
|\mathbb{H}_2(\tau,u,v)-\mathbb{H}_2(\tau,\overline{u},\overline{v})| &\le \tfrac{\sqrt{2}\pi}{164e^{\tau^2}}\|u-\overline{u}\|_{\mathcal{C}_\delta}+\tfrac{\sin^2(2\pi\tau)}{(9+e^{\tau})}\|v-\overline{v}\|_{\mathcal{C}_\delta}\,,\\
|\mathbb{F}_1^*(\tau,u,v)-\mathbb{F}_1^*(\tau,\overline{u},\overline{v})| &\le \tfrac{1}{2(4+e^{\tau})}\|u-\overline{u}\|_{\mathcal{C}_\delta}+\tfrac{\tau}{9+\sqrt{e^{\tau}}}\|v-\overline{v}\|_{\mathcal{C}_\delta}\,,\\
|\mathbb{G}_1(\tau,u,v)-\mathbb{G}_1(\tau,\overline{u},\overline{v})| &\le \tfrac{1}{8(1+e^{\tau^2})}\|u-\overline{u}\|_{\mathcal{C}_\delta}+\tfrac{\tau}{10(1+e^{\tau^2})}\|v-\overline{v}\|_{\mathcal{C}_\delta}\,.
\end{aligned}$$

Hence, the hypothesis (H2) is satisfied with $\|L_1\|=\frac{\sqrt{3}}{78}$, $\|K_1\|=\frac{\sqrt{2}}{49\pi}$, $\|L_2\|=\frac{\sqrt{2}\pi}{164}$, $\|K_2\|=\frac{1}{10}$, $\|\omega_{1,1}\|=\frac{1}{10}$, $\|\varpi_{1,1}\|=\frac{1}{10}$, $\|\omega_{1,2}\|=\frac{1}{16}$ *and* $\|\varpi_{1,2}\|=\frac{1}{20}$. *We shall show that condition* (3.7) *holds with* $\mathrm{J}=[0,1]$. *Indeed, for any* $\tau\in[0,1]$. *Then,* $\mathbb{H}_i, i=1,2$, $\mathbb{F}_1^*$ *and* $\mathbb{G}_1$ *satisfying* $(H1)$ *and* $(H2)$. *We find that*

$$\Theta_1=0.1670,\quad \Theta_2=0.1335,$$

$$\overline{\Theta}_1=0.1477,\quad \overline{\Theta}_2=0.2058.$$

Hence $\Omega=0.3728<1$.
Then, the conditions of Theorem 3.9 are satisfied. Then, there exists a unique solution for (4.2) in $[0,1]$. *Moreover, Theorem 3.14 ensures the UH and generalized UH stability for problem (4.2). Furthermore, as shown in Theorem 3.14, for every* $\epsilon=\max(\epsilon_1,\epsilon_2)>0$, *if* $(\overline{\omega},\overline{\varpi})\in\mathcal{C}_b\times\mathcal{C}_b$ *satisfies*

$$\begin{cases}\left|\,{}^{c}\mathbb{D}_{a^+}^{\nu;\psi}\left[\overline{\omega}(\tau)-\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k^*(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)\right]-\mathbb{H}_1(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)\right|\le\epsilon_1,\\[2mm] \left|\,{}^{c}\mathbb{D}_{a^+}^{\varsigma;\psi}\left[\overline{\varpi}(\tau)-\sum_{k=1}^{m}\mathbb{I}_{a^+}^{\xi_k;\psi}\mathbb{G}_k(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)\right]-\mathbb{H}_2(\tau,\overline{\omega}_\tau,\overline{\varpi}_\tau)\right|\le\epsilon_1.\end{cases}\tag{4.3}$$

there exists a unique solution $(\varpi,\omega)\in\mathcal{C}_b\times\mathcal{C}_b$ *such that*

$$\|(\overline{\omega},\overline{\varpi})-(\omega,\varpi)\|_{\mathcal{C}_b\times\mathcal{C}_b}\le c\epsilon.$$

where $c=\left(\frac{(1-\Theta_1+\Theta_2)\mathcal{A}_1+(\overline{\Theta}_1+1-\overline{\Theta}_2)\mathcal{A}_2}{\Delta}\right)\approx 4.2295>0.$

$\mathcal{A}_1=1.4791$, $\mathcal{A}_2=1.2631$ *and* $(1-\Theta_1)\left(1-\overline{\Theta}_2\right)-\Theta_2\overline{\Theta}_1=0.6419\neq 0$. *Hence coupled system (4.2) is UH and generalized UH stable.*

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