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Some results on a system of multiterm fractional integro-differential equations

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Abstract: In present study, we investigate the existence of solution for a multiterm fractional integro-differential system with special boundary conditions under some different conditions. In this way, we provide different results for the existence of solutions for the system and also for obtaining unique solution for the system under different conditions. We also present three numerical examples in which by using the Legendre multiwavelet method, we approximate solutions of the system. These examples illustrate our main results.

Key words: Caputo derivation, Legendre multiwavelet, system of multiterm fractional integro-differential equations

1. Introduction

It is known that fractional calculus can describe most natural phenomena and has been published a lot of works in this field from analytical view up to applied one (see for example, [1], [4], [5], [7]-[27], [29]–[37], [39], [40], [42]–[46], [49] and [51]). Some researchers have been considered multiterm fractional differential equations (see for example, [28] and [38]). It is known that the Caputo fractional derivative is defined by ${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(t)}{(t-s)^{\alpha-n+1}} dt$, where $n-1 < \alpha \leq n$ ([41]). Also, the fractional integral of order α is defined by $I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds$, when $\alpha > 0$ ([41]). We need the following results.

Lemma 1.1 *([41]) Let $\alpha > 0$ and $n = \lceil \alpha \rceil + 1$. Then, $I^{\alpha c}D^\alpha f(t) = f(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, where $c_0, c_1, \ldots, c_{n-1}$ are some real numbers.*

Theorem 1.2 *([48]) Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the Operators such that (i) $Ax + By \in M$ whenever $x, y \in M$, (ii) A is compact and continuous, (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

In 2011, Ahmed and Nieto studied the existence of solution for the fractional equation ${}^cD^q x(t) = f(t, x(t))$ with boundary conditions $x(0) = -x(T)$, ${}^cD^p x(0) = -{}^cD^p x(T)$, where $t \in [0, T]$, $T > 0$, $1 < q \leq 2$, $0 < p < 1$ and f is a continuous function ([2]). In 2013, Wang, Guo and Tang reviewed the antiperiodic fractional

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boundary value problem ${}^cD^\alpha x(t) = f(t, x(t))$ with boundary condition $x(0) = -x(T)$, ${}^cD^p x(0) = -{}^cD^p x(T)$ and ${}^cD^q x(0) = -{}^cD^q x(T)$, where $t \in J = [0, T]$, $T > 0$, $2 < \alpha \le 3$, $0 < p < 1 < q < 2$ and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function ([50]). By use and mixing main idea of [2], [28], [38] and [50]), we investigate the existence of solutions for the k-dimensional system of fractional integro-differential equations

$$\begin{cases} {}^cD^{\alpha_1} x_1(t) = f_1\Big(t, x_1(t), \dots, x_k(t), I^{\beta_{11}} x_1(t), ..., I^{\beta_{1k}} x_k(t), {}^cD^{\gamma_{11}} x_1(t), \dots, {}^cD^{\gamma_{1k}} x_k(t)\Big), \\ {}^cD^{\alpha_2} x_2(t) = f_2\Big(t, x_1(t), \dots, x_k(t), I^{\beta_{21}} x_1(t), ..., I^{\beta_{2k}} x_k(t), {}^cD^{\gamma_{21}} x_1(t), \dots, {}^cD^{\gamma_{2k}} x_k(t)\Big), \\ \vdots \\ {}^cD^{\alpha_k} x_k(t) = f_k\Big(t, x_1(t), \dots, x_k(t), I^{\beta_{k1}} x_1(t), ..., I^{\beta_{kk}} x_k(t), {}^cD^{\gamma_{k1}} x_1(t), \dots, {}^cD^{\gamma_{kk}} x_k(t)\Big), \end{cases}$$

with boundary condition $x_i(0) + x_i(1) = \sum_{j=1}^k a_i x_i(\xi_j)$ and

$$\sum_{j=1}^k {}^cD^{\gamma_{ij}} x_i(0) + \sum_{j=1}^k {}^cD^{\gamma_{ij}} x_i(1) = \sum_{j=1}^k b_i x_i'(\eta_j),$$

where a_i, $b_i \in \mathbb{R}$, $1 \ne b_i\Gamma(2 - \gamma_{ij})$, $a_i \ne 2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_k < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_k < 1$, $1 < \alpha_i \le 2$, $0 < \gamma_{ij} \le 1$, $\beta_{ij} > 0$ for $i = 1, \dots, k$, $t \in I = [0, 1]$ and $f_1, \dots, f_k \in C(I \times \mathbb{R}^{3k}, \mathbb{R})$. Consider the Banach spaces

$$X = \{x : \ x \in C(I) \ and \ {}^cD^{\gamma_{ij}} x \in C(I) \ for \ i, j = 1, 2, \dots, k\}$$

endowed with the norm $\|x\| = max_{t\in I}|x(t)| + max_{t\in I}|{}^cD^\gamma x(t)|$ and $X^k = X \times X \times \cdots \times X$ endowed with the norm $\| (x_1, x_2, \dots, x_k) \|_* = \| x_1 \| + \| x_2 \| + \cdots + \| x_k \|$.

2. Main results

Now, we are ready to state and prove our main results.

Lemma 2.1 *Let $y \in C(I)$, $1 \ne b\Gamma(2 - \gamma_j)$, $a \ne 2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_k < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_k < 1$, $1 < \alpha \le 2$ and $0 < \gamma_j \le 1$ for $j = 1, \dots, k$. Then the problem ${}^cD^\alpha x(t) = y(t)$ with boundary conditions $x(0) + x(1) = a \sum_{j=1}^k x(\xi_j)$ and $\sum_{j=1}^k {}^cD^{\gamma_j} x(0) + \sum_{j=1}^k {}^cD^{\gamma_j} x(1) = b \sum_{j=1}^k x'(\eta_j)$ has the unique solution*

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{1}{(a-2)\Gamma(\alpha)} (\int_0^1 (1-s)^{\alpha-1} y(s) - a \sum_{j=1}^k \int_0^{\xi_j} (\xi_j - s)^{\alpha-1} y(s) ds)$$

$$- \sum_{j=1}^k \frac{\Gamma(2-\gamma_j)(1 - a\xi_j + t(a-2))}{(a-2)(1 - b\Gamma(2-\gamma_j))} (\frac{1}{\Gamma(\alpha-\gamma_j)} \int_0^1 (1-s)^{\alpha-\gamma_j-1} y(s) ds - \frac{b}{\Gamma(\alpha-1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha-2} y(s) ds).$$

Proof By using some calculations, one can check that the given x is a solution for the problem satisfying the boundary conditions. On the other hand by using Lemma 1.1, there exist $b_1, b_2 \in \mathbb{R}$ such that

$$x(t) = I^{\alpha} y(t) - b_1 - b_2 t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds - b_1 - b_2 t$$

for all $t \in I$. Thus, ${}^cD^{\gamma_j} x(t) = \frac{1}{\Gamma(\alpha-\gamma_j)} \int_0^t (t-s)^{\alpha-\gamma_j-1} y(s) ds - b_2 \frac{t^{1-\gamma_j}}{\Gamma(2-\gamma_j)}$ and

$$x'(\eta_j) = \frac{1}{\Gamma(\alpha-1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha-2} y(s) ds - b_2.$$

Hence, $x(0) + x(1) = a \sum_{j=1}^k x(\xi_j)$ and $\sum_{j=1}^k {}^cD^{\gamma_j} x(0) + \sum_{j=1}^k {}^cD^{\gamma_j} x(1) = b \sum_{j=1}^k x'(\eta_j)$ and so

$$b_2 = -\sum_{j=1}^{k} \frac{\Gamma(2-\gamma_j)}{(1 - b\Gamma(2-\gamma_j))} (\frac{1}{\Gamma(\alpha-\gamma_j)} \int_0^1 (1-s)^{\alpha-\gamma_j-1} y(s) ds - \frac{b}{\Gamma(\alpha-1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha-2} y(s) ds)$$

and

$$b_1 = \sum_{j=1}^{k} \frac{a}{(a-2)\Gamma(\alpha)} \int_0^{\xi_j} (\xi_j - s)^{\alpha-1} y(s) ds - \frac{1}{(a-2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds + \sum_{j=1}^{k} \frac{(1-a\xi)\Gamma(2-\gamma_j)}{(1-b\Gamma(2-\gamma_j))(a-2)}$$

$$(\frac{1}{\Gamma(\alpha-\gamma_j)} \int_0^1 (1-s)^{\alpha-\gamma_j-1} y(s) ds - \frac{b}{\Gamma(\alpha-1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha-2} y(s) ds).$$

Therefore, we obtain

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{1}{(a-2)\Gamma(\alpha)} (\int_0^1 (1-s)^{\alpha-1} y(s) - a \sum_{j=1}^{k} \int_0^{\xi_j} (\xi_j - s)^{\alpha-1} y(s) ds)$$

$$-\sum_{j=1}^{k} \frac{\Gamma(2-\gamma_j)(1 - a\xi_j + t(a-2))}{(a-2)(1-b\Gamma(2-\gamma_j))} (\frac{1}{\Gamma(\alpha-\gamma_j)} \int_0^1 (1-s)^{\alpha-\gamma_j-1} y(s) ds - \frac{b}{\Gamma(\alpha-1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha-2} y(s) ds).$$

This completes the proof. □

Define the operator $E : X^k \to X^k$ by $Ex(t) = \begin{pmatrix} E_1 x(t) \\ E_2 x(t) \\ \vdots \\ E_k x(t) \end{pmatrix}$, where $x = (x_1, x_2, \ldots, x_k)$ and

$$E_i x(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} y(s) ds + \frac{1}{(a_i-2)\Gamma(\alpha_i)} (\int_0^1 (1-s)^{\alpha_i-1} y(s) - a_i \sum_{j=1}^{k} \int_0^{\xi_j} (\xi_j - s)^{\alpha_i-1} y(s) ds)$$

$$-\sum_{j=1}^{k} \frac{\Gamma(2-\gamma_{ij})(1 - a_i\xi_j + t(a-2))}{(a-2)(1-b_i\Gamma(2-\gamma_{ij}))} (\frac{1}{\Gamma(\alpha_i-\gamma_{ij})} \int_0^1 (1-s)^{\alpha_i-\gamma_{ij}-1} y(s) ds - \frac{b_i}{\Gamma(\alpha_i-1)} \int_0^{\eta_j} (\eta_j - s)^{\alpha-i-2} y(s) ds).$$

Observe that the main problem has a solution u_0 if and only if u_0 is a fixed point of the operator E. Put

$$\Delta_{i1} = \sum_{j=1}^{k} (a_i - 1 + a_i \xi_j^{\alpha_i})(1 - b_i \Gamma(2 - \gamma_{ij}))\Gamma(\alpha_i - \gamma_{ij} + 1),$$

$$\Delta_{i2} = \sum_{j=1}^{k} \Gamma(2 - \gamma_{ij})(3 + a_i \xi_j + a_i)(\Gamma(\alpha_i + 1) + b_i \eta_j^{\alpha_i} \Gamma(\alpha_i - \gamma_{ij} + 1)),$$

$$\Omega_{i1} = \sum_{j=1}^{k} (a_i - 1 + a_i \xi_j^{\alpha_i - \gamma_{ij}})(1 - b_i \Gamma(2 - \gamma_{ij}))\Gamma(\alpha_i - 2\gamma_{ij} + 1)$$

and $\Omega_{i2} = \sum_{j=1}^{k} \Gamma(2 - \gamma_{ij})(3 + a_i \xi_j + a_i)(\Gamma(\alpha_i - \gamma_{ij} + 1) + b_i \eta_j^{\alpha_i - \gamma_{ij}} \Gamma(\alpha_i - 2\gamma_{ij} + 1))$ for $i = 1, \ldots, k$.

Theorem 2.2 *Suppose that $f_1, \ldots, f_k : I \times \mathbb{R}^{3k} \to \mathbb{R}$ are continuous and there exist $l_1, \ldots, l_k \in C([0,1], (0,\infty))$, nondecreasing maps $\lambda_1, \ldots, \lambda_k \in C([0,\infty), (0,\infty))$ and a positive constant K such that*

$$|f_i(t, x_1(t), \ldots, x_k(t), z_1(t), \ldots, z_k(t), y_1(t), \ldots, y_k(t))| \leq \sum_{j=1}^{k} l_i(t)(\lambda_i(|x_j(t)| + |y_j(t)|)),$$

$$|f_i(t, x_1(t), \ldots, x_{3k}(t)) - f_i(t, y_1(t), \ldots, y_{3k}(t))| \leq K \sum_{j=1}^{2k} |x_j(t) - y_j(t)|$$

for all $t \in I$ and

$$K \leq \sum_{j=1}^{k} (\frac{\Delta_{i1} + \Delta_{i2}}{2(a_i - 2)\Gamma(\alpha_i + 1)(1 - b\Gamma(2 - \gamma_{ij}))\Gamma(\alpha_i - \gamma_{ij} + 1)}$$

$$+ \frac{\Omega_{i1} + \Omega_{i2}}{2(a_i - 2)\Gamma(\alpha_i - \gamma_{ij} + 1)(1 - b_i\Gamma(2 - \gamma_{ij}))\Gamma(\alpha_i - 2\gamma_{ij} + 1)})^{-1}.$$

Then the main problem has a solution.

Proof Choose a real number

$$r \geq \sum_{j=1}^{k} 2M(\frac{\Delta_{i1} + \Delta_{i2}}{(a_i - 2)\Gamma(\alpha_i + 1)(1 - b_i\Gamma(2 - \gamma_{ij}))\Gamma(\alpha_i - \gamma_{ij} + 1)}$$

$$+ \frac{\Omega_{i1} + \Omega_{i2}}{(a_i - 2)\Gamma(\alpha_i - \gamma_{ij} + 1)(1 - b_i\Gamma(2 - \gamma_{ij}))\Gamma(\alpha_i - 2\gamma_{ij} + 1)}).$$

Let $B_r = \{x = (x_1, x_2, \ldots, x_k) \in X^k : \|x\| \leq r\}$ and $M = sup_{t\in[0,1]} \|f(t, 0, \ldots, 0)\|$. We show that $E(B_r) \subset B_r$. For each $x = (x_1, x_2, \ldots, x_k) \in B_r$, we have

$$|E_i x(t)| = |\frac{1}{\Gamma(\alpha_i)} \int_0^t (t - s)^{\alpha_i - 1} f_i(s, x_1, \ldots, x_k, I^{\beta_{i1}} x_1, \ldots, I^{\beta_{ik}} x_k, {}^cD^{\gamma_{i1}} x_1, \ldots, {}^cD^{\gamma_{ik}} x_k) ds$$

$$+\frac{1}{(a_i-2)\Gamma(\alpha_i)}(\int_0^1(1-s)^{\alpha_i-1}f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)ds$$

$$-a_i\sum_{j=1}^k\int_0^{\xi_j}(\xi_j-s)^{\alpha-1}f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)ds)$$

$$-\sum_{j=1}^k\frac{\Gamma(2-\gamma_{ij})(1-a_i\xi_j+t(a_i-2))}{(a-2)(1-b_i\Gamma(2-\gamma_{ij}))}(\frac{1}{\Gamma(\alpha_i-\gamma_{ij})}$$

$$\int_0^1(1-s)^{\alpha_i-\gamma_{ij}-1}f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)ds$$

$$-\frac{b_i}{\Gamma(\alpha_i-1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_i-2}f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)ds)|$$

$$\leq\frac{1}{\Gamma(\alpha_i)}\int_0^t(t-s)^{\alpha_i-1}(|f_i(s,x_1,\dots,x_kI^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)-f_i(s,0,\dots,0)|+$$

$$|f_i(s,0,\dots,0)|)ds+\frac{1}{(a_i-2)\Gamma(\alpha_i)}(\int_0^1(1-s)^{\alpha_i-1}(|f_i(x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)$$

$$-f_i(s,0,\dots,0)|+|f_i(s,0,\dots,0)|)ds+a_i\sum_{j=1}^k\int_0^{\xi_j}(\xi_j-s)^{\alpha_i-1}(|f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1$$

$$,\dots,{}^cD^{\gamma_{ik}}x_k)+f_i(s,0,\dots,0)|+|f_i(s,0,\dots,0)|)ds)+\sum_{j=1}^k\frac{\Gamma(2-\gamma_{ij})(1-a_i\xi_j+t(a_i-2))}{(a_i-2)(1-b_i\Gamma(2-\gamma_{ij}))}$$

$$(\frac{1}{\Gamma(\alpha_i-\gamma_{ij})}\int_0^1(1-s)^{\alpha_i-\gamma_{ij}-1}(|f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)-f_i(s,0,\dots,0)|$$

$$+|f_i(s,0,\dots,0)|)ds+\frac{b_i}{\Gamma(\alpha_i-1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_i-2}(|f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)$$

$$-f_i(s,0,\dots,0)|+|f_i(s,0,\dots,0)|)ds)\leq\frac{1}{\Gamma(\alpha_i)}\int_0^t(t-s)^{\alpha_i-1}(K\|x\|+M)ds+\frac{1}{(a_i-2)\Gamma(\alpha_i)}(\int_0^1(1-s)^{\alpha_i-1}$$

$$(K\|x\|+M)ds+a\sum_{j=1}^k\int_0^{\xi_j}(\xi_j-s)^{\alpha_i-1}(K\|x\|+M)ds)+\sum_{j=1}^k\frac{\Gamma(2-\gamma_{ij})(1-a_i\xi_j+t(a_i-2))}{(a_i-2)(1-b_i\Gamma(2-\gamma_{ij}))}(\frac{1}{\Gamma(\alpha_i-\gamma_{ij})}$$

$$\int_0^1(1-s)^{\alpha_i-\gamma_{ij}-1}(K\|x\|+M)ds+\frac{b_i}{\Gamma(\alpha_i-1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_i-2}(K\|x\|+M)ds)\leq(K\|x\|+M)$$

$$[\frac{1}{\Gamma(\alpha_i+1)}+\sum_{j=1}^k(\frac{1+a_i\xi_j}{(a_i-2)\Gamma(\alpha_i+1)}+\frac{\Gamma(2-\gamma_{ij})(3+a_i\xi_j+a_i)}{(a_i-2)(1-b_i\Gamma(2-\gamma_{ij}))}(\frac{1}{\Gamma(\alpha_i-\gamma_{ij})+1}+\frac{b_i\eta_j^{\alpha_i}}{\Gamma(\alpha_i+1)}))].$$

Also, we have

$$|{}^cD^{\gamma_{ij}}T_ix(t)| = |\frac{1}{\Gamma(\alpha_i-\gamma_{ij})}\int_0^t (t-s)^{\alpha_i-\gamma_{ij}-1} f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)ds$$

$$+\frac{1}{(a_i-2)\Gamma(\alpha_i-\gamma_{ij})}(\int_0^1 (1-s)^{\alpha_i-1-\gamma_{ij}} f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)ds$$

$$-a\sum_{j=1}^k \int_0^{\xi_j} (\xi_j-s)^{\alpha-1-\gamma_{ij}} f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)ds)$$

$$-\sum_{j=1}^k \frac{\Gamma(2-\gamma_{ij})(1-a_i\xi_j+t(a_i-2))}{(a_i-2)(1-b_i\Gamma(2-\gamma_{ij}))}(\frac{1}{\Gamma(\alpha_i-2\gamma_{ij})}\int_0^1 (1-s)^{\alpha_i-2\gamma_{ij}-1} f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,$$

$${}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)ds - \frac{b_i}{\Gamma(\alpha_i-1-\gamma_{ij})}\int_0^{\eta_j} (\eta_j-s)^{\alpha_i-2-\gamma_{ij}} f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,$$

$${}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)ds)| \le \frac{1}{\Gamma(\alpha_i-\gamma_{ij})}\int_0^t (t-s)^{\alpha_i-1}(|f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,$$

$${}^cD^{\gamma_{ik}}x_k) - f_i(s,0,\dots,0)| + |f_i(s,0,\dots,0)|)ds + \frac{1}{(a_i-2)\Gamma(\alpha_i-\gamma_{ij})}(\int_0^1 (1-s)^{\alpha_i-\gamma_{ij}-1}(|f_i(s,x_1,\dots,x_k,$$

$$I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k) - f_i(s,0,\dots,0)| + |f_i(s,0,\dots,0)|)ds + a_i\sum_{j=1}^k \int_0^{\xi_j} (\xi_j-s)^{\alpha_i-\gamma_{ij}-1}$$

$$(|f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k) + f_i(s,0,\dots,0)| + |f_i(s,0,\dots,0)|)ds)$$

$$+\sum_{j=1}^k \frac{\Gamma(2-\gamma_{ij})(1-a_i\xi_j+t(a_i-2))}{(a_i-2)(1-b_i\Gamma(2-\gamma_{ij}))}(\frac{1}{\Gamma(\alpha_i-2\gamma_{ij})}\int_0^1 (1-s)^{\alpha_i-2\gamma_{ij}-1}(|f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,$$

$$I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k) - f_i(s,0,\dots,0)| + |f_i(s,0,\dots,0)|)ds + \frac{b_i}{\Gamma(\alpha_i-\gamma_{ij}-1)}\int_0^{\eta_j} (\eta_j-s)^{\alpha_i-\gamma_{ij}-2}$$

$$(|f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k) - f_i(s,0,\dots,0)| + |f_i(s,0,\dots,0)|)ds)$$

$$\le (K\|x\|+M)[\frac{1}{\Gamma(\alpha_i-\gamma_{ij})}\int_0^t (t-s)^{\alpha_i-\gamma_{ij}-1}ds + \frac{1}{(a_i-2)\Gamma(\alpha_i)}(\int_0^1 (1-s)^{\alpha_i-1-\gamma_{ij}}ds$$

$$-a_i\sum_{j=1}^k \int_0^{\xi_j} (\xi_j-s)^{\alpha_i-1-\gamma_{ij}}ds) - \sum_{j=1}^k \frac{\Gamma(2-\gamma_{ij})(1-a_i\xi_j+t(a_i-2))}{(a_i-2)(1-b_i\Gamma(2-\gamma_{ij}))}(\frac{1}{\Gamma(\alpha_i-2\gamma_{ij})}\int_0^1 (1-s)^{\alpha_i-2\gamma_{ij}-1}ds$$

$$-\frac{b_i}{\Gamma(\alpha_i-1-\gamma_{ij})}\int_0^{\eta_j} (\eta_j-s)^{\alpha_i-2-\gamma_{ij}}ds)] \le (Kr+M)[\frac{1}{\Gamma(\alpha_i-\gamma_{ij}+1)} + \sum_{j=1}^k (\frac{1+a_i\xi_j^{\alpha_i-\gamma_{ij}}}{(a-2)\Gamma(\alpha_i-\gamma_{ij}+1)}$$

$$+\frac{\Gamma(2-\gamma_{ij})(3+a_i\xi_j+a_i)}{(a_i-2)(1-b_i\Gamma(2-\gamma_{ij}))}(\frac{1}{\Gamma(\alpha_i-2\gamma_{ij}+1)} + \frac{b_i\eta_j^{\alpha_i-\gamma_{ij}}}{\Gamma(\alpha_i-\gamma_{ij}+1)})].$$

Thus,

$$\|Ex(t)\|_* = \|E(x_1,x_2,x_k)(t)\|_* \leq \sum_{j=1}^{k}(Kr+M)(\frac{\Delta_{i1}+\Delta_{i2}}{(a_i-2)\Gamma(\alpha_i+1)(1-b_i\Gamma(2-\gamma_{ij}))\Gamma(\alpha_i-\gamma_{ij}+1)}$$

$$+\frac{\Omega_{i1}+\Omega_{i2}}{(a_i-2)\Gamma(\alpha_i-\gamma_{ij}+1)(1-b_i\Gamma(2-\gamma_{ij}))\Gamma(\alpha_i-2\gamma_{ij}+1)}) \leq r(\frac{1}{2})+\frac{r}{2}=r$$

and so $E(B_r)\subseteq B_r$. Now, define the operators F_i and G_i on B_r by

$$F_ix(t)=\frac{1}{\Gamma(\alpha_i)}\int_0^t(t-s)^{\alpha_i-1}f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)ds$$

and

$$G_ix(t)=\frac{1}{(a-2)\Gamma(\alpha_i)}(\int_0^1(1-s)^{\alpha_i-1}f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)ds$$

$$-a\sum_{j=1}^{k}\int_0^{\xi_j}(\xi_j-s)^{\alpha-1}f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)ds)$$

$$-\sum_{j=1}^{k}\frac{\Gamma(2-\gamma_{ij})(1-a\xi_j+t(a-2))}{(a-2)(1-b\Gamma(2-\gamma_{ij}))}$$

$$\times(\frac{1}{\Gamma(\alpha_i-\gamma_{ij})}\int_0^1(1-s)^{\alpha_i-\gamma_{ij}-1}f_i(s,x_1,\dots,x_kI^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)ds$$

$$-\frac{b}{\Gamma(\alpha_i-1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_i-2}f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)ds).$$

Now, we prove that G_i is a contraction with the constant

$$\Lambda'=\sum_{j=1}^{k}K[\frac{(1+a_i\xi^{\alpha_i})\Gamma(\alpha_i-\gamma_{ij}+1)+(1+a_i\xi^{\alpha_i-\gamma_{ij}})\Gamma(\alpha_i+1)}{(a_i-2)\Gamma(\alpha_i+1)\Gamma(\alpha_i-\gamma_{ij}+1)}+\frac{\Gamma(2-\gamma_{ij})(3+a_i\xi_j+a_i)}{(a_i-2)(1-b_i\Gamma(2-\gamma_{ij}))}$$

$$(\frac{\Gamma(\alpha_i-2\gamma_{ij}+1)+\Gamma(\alpha_i-\gamma_{ij}+1)}{\Gamma(\alpha_i-\gamma_{ij}+1)\Gamma(\alpha_i-2\gamma_{ij}+1)}+\frac{b_i\eta^{\alpha_i}\Gamma(\alpha_i-\gamma_{ij}+1)+b\eta^{\alpha_i-\gamma_{ij}}\Gamma(\alpha_i+1)}{\Gamma(\alpha_i+1)\Gamma(\alpha_i-\gamma_{ij}+1)})]<1.$$

Note that,

$$|G_ix(t)-G_iy(t)|\leq\frac{1}{(a_i-2)\Gamma(\alpha_i)}(\int_0^1(1-s)^{\alpha_i-1}|f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)$$

$$-f_i(s,y_1,\dots,y_k,I^{\beta_{i1}}y_1,\dots,I^{\beta_{ik}}y_k,{}^cD^{\gamma_{i1}}y_1,\dots,{}^cD^{\gamma_{ik}}y_k)|ds+a\sum_{j=1}^{k}\int_0^{\xi_j}(\xi_j-s)^{\alpha-1}|f_i(s,x_1,\dots,x_k,$$

$$I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)-f_i(s,y_1,\dots,y_k,I^{\beta_{i1}}y_1,\dots,I^{\beta_{ik}}y_k,{}^cD^{\gamma_{i1}}y_1,\dots,{}^cD^{\gamma_{ik}}y_k)|ds)$$

$$+\sum_{j=1}^{k}\frac{\Gamma(2-\gamma_{ij})(1-a_i\xi_j+t(a-2))}{(a_i-2)(1-b_i\Gamma(2-\gamma_{ij}))}(\frac{1}{\Gamma(\alpha_i-\gamma_{ij})}\int_0^1(1-s)^{\alpha_i-\gamma_{ij}-1}|f_i(s,x_1,\dots,x_k,$$

$$I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)-f_i(s,y_1,\dots,y_k,I^{\beta_{i1}}y_1,\dots,I^{\beta_{ik}}y_k,{}^cD^{\gamma_{i1}}y_1,\dots,{}^cD^{\gamma_{ik}}y_k)|ds$$

$$+\frac{b_i}{\Gamma(\alpha_i-1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_i-2}|f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)-$$

$$f_i(s,y_1,\dots,y_k,I^{\beta_{i1}}y_1,\dots,I^{\beta_{ik}}y_k,{}^cD^{\gamma_{i1}}y_1,\dots,{}^cD^{\gamma_{ik}}y_k)|ds)\leq K\|x-y\|\big(\frac{1}{(a_i-2)\Gamma(\alpha_i)}(\int_0^1(1-s)^{\alpha_i-1}ds+$$

$$a_i\sum_{j=1}^{k}\int_0^{\xi_j}(\xi_j-s)^{\alpha_i-1}ds)+\sum_{j=1}^{k}\frac{\Gamma(2-\gamma_{ij})(3+a_i\xi_j+a_i)}{(a_i-2)(1-b_i\Gamma(2-\gamma_{ij}))}(\frac{1}{\Gamma(\alpha_i-\gamma_{ij})}\int_0^1(1-s)^{\alpha_i-\gamma_{ij}-1}ds$$

$$+\frac{b_i}{\Gamma(\alpha_i-1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_i-2}ds)\big)\leq\sum_{j=1}^{k}K\|x-y\|(\frac{1+a_i\xi_j^{\alpha_i}}{(a_i-2)\Gamma(\alpha_i+1)}+\frac{\Gamma(2-\gamma_{ij})(3+a_i\xi_j+a_i)}{(a_i-2)(1-b_i\Gamma(2-\gamma_{ij}))}$$

$$(\frac{1}{\Gamma(\alpha_i-\gamma_{ij}+1)}+\frac{b_i\eta^{\alpha_i}}{\Gamma(\alpha_i+1)})).$$

Hence,

$$|{}^cD^{\gamma_{ij}}G_ix(t)-{}^cD^{\gamma_{ij}}G_iy(t)|\leq\sum_{j=1}^{k}K\|x-y\|(\frac{1+a_i\xi^{\alpha_i-\gamma_{ij}}}{(a-2)\Gamma(\alpha_i-\gamma_{ij}+1)}+\frac{\Gamma(2-\gamma_{ij})(3+a_i\xi_j+a_i)}{(a_i-2)(1-b_i\Gamma(2-\gamma_{ij}))}$$

$$(\frac{1}{\Gamma(\alpha_i-2\gamma_{ij}+1)}+\frac{b_i\eta^{\alpha_i-\gamma_{ij}}}{\Gamma(\alpha_i+1-\gamma_{ij})}))$$

and so

$$\|Gx(t)-Gy(t)\|_*\leq\sum_{j=1}^{k}K\|x-y\|_*[\frac{(1+a_i\xi^{\alpha_i})\Gamma(\alpha_i-\gamma_{ij}+1)+(1+a_i\xi^{\alpha_i-\gamma_{ij})\Gamma(\alpha_i+1)}}{(a_i-2)\Gamma(\alpha_i+1)\Gamma(\alpha_i-\gamma_{ij}+1)}+\frac{\Gamma(2-\gamma_{ij})(3+a\xi+a)}{(a-2)(1-b\Gamma(2-\gamma_{ij}))}$$

$$(\frac{\Gamma(\alpha_i-2\gamma_{ij}+1)+\Gamma(\alpha_i-\gamma_{ij}+1)}{\Gamma(\alpha_i-\gamma_{ij}+1)\Gamma(\alpha_i-2\gamma_{ij}+1)}+\frac{b_i\eta^{\alpha_i}\Gamma(\alpha_i-\gamma_{ij}+1)+b\eta^{\alpha_i-\gamma_{ij}}\Gamma(\alpha_i+1)}{\Gamma(\alpha_i+1)\Gamma(\alpha_i-\gamma_{ij}+1)})]=\Lambda'\|x-y\|_*.$$

Thus, G is a contraction. Since $f_1,\dots,f_k$ are continuous, It is easy to check that $F_1,\dots,F_k$ are so. We show that $F_1,\dots,F_k$ are uniformly bounded on B_r. Let $x\in B_r$. Then, we have

$$|F_ix(t)|\leq\frac{1}{\Gamma(\alpha_i)}\int_0^t(t-s)^{\alpha_i-1}|f_i(s,x_1(s),\dots,x_k(s),I^{\beta_{i1}}x(s),\dots,I^{\beta_{ik}}x(s),{}^cD^{\gamma_{i1}}x_1(s),\dots,{}^cD^{\gamma_{ik}}x_k(s))|ds$$

$$\leq\frac{l_i(t)}{\Gamma(\alpha)}\sum_{j=1}^{k}\lambda_i(|x_j|+|{}^cD^{\gamma_{ij}}x_j|)\int_0^t(t-s)^{\alpha_i-1}ds,$$

$$|{}^cD^{\gamma_{ij}}F_ix(t)|\leq\frac{1}{\Gamma(\alpha_i+\beta_{ij})}\int_0^t(t-s)^{\alpha_i-\gamma_{ij}-1}|f_i(s,x_1(s),\dots,x_k(s),I^{\beta_{i1}}x_1(s),\dots,I^{\beta_{ik}}x_k(s),$$

$$^{c}D^{\gamma_{i1}}x_1(s),\ldots,{}^{c}D^{\gamma_{ik}}x_k(s))|ds \leq \frac{l_i(t)}{\Gamma(\alpha_i-\gamma_{ij})}\sum_{j=1}^{k}\lambda_i(|x_j|+|^{c}D^{\gamma_{ij}}x_j|)\int_0^t (t-s)^{\alpha_i-\gamma_{ij}-1}ds.$$

Hence, $\|Fx\|_* \leq \sum_{i=1}^{k}(\frac{1}{\Gamma(\alpha_i)}+\frac{1}{\Gamma(\alpha_i-\gamma_{ij})})\|l_i\|(\sum_{j=1}^{k}\lambda_i(r))$. By using the Arzela-Ascoli Theorem, we conclude that the map F is compact. For $0 \leq t_1 < t_2 \leq 1$, we have

$$\|Fx(t_1)-Fx(t_2)\|_* \leq \sum_{i=1}^{k}\|l_i\|\lambda_i(r)|\frac{2(t_2-t_1)^{\alpha_i}+t_1^{\alpha_i}-t_2^{\alpha_i}}{\Gamma(\alpha_i+1)}-\sum_{j=1}^{k}\frac{2(t_2-t_1)^{\alpha_i-\gamma_{ij}}+t_1^{\alpha_i-\gamma_{ij}}-t_2^{\alpha_i-\gamma_{ij}}}{\Gamma(\alpha_i-\gamma_{ij}+1)}|$$

and so right side of the inequality tends to zero whenever $t_2 \to t_1$. Now by using Theorem 1.2, the operator T has at least one fixed point which is a solution for the problem. □

Theorem 2.3 *Suppose that* $f_1,\ldots,f_k : I\times\mathbb{R}^{3k}\to\mathbb{R}$ *are continuous functions and there exist a positive constant* K *such that* $|f_i(t,x_1(t),\ldots,x_{3k}(t))-f_i(t,y_1(t),\ldots,y_{3k}(t))| \leq K\sum_{j=1}^{2k}|x_j(t)-y_j(t)|$ *for all* $t\in I$ *and* $i=1,\ldots,k$. *If* $\Lambda=\sum_{i=1}^{k}K[\sum_{j=1}^{k}\frac{\Gamma(\alpha_i-\gamma_{ij}+1)(\Delta_{i1}+\Delta_{i2})\Gamma(\alpha_i-2\gamma_{ij}+1)+\Gamma(\alpha_i+1)(\Omega_{i1}+\Omega_{i2})\Gamma(\alpha_i-\gamma_{ij}+1)}{(a_i-2)\Gamma(\alpha_i+1)\Gamma(\alpha_i-\gamma_{ij}+1)(1-b\Gamma(2-\gamma_{ij}))\Gamma(\alpha_i-\gamma_{ij}+1)\Gamma(\alpha_i-2\gamma_{ij}+1)}]<1$, *then the main problem has a unique solution.*

Proof Let $x=(x_1,\ldots,x_k), y=(y_1,\ldots,y_k)\in X^k$. Then, we have

$$|E_i(x_1,x_2,\ldots,x_k)(t)-E_i(y_1,y_2,\ldots,y_k)(t)| \leq \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}|f_i(s,x_1,\ldots,x_k,I^{\beta_{i1}}x_1,\ldots,I^{\beta_{ik}}x_k,$$

$$^{c}D^{\gamma_{i1}}x_1,\ldots,{}^{c}D^{\gamma_{ik}}x_k)-f_i(s,y_1,\ldots,y_k,I^{\beta_{i1}}y_1,\ldots,I^{\beta_{ik}}y_k,{}^{c}D^{\gamma_{i1}}y_1,\ldots,{}^{c}D^{\gamma_{ik}}y_k)|ds+\frac{1}{(a_i-2)\Gamma(\alpha_i)}$$

$$(\int_0^1 (1-s)^{\alpha_i-1}|f_i(s,x_1,\ldots,x_k,I^{\beta_{i1}}x_1,\ldots,I^{\beta_{ik}}x_k,{}^{c}D^{\gamma_{i1}}x_1,\ldots,{}^{c}D^{\gamma_{ik}}x_k)-f_i(s,y_1,\ldots,y_k,I^{\beta_{i1}}y_1,\ldots,I^{\beta_{ik}}y_k,$$

$$^{c}D^{\gamma_{i1}}y_1,\ldots,{}^{c}D^{\gamma_{ik}}y_k)|ds+a_i\sum_{j=1}^{k}\int_0^{\xi_j}(\xi_j-s)^{\alpha-1}|f_i(s,x_1,\ldots,x_k,I^{\beta_{i1}}x_1,\ldots,I^{\beta_{ik}}x_k,{}^{c}D^{\gamma_{i1}}x_1,\ldots,{}^{c}D^{\gamma_{ik}}x_k)$$

$$-f_i(s,y_1,\ldots,y_k,I^{\beta_{i1}}y_1,\ldots,I^{\beta_{ik}}y_k,{}^{c}D^{\gamma_{i1}}y_1,\ldots,{}^{c}D^{\gamma_{ik}}y_k)|ds)+\sum_{j=1}^{k}\frac{\Gamma(2-\gamma_{ij})(1-a_i\xi_j+t(a_i-2))}{(a_i-2)(1-b_i\Gamma(2-\gamma_{ij}))}$$

$$(\frac{1}{\Gamma(\alpha_i-\gamma_{ij})}\int_0^1 (1-s)^{\alpha_i-\gamma_i j-1}|f_i(s,x_1,\ldots,x_k,I^{\beta_{i1}}x_1,\ldots,I^{\beta_{ik}}x_k,{}^{c}D^{\gamma_{i1}}x_1,\ldots,{}^{c}D^{\gamma_{ik}}x_k)-f_i(s,y_1,\ldots,y_k,$$

$$I^{\beta_{i1}}y_1,\ldots,I^{\beta_{ik}}y_k,{}^{c}D^{\gamma_{i1}}y_1,\ldots,{}^{c}D^{\gamma_{ik}}y_k)|ds-\frac{b_i}{\Gamma(\alpha_i-1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_i-2}|f_i(s,x_1,\ldots,x_k,I^{\beta_{i1}}x_1,\ldots,I^{\beta_{ik}}x_k,$$

$$^{c}D^{\gamma_{i1}}x_1,\ldots,{}^{c}D^{\gamma_{ik}}x_k)-f_i(s,y_1,\ldots,y_k,I^{\beta_{i1}}y_1,\ldots,I^{\beta_{ik}}y_k,{}^{c}D^{\gamma_{i1}}y_1,\ldots,{}^{c}D^{\gamma_{ik}}y_k)|ds) \leq \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}$$

$$(K\sum_{j=1}^{k}(|x_j-y_j|+|^{c}D^{\gamma_{ij}}x_j-{}^{c}D^{\gamma_{ij}}y_j|))ds+\frac{1}{(a_i-2)\Gamma(\alpha_i)}(\int_0^1 (1-s)^{\alpha_i-1}(K\sum_{j=1}^{k}(|x_j-y_j|+|^{c}D^{\gamma_{ij}}x_j-{}^{c}D^{\gamma_{ij}}y_j|))ds+$$

$$a_i\sum_{j=1}^{k}\int_0^{\xi_j}(\xi_j-s)^{\alpha-1}(K\sum_{j=1}^{k}(|x_j-y_j|+|^cD^{\gamma_{ij}}x_j-{}^cD^{\gamma_{ij}}y_j|))ds)+\sum_{j=1}^{k}\frac{\Gamma(2-\gamma_{ij})(1-a_i\xi_j+t(a_i-2))}{(a_i-2)(1-b_i\Gamma(2-\gamma_{ij}))}(\frac{1}{\Gamma(\alpha_i-\gamma_{ij})}$$

$$\int_0^1(1-s)^{\alpha_i-\gamma_{ij}-1}(K\sum_{j=1}^{k}(|x_j-y_j|+|^cD^{\gamma_{ij}}x_j-{}^cD^{\gamma_{ij}}y_j|))ds+\frac{b_i}{\Gamma(\alpha_i-1)}\int_0^{\eta_j}(\eta_j-s)^{\alpha_i-2}(K\sum_{j=1}^{k}(|x_j-y_j|+$$

$$|^cD^{\gamma_{ij}}x_j-{}^cD^{\gamma_{ij}}y_j|))ds)\leq\sum_{j=1}^{k}K\|x_j-y_j\|(\frac{\Delta_{i1}+\Delta_{i2}}{(a_i-2)\Gamma(\alpha_i+1)(1-b_i\Gamma(2-\gamma_{ij}))\Gamma(\alpha_i-\gamma_{ij}+1)})$$

for all $t\in I$. Also, we have

$$|^cD^{\gamma_{ij}}E_i(x_1,x_2,\dots,x_k)(t)-{}^cD^{\gamma_{ij}}E_i(y_1,y_2,\dots,y_k)(t)|\leq\frac{1}{\Gamma(\alpha_i-\gamma_{ij})}\int_0^t(t-s)^{\alpha_i-\gamma_{ij}-1}|f_i(s,x_1,\dots,x_k,$$

$$I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)-f_i(s,y_1,\dots,y_k,I^{\beta_{i1}}y_1,\dots,I^{\beta_{ik}}y_k,{}^cD^{\gamma_{i1}}y_1,\dots,{}^cD^{\gamma_{ik}}y_k)|ds+$$

$$\frac{1}{(a-2)\Gamma(\alpha_i-\gamma_{ij})}(\int_0^1(1-s)^{\alpha_i-\gamma_{ij}-1}|f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)$$

$$-f_i(s,y_1,\dots,y_k,I^{\beta_{i1}}y_1,\dots,I^{\beta_{ik}}y_k,{}^cD^{\gamma_{i1}}y_1,\dots,{}^cD^{\gamma_{ik}}y_k)|ds+a\sum_{j=1}^{k}\int_0^{\xi_j}(\xi_j-s)^{\alpha-\gamma_{ij}-1}|f_i(s,x_1,\dots,x_k,$$

$$I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)-f_i(s,y_1,\dots,y_k,I^{\beta_{i1}}y_1,\dots,I^{\beta_{ik}}y_k,{}^cD^{\gamma_{i1}}y_1,\dots,{}^cD^{\gamma_{ik}}y_k)|ds)+$$

$$\sum_{j=1}^{k}\frac{\Gamma(2-\gamma_{ij})(1-a_i\xi_j+t(a_i-2))}{(a_i-2)(1-b_i\Gamma(2-\gamma_{ij}))}(\frac{1}{\Gamma(\alpha_i-2\gamma_{ij})}\int_0^1(1-s)^{\alpha_i-2\gamma_{ij}-1}|f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,$$

$${}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)-f_i(s,y_1,\dots,y_k,I^{\beta_{i1}}y_1,\dots,I^{\beta_{ik}}y_k,{}^cD^{\gamma_{i1}}y_1,\dots,{}^cD^{\gamma_{ik}}y_k)|ds-\frac{b_i}{\Gamma(\alpha_i-\gamma_{ij}-1)}$$

$$\int_0^{\eta_j}(\eta_j-s)^{\alpha_i-\gamma_{ij}-2}|f_i(s,x_1,\dots,x_k,I^{\beta_{i1}}x_1,\dots,I^{\beta_{ik}}x_k,{}^cD^{\gamma_{i1}}x_1,\dots,{}^cD^{\gamma_{ik}}x_k)-f_i(s,y_1,\dots,y_k,I^{\beta_{i1}}y_1,\dots,$$

$$I^{\beta_{ik}}y_k,{}^cD^{\gamma_{i1}}y_1,\dots,{}^cD^{\gamma_{ik}}y_k)|ds)\leq\sum_{j=1}^{k}K\|x_j-y_j\|(\frac{\Omega_{i1}+\Omega_{i2}}{(a_i-2)\Gamma(\alpha_i-\gamma_{ij}+1)(1-b_i\Gamma(2-\gamma_{ij}))\Gamma(\alpha_i-2\gamma_{ij}+1)})$$

and so

$$\|E_i(x_1,x_2,\dots,x_k)-E_i(y_1,y_2,\dots,y_k)\|\leq\sum_{j=1}^{k}K\|x_j-y_j\|[\frac{\Delta_{i1}+\Delta_{i2}}{(a_i-2)\Gamma(\alpha_i+1)(1-b\Gamma(2-\gamma_{ij}))\Gamma(\alpha_i-\gamma_{ij}+1)}$$

$$+\frac{\Omega_{i1}+\Omega_{i2}}{(a_i-2)\Gamma(\alpha_i-\gamma_{ij}+1)(1-b_i\Gamma(2-\gamma_{ij}))\Gamma(\alpha_i-2\gamma_{ij}+1)}]$$

for $i=1,\dots,k$. Thus, we obtain

$$\|E(x_1,x_2,\dots,x_k)-E(y_1,y_2,\dots,y_k)\|_*\leq K\|x-y\|_*\sum_{i=1}^{k}[\sum_{j=1}^{k}\frac{\Delta_{i1}+\Delta_{i2}}{(a_i-2)\Gamma(\alpha_i+1)(1-b_i\Gamma(2-\gamma_{ij}))\Gamma(\alpha_i-\gamma_{ij}+1)}$$

$$+\sum_{j=1}^{k}\frac{\Omega_{i1}+\Omega_{i2}}{(a_i-2)\Gamma(\alpha_i+\beta_{ij}+1)(1-b_i\Gamma(2-\gamma_{ij}))\Gamma(\alpha_i+\beta_{ij}-\gamma_{ij}+1)}]=\Lambda\|x-y\|_*.$$

Since $\Lambda<1$, E has a unique fixed point which is unique solution for the main problem. □

Now, we introduced a class of wavelet basis constructed by Alpert for $L^2[0,1]$ ([3]). First, we review Legendre multiwavelets briefly ([6]). For functions $\varphi^m\in L^2(\mathbb{R})$ $(m=0,1,\ldots,r)$, consider a reference subspace $V_0=\overline{\langle\varphi^m(.-k):\ k\in\mathbb{Z},\ m=0,1,\ldots,r\rangle}$ be generated as the L^2-closure of the linear span of the integer translation of the functions. Also, consider other subspaces $V_j=\overline{\langle\varphi^m_{j,k}=\varphi^m(2^jx-k):\ k\in\mathbb{Z},\ m=0,1,\ldots,r,\ j\in\mathbb{Z}\rangle}$. We say that the functions $\varphi^m\in L^2(\mathbb{R})$ $(m=0,1,\ldots,r)$ generate a multiresolution analysis (MRA) whenever generate a nested sequence of closed subspaces $\cdots\subset V_{-2}\subset V_{-1}\subset V_0\subset V_1\subset V_2\subset\ldots$ such that $\cup_{j\in\mathbb{Z}}V_j$ is dense in $L^2(\mathbb{R})$, $\cap_{j\in\mathbb{Z}}V_j=\{0\}$, $f(x)\in V_j\iff f(x+2^{-j})\in V_j\iff f(2x)\in V_{j+1}$ and $\{\varphi^m(.-k)\}_{k\in\mathbb{Z}}$ is an Riesz basis of V_0 ([6]). If φ^m generate an MRA, then φ^m are nominated scaling functions ([6]). The scaling functions are called orthogonal whenever $\varphi^m(.-k)\perp\varphi^{m'}(.-k')$ for $m\neq m'$ and $k\neq k'$ ([6]). Since the subspaces V_j are nested, there exist complementary orthogonal subspaces W_j such that $V_{j+1}=V_j\bigoplus W_j$ for all j ([6]). This gives an orthogonal decomposition of $L^2(\mathbb{R})$ as $L^2(\mathbb{R})=\bigoplus_{j\in F}V_j=\bigoplus_{j\in F}W_j$ ([6]). We say that $\psi^0,\psi^1,\ldots,\psi^r\in L^2(\mathbb{R})$ are wavelets whenever those supply the complementary orthogonal subspaces W_j of an MRA, that is, $W_j=\overline{\langle\psi^m_{j,k}=\psi^m(2^jx-k):\ k\in\mathbb{Z},\ m=0,1,\ldots,r,\ j\in\mathbb{Z}\rangle}$ ([6]). If $\psi^m_{j,k}\perp\psi'^{m'}_{j',k'}$ for $j\neq j'$, $k\neq k'$ and $m\neq m'$, then $\psi^0,\psi^1,\ldots,\psi^r$ are called orthonormal wavelets ([6]). It is known that Alpert multiwavelets systems with multiplicity r consist of $r+1$ scaling functions and $r+1$ wavelets ([6]). The rth order scaling functions are the $r+1$ functions $\varphi^0(x),\ldots,\varphi^r(x)$, where $\varphi^i(x)$ is a polynomial of ith order and allφ's form orthonormal basis (see [3], [6]), that is, $\varphi^i(x)=\sum_{k=0}^{i}a_{ik}x^k$ for some $a_{ik}\geq 0$ and $\int_0^1\varphi^i(x)\varphi^k(x)=\delta_{i,k}$ for all i and k ([6]). The two-scale relation for scaling functions of order r are in the form $\varphi_i(x)=\sum_{k=0}^{r}c_{i,j}\varphi^j(2x)+\sum_{k=0}^{r}c_{i,r+j+1}\varphi^j(2x-1)$ ([6]). The coefficients $\{c_{i,j}\}$ could be obtained uniquely ([6]). Suppose that Q^k_m are the orthonormal projections from $L^2[0,1]$ onto S^k_m ([6]). If $f\in C^k[0,1]$, then $\parallel Q^k_mf-f\parallel\leq 2^{-mk}\frac{2}{4^kk!}sup_{x\in[0,1]}\mid f^{(k)}(x)\mid$ ([6]). It is known that each function $x(t)$ which is square integrable on the interval $[0,1]$ can be expanded by the scaling functions, that is,

$$x(t)\approx\sum_{k=0}^{2^J-1}\sum_{m=0}^{r}c_{J,k}\varphi^m_{J,k}(t)=C^T\Phi_J(t)$$

and the corresponding wavelet functions, $x(t)\approx\sum_{m=0}^{r}\{c^m_{0,0}\varphi^m_{0,0}(t)+\sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}d^m_{j,k}\psi^m_{j,k}(t)\}=D^T\Psi_J(t)$, where $c^m_{J,k}=\int_0^1x(t)\varphi^m_{J,k}(t)dt$, $d^m_{j,k}=\int_0^1x(t)\psi^m_{j,k}(t)dt$ and C and D are $n\times 1$ $(n=(r+1)2^J)$ matrices given by $C=[c^0_{J,0},\ldots,c^r_{J,0},\ldots,c^0_{J,2^J-1},\ldots,c^r_{J,2^J-1}]^T$ and

$$D=[c^0_{0,0},\ldots,c^r_{0,0},d^0_{0,0},\ldots,d^r_{0,0},\ldots,d^0_{J-1,0},\ldots,d^r_{J-1,0},\ldots,d^0_{J-1,2^{J-1}-1},\ldots,d^r_{J-1,2^{J-1}-1}]^T$$

(see [3], [6] and [47]). By approximating $x_i(t)\approx\sum_{k'=0}^{r}d_{ik'}\psi_{k'}(t)=D^T\Psi(t)$ and replacing in the system, we obtain

$$\begin{cases} \sum_{k'=\lceil\alpha_1\rceil}^{r} d_{1k'}\,{}^cD^{\alpha_1}(\psi_{k'}(t)) = f_1\Big(t, \sum_{k'=0}^{r} d_{1k'}\psi'_k(t), \dots, \sum_{k'=0}^{r} d_{kk'}\psi_{k'}(t), \sum_{k'=0}^{r} d_{1k'}I^{\beta_{11}}(\psi_{k'}(t)), \\ \dots, \sum_{k'=0}^{r} d_{kk'}I^{\beta_{1k}}(\psi_{k'}(t)), \sum_{k'=\lceil\gamma_{11}\rceil}^{r} d_{1k'}\,{}^cD^{\gamma_{11}}(\psi_{k'}(t)), \dots, \sum_{k'=\lceil\gamma_{1k}\rceil}^{r} d_{kk'}\,{}^cD^{\gamma_{1k}}(\psi_{k'}(t))\Big), \\ \sum_{k'=\lceil\alpha_2\rceil}^{r} d_{2k'}\,{}^cD^{\alpha_2}(\psi_{k'}(t)) = f_2\Big(t, \sum_{k'=0}^{r} d_{1k'}\psi_{k'}(t), \dots, \sum_{k'=0}^{r} d_{kk'}\psi_{k'}(t), \sum_{k'=0}^{r} d_{1k'}I^{\beta_{21}}(\psi_{k'}(t)), \\ \dots, \sum_{k'=0}^{r} d_{kk'}I^{\beta_{2k}}(\psi_{k'}(t)), \sum_{k'=\lceil\gamma_{21}\rceil}^{r} d_{1k'}\,{}^cD^{\gamma_{21}}(\psi_{k'}(t)), \dots, \sum_{k'=\lceil\gamma_{2k}\rceil}^{r} d_{kk'}\,{}^cD^{\gamma_{2k}}(\psi_{k'}(t))\Big), \\ \vdots \\ \sum_{k'=\lceil\alpha_k\rceil}^{r} d_{kk'}\,{}^cD^{\alpha_k}(\psi_{k'}(t)) = f_1\Big(t, \sum_{k'=0}^{r} d_{1k'}\psi'_k(t), \dots, \sum_{k'=0}^{r} d_{kk'}\psi'_{k'}(t), \sum_{k'=0}^{r} d_{1k'}I^{\beta_{k1}}(\psi_{k'}(t)), \\ \dots, \sum_{k'=0}^{r} d_{kk'}I^{\beta_{kk}}(\psi_{k'}(t)), \sum_{k'=\lceil\gamma_{k1}\rceil}^{r} d_{1k'}\,{}^cD^{\gamma_{k1}}(\psi_{k'}(t)), \dots, \sum_{k'=\lceil\gamma_{kk}\rceil}^{r} d_{kk'}\,{}^cD^{\gamma_{kk}}(\psi_{k'}(t))\Big). \end{cases}$$

and

$$\begin{cases} \sum_{k'=0}^{r} d_{1k'}\psi_{k'}(0) + \sum_{k'=0}^{r} d_{1k'}\psi_{k'}(1) = \sum_{j=1}^{k}\sum_{k'=0}^{r} d_{1k'}\psi_{k'}(\xi_j), \\ \sum_{j=1}^{k}\sum_{k'=\lceil\gamma_{ij}\rceil}^{r} d_{1k'}\,{}^cD^{\gamma_{ij}}\psi_{k'}(0) + \sum_{j=1}^{k}\sum_{k'=\lceil\gamma_{ij}\rceil}^{r} d_{1k'}\,{}^cD^{\gamma_{ij}}\psi_{k'}(1) = \sum_{j=1}^{k}\sum_{k'=0}^{r} b_i d_{ik'}\psi'_{k'}(\eta_j). \end{cases}$$

for all $j = 1, 2, \dots, k$. By applying the Newton iterated method, we can calculate the coefficients d_{ij}. Here, we give three examples to illustrate our main results by using the method.

Example 2.4 *Consider the system of fractional integro-differential equations*

$$\begin{cases} {}^cD^{\frac{4}{3}}x_1(t) = f(t) + 0.0078(x_1(t) + x_2(t) + I^{(2)}x_1(t)), \\ {}^cD^{\frac{5}{4}}x_2(t) = g(t) + 0.0078(x_1(t) + x_2(t) + I^{(3)}x_2(t)), \end{cases}$$

with boundary conditions ${}^cD^{\frac{1}{4}}x_1(0) +^c D^{\frac{3}{4}}x_1(0) +^c D^{\frac{1}{4}}x_1(1) +^c D^{\frac{3}{4}}x_1(1) = 1.82955(x'_1(\frac{1}{3}) + x'_1(\frac{2}{3})$, $x_1(0) + x_1(1) = 8.39757(x_1(\frac{1}{5}) + x_1(\frac{2}{5}))$, ${}^cD^{\frac{1}{5}}x_2(0) +^c D^{\frac{1}{3}}x_2(0) +^c D^{\frac{1}{5}}x_2(1) +^c D^{\frac{1}{3}}x_2(1) = 1.0907(x'_2(\frac{1}{3}) + x'_2(\frac{2}{3}))$ *and* $x_2(0) + x_2(1) = 1.66667(x_2(\frac{1}{5}) + x_2(\frac{2}{5}))$. *Put* $f(t) = 0.0078(-t - t^{\frac{5}{2}} - \frac{\Gamma(\frac{7}{2})t^{\frac{9}{2}}}{\Gamma(\frac{11}{2})}) + \frac{\Gamma(\frac{7}{2})t^{\frac{7}{6}}}{\Gamma(\frac{13}{6})}$, $g(t) = 0.0078(-t - t^{\frac{5}{2}} - \frac{t^4}{24}) + \frac{t^{-\frac{1}{4}}}{\Gamma(\frac{3}{4})}$, $\alpha_1 = \frac{4}{3}$, $\alpha_2 = \frac{5}{4}$, $\beta_{11} = 2$, $\beta_{22} = 3$, $\gamma_{11} = \frac{1}{4}$, $\gamma_{12} = \frac{3}{4}$, $\gamma_{21} = \frac{1}{5}$, $\gamma_{22} = \frac{1}{3}$, $\xi_1 = 0.01$, $\xi_2 = 0.03$, $\eta_1 = 0.02$, $\eta_2 = 0.04$, $a_1 = 8.39757$, $a_2 = 1.66667$, $b_1 = 1.82955$, $b_2 = 1.0907$, $f_1(t, x_1, x_2, x_3, x_4) = f_1(t) + (x_1(t) + x_2(t) + I^{(2)}x_1(t))$ *and* $f_2(t, y_1, y_2, y_3, y_4) = f_2(t) + (y_1(t) + y_2(t) + I^{(3)}y_2(t))$. *If* $l_1(t) = 2t + 2t^{\frac{5}{2}} + \frac{8t^{9/2}}{63} + \frac{15\sqrt{\pi}}{4\Gamma(13/6)}t^{7/6}$, $l_2(t) = 2t + 2t^{5/2} + \frac{t^4}{12} + 0.816049t^{-1/4}$, $\lambda_1 = \lambda_2 = 1$, $\Delta_{11} = -11.8103$, $\Delta_{12} = 25.4806$, $\Delta_{21} = -0.00394216$, $\Delta_{22} = 9.88335$, $\Omega_{11} = -38.0068$, $\Omega_{12} = 110.528$, $\Omega_{21} = 0.206273$, $\Omega_{22} = 154.908$ *and* $K = 0.0078$, *then by using Theorem 2.2 the system has a solution. Note that, the exact solutions for the system are* $x_1(t) = t^{\frac{5}{2}}$ *and* $x_2(t) = t$. *Check the following table for numerical errors.*

t_i	*The coefficient value of* $x_1(t)$	*Absolute error with Alpert's multiwavelets*
0	*0.284359*	*6.960027e-06*
0.2	*0.275145*	*4.153067e-05*
0.4	*0.096792*	*4.922978e-05*
0.6	*0.008913*	*6.0522151e-05*
0.8	*0.000627*	*5.932623e-05*
1	*0.000174*	*1.516906e-04*

t_i	*The coefficient value of* $x_2(t)$	*Absolute error with Alpert's multiwavelets*
0	*0.500000*	*3.9350240e-08*
0.2	*0.288675*	*2.146224e-07*
0.4	*–2.311520e-08*	*2.696232e-07*
0.6	*2.287753e-08*	*3.150615e-07*
0.8	*–6.240553e-09*	*3.4434256e-07*
1	*9.0859893e-09*	*4.463759e-07*

Example 2.5 *Consider the system of fractional integro-differential equations*

$$\begin{cases} {}^cD^{\frac{7}{4}}x_1(t) = f(t) + 0.1(x_1(t) + x_2(t) + I^{(2)}x_1(t)), \\ {}^cD^{\frac{3}{2}}x_2(t) = g(t) + 0.1(x_1(t) + x_2(t) + I^{(1)}x_2(t)), \end{cases}$$

with boundary conditions ${}^cD^{\frac{1}{2}}x_1(0) +^c D^{\frac{1}{3}}x_1(0) +^c D^{\frac{1}{2}}x_1(1) +^c D^{\frac{1}{3}}x_1(1) = -1.19536(x_1'(\frac{1}{4}) + x_1'(\frac{1}{2}))$, $x_1(0) + x_1(1) = 0(x_1(\frac{1}{5}) + x_1(\frac{2}{5}))$, ${}^cD^{\frac{2}{3}}x_2(0) +^c D^{\frac{4}{5}}x_2(0) +^c D^{\frac{2}{3}}x_2(1) +^c D^{\frac{4}{5}}x_2(1) = 4.94398(x_2'(\frac{1}{3}) + x_2'(\frac{2}{3}))$ *and* $x_2(0) + x_2(1) = 6.17143(x_2(\frac{1}{3}) + x_2(\frac{1}{2}))$*. Put* $f(t) = 0.1(t - t^2 - \frac{5t^3}{6} - \frac{t^4}{12}) - \frac{t^{-\frac{3}{4}}}{\Gamma(\frac{1}{4})} + \frac{2t^{\frac{1}{4}}}{\Gamma(\frac{5}{4})}$, $g(t) = 0.1(t - t^2 - t^3 - \frac{t^4}{4}) + \frac{6t^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}$, $\alpha_1 = \frac{7}{4}$, $\alpha_2 = \frac{3}{2}$, $\beta_{11} = 2$, $\beta_{22} = 1$, $\gamma_{11} = \frac{1}{2}$, $\gamma_{12} = \frac{1}{3}$, $\gamma_{21} = \frac{2}{3}$, $\gamma_{22} = \frac{4}{5}$, $\xi_1 = \frac{1}{5}$, $\xi_2 = \frac{2}{5}$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{2}$, $a_1 = 0$, $a_2 = 6.17143$, $b_1 = -1.19536$, $b_2 = 4.94398$, $f_1(t, x_1, x_2, x_3, x_4) = f_1(t) + (x_1(t) + x_2(t) + I^{(2)}x_1(t))$ *and* $f_2(t, y_1, y_2, y_3, y_4) = f_2(t) + (y_1(t) + y_2(t) + I^{(1)}y_2(t))$*. If* $\Delta_{11} = -4.94431$, $\Delta_{21} = -40.0375$, $\Delta_{12} = 7.10496$, $\Delta_{22} = 48.9359$, $\Omega_{11} = -4.05109$, $\Omega_{21} = -53.3475$, $\Omega_{12} = 8.17966$, $\Omega_{22} = 91.2142$ *and* $K = 0.1$*, then by using Theorem 2.3, the system has a unique solution. In fact, the exact solutions of the system are* $x_(t) = t^2 - t$ *and* $x_2(t) = t^3$*. Check the following table for numerical errors.*

t_i	*The coefficient value of* $x_1(t)$	*Absolute error with Alpert's multiwavelets*
0	*–0.166667*	*8.1740945e-07*
0.2	*4.32993e-07*	*3.495237e-07*
0.4	*0.0745356*	*3.768863e-08*
0.6	*2.263278e-08*	*2.332886e-07*
0.8	*–4.69e-09*	*5.050237e-07*
1	*2.27e-09*	*8.1740945e-07*

t_i	*The coefficient value of* $x_2(t)$	*Absolute error with Alpert's multiwavelets*
0	*0.25*	*7.9843424e-07*
0.2	*0.2598079*	*2.466749e-07*
0.4	*0.1118033*	*3.70490706e-08*
0.6	*0.01889827*	*1.40109070e-07*
0.8	*–1.117e-08*	*3.5657498e-07*
1	*9.29385839e-09*	*7.4183424e-07*

Example 2.6 *Consider the system of fractional integro-differential equations*

$$\begin{cases} {}^cD^{\frac{6}{5}}x_1(t) = f(t) + 68(x_1(t) + x_2(t) + I^{(1)}x_1(t)), \\ {}^cD^{\frac{5}{4}}x_2(t) = g(t) + 68(x_1(t) + x_2(t) + I^{(2)}x_2(t)), \end{cases}$$

with boundary condition ${}^cD^{\frac{2}{3}}x_1(0) +^c D^{\frac{1}{5}}x_1(0) +^c D^{\frac{2}{3}}x_1(1) +^c D^{\frac{1}{5}}x_1(1) = 1.2361(x_1'(\frac{1}{3}) + x_1'(\frac{2}{3}))$, $x_1(0) + x_1(1) = 1.09373(x_1(\frac{2}{5}) + x_1(\frac{3}{5}))$, ${}^cD^{\frac{5}{6}}x_2(0) +^c D^{\frac{4}{6}}x_2(0) +^c D^{\frac{5}{6}}x_2(1) +^c D^{\frac{4}{6}}x_2(1) = 2.0758(x_2'(\frac{1}{3}) + x_2'(\frac{2}{3}))$ *and* $x_2(0) + x_2(1) = 1.5479(x_2(\frac{2}{5}) + x_2(\frac{3}{5}))$. *Put* $f(t) = \sum_{k=0}^{\infty} \frac{t^{k+\frac{4}{5}}}{\Gamma(k+\frac{9}{5})} + \frac{t^{-\frac{1}{5}}}{\Gamma(\frac{4}{5})} - 68(-1 + 2e^t + t + \frac{3t^2}{2} + e^{2t})$, $g(t) = \frac{2^{\frac{5}{4}}e^{2t}(\Gamma(\frac{3}{4}) - \Gamma(\frac{3}{4}, 2t))}{\Gamma(\frac{3}{4})} + \frac{8t^{\frac{3}{4}}}{3\Gamma(\frac{3}{4})} - 68(\frac{1}{4} + \frac{t}{2} + t^2 + \frac{t^4}{12} + e^t + \frac{5e^{2t}}{4})$, $\alpha_1 = \frac{6}{5}$, $\alpha_2 = \frac{5}{4}$, $\beta_{11} = 1$, $\beta_{22} = 2$, $\gamma_{11} = \frac{2}{3}$, $\gamma_{12} = \frac{1}{5}$, $\gamma_{21} = \frac{5}{6}$, $\gamma_{22} = \frac{4}{6}$, $\xi_1 = \frac{2}{5}$, $\xi_2 = \frac{3}{5}$, $\eta_1 = \frac{1}{3}$, $\eta_2 = \frac{2}{3}$, $a_1 = 1.09373$, $a_2 = 1.5479$, $b_1 = 1.2361$, $b_2 = 2.0757$, $f_1(t, x_1, x_2, x_3, x_4) = f_1(t) + (x_1(t) + x_2(t) + I^{(1)}x_1(t))$ *and* $f_2(t, y_1, y_2, y_3, y_4) = f_2(t) + (y_1(t) + y_2(t) + I^{(2)}y_2(t))$. *If* $K = 68$, *then by using Theorem 2.3, the system has a unique solution. In fact, the exact solutions of the system are* $x_(t) = e^t + t$ *and* $x_2(t) = e^{2t} + t^2$. *Check the following table for numerical errors.*

t_i	*The coefficient value of* $x_1(t)$	*Absolute error with Alpert's multiwavelets*
0	*2.2153067*	*3.8596012e-03*
0.2	*0.77649721*	*3.3994125e-04*
0.4	*0.063107775*	*2.30481104e-03*
0.6	*0.00562384*	*2.67062026e-03*
0.8	*−0.000549966*	*7.349461374e-04*
1	*0.00014101506*	*1.58216779e-03*

t_i	*The coefficient value of* $x_2(t)$	*Absolute error with Alpert's multiwavelets*
0	*3.561401498*	*2.203544165e-02*
0.2	*2.01399322*	*2.98524144e-03*
0.4	*0.50864556*	*4.11330544e-03*
0.6	*0.070565292*	*1.364426553e-03*
0.8	*0.0110211123*	*7.15643916e-03*
1	*0.0005111546*	*1.35564386e-02*

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