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Generalized π-Baer rings

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Abstract: We call a ring R generalized right π-Baer, if for any projection invariant left ideal Y of R, the right annihilator of Y^n is generated, as a right ideal, by an idempotent, for some positive integer n, depending on Y. In this paper, we investigate connections between the generalized π-Baer rings and related classes of rings (e.g., π-Baer, generalized Baer, generalized quasi-Baer, etc.) In fact, generalized right π-Baer rings are special cases of generalized right quasi-Baer rings and also are a generalization of π-Baer and generalized right Baer rings. The behavior of the generalized right π-Baer condition is investigated with respect to various constructions and extensions. For example, the trivial extension of a generalized right π-Baer ring and the full matrix ring over a generalized right π-Baer ring are characterized. Also, we show that this notion is well-behaved with respect to certain triangular matrix extensions. In contrast to generalized right Baer rings, it is shown that the generalized right π-Baer condition is preserved by various polynomial extensions without any additional requirements. Examples are provided to illustrate and delimit our results.

Key words: Generalized π-Baer ring, π-Baer ring, generalized quasi-Baer ring, generalized Baer ring, generalized p.p. ring, skew polynomial ring

1. Introduction

The study of Baer rings has its roots in operator theory in the sense of Kaplansky [17]. Kaplansky introduced Baer rings to abstract various properties of von Neumann algebras. Recall that a ring is *Baer* if the right annihilator of any nonempty subset is generated by an idempotent. The class of Baer rings includes the von Neumann algebras (e.g., the algebra of all bounded operators on a Hilbert space), the commutative C^*-algebra $C(T)$ of continuous complex valued functions on a Stonian space T, and the regular rings whose lattice of principal right ideals is complete.

Various weaker versions of Baer rings have been studied. In [13], Clark defined a ring to be a *quasi-Baer* ring if the left annihilator of every ideal is generated by an idempotent. He has proved that the quasi-Baer ring property is left-right symmetric. He then uses the quasi-Baer concept to characterize when a finite-dimensional algebra with identity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. The theory of quasi-Baer rings is studied in [5–8].

Another generalization of Baer rings are p.p. rings. A ring R is called right (left) *p.p.* if the right (left) annihilator of any element of R is generated, as a right (left) ideal, by an idempotent of R. The p.p. property is not left-right symmetric.

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In [4], Birkenmeier et al. introduced another interesting generalization of Baer rings. Recall that a ring R is said to be a *π-Baer* ring if the right annihilator of every projection invariant left ideal Y (i.e. $Ye \subseteq Y$ for all $e = e^2 \in R$) is generated by an idempotent. The π-Baer condition is strictly between the Baer and quasi-Baer conditions. Like the Baer and quasi-Baer properties, the π-Baer property is left-right symmetric. The class of such rings have been studied in [4]. In this trend we take attention of the readers to look at the related papers [10, 11].

From [20], a ring R is called *generalized right Baer* if for any nonempty subset S of R, the right annihilator of S^n is generated by an idempotent for some positive integer n, where S^n contains elements $a_1a_2\dots a_n$ such that $a_i \in S$ for $1 \leqslant i \leqslant n$.

In [19], a ring was called a *generalized right (principally) quasi-Baer* ring if for any (principal) right ideal I of R, the right annihilator of I^n is generated by an idempotent for some positive integer n, depending on I.

To transfer the generalized quasi-Baer condition from a base ring R to various extensions (e.g., full matrix rings over R or $R[x]$ or $R[[x]]$) one needs no additional conditions which is certainly not the case for the generalized Baer condition (see [1, Theorem 3.12]). Thus, it is natural to ask: is there a condition strictly between the generalized Baer and generalized quasi-Baer conditions, which is able to combine some of the notable features of the generalized Baer and generalized quasi-Baer conditions?

In this paper, we say that a ring R is a *generalized right (left) π-Baer* ring if for any projection invariant left (right) ideal Y, the right (left) annihilator of Y^n is generated, as a right (left) ideal, by an idempotent for some positive integer n, depending on Y. We have some motivations to study such rings. These rings are generalizations of π-Baer and generalized right Baer rings, and there are examples distinguishing these classes. From another point of view, there are subclasses of triangular matrix rings which are generalized right π-Baer rings but are not π-Baer rings. As an example, consider the subring $S_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \Big| a, b \in \mathbb{Z} \right\}$ of $\mathrm{M}_2(\mathbb{Z})$. Then $S_2(\mathbb{Z})$ is a generalized right π-Baer ring but is not a π-Baer ring (see Theorem 3.8).

The structure of the paper is as follows. In Section 2, we introduce generalized right π-Baer rings and we study their properties, and relations with other Baer-type rings. We provide many examples to distinguish these classes. We also study the singular ideal of such rings. We show that the notion of a generalized right π-Baer ring passes to the corners, the center and certain overrings.

In Section 3, we characterize the trivial extension of generalized right π-Baer rings (Theorem 3.3). We investigate the triangular matrix rings $T_n(R)$, $S_n(R)$, $A_n(R)$, $B_n(R)$, $U_n(R)$, and $V_n(R)$. We prove that if $\mathbf{S}_\ell(R) = \mathbf{B}(R)$, then R is a generalized right π-Baer ring if and only if so are these matrix rings, for $n \geq 2$ (Theorem 3.8). We also show that the class of generalized right π-Baer rings is closed with respect to full matrix rings (Proposition 3.1).

In Section 4, we show that being a generalized right π-Baer ring and being a generalized left π-Baer ring is preserved by various polynomial extensions (Theorems 4.2 and 4.4).

Throughout this paper all rings are associative with unity and R denotes such a ring. Subrings and overrings preserve the unity of the base ring. An idempotent $e \in R$ is called *right* (resp., *left*) *semicentral* if $ex = exe$ (resp., $xe = exe$), for all $x \in R$ [3]. We denote by $\mathbf{S}_r(R)$ (resp., $\mathbf{S}_\ell(R)$) the set of all right (resp., left) semicentral idempotents of R. For any nonempty subset X of R, $r_R(X)$ (resp. $\ell_R(X)$) is used for the right (resp., left) annihilator of X over R. We use $\mathbf{B}(R)$, $\mathbf{I}(R)$, $\mathrm{C}(R)$, $\mathrm{M}_n(R)$, $T_n(R)$, I_n, $R[x]$, $R[[x]]$, $R[x;\alpha]$, $R[x;\alpha,\delta]$, $R[[x;\alpha]]$, $R[x;x^{-1};\alpha]$, and $R[[x;x^{-1};\alpha]]$ for the set of all central idempotents of R, the subring of R generated by idempotents, the center of R, the $n \times n$ matrix ring over R, the $n \times n$ triangular matrix ring over R, the $n \times n$ identity matrix, the ring of polynomials, the ring of formal power series, the skew polynomial ring, the ore extension of R, the skew power series ring, the skew Laurent polynomial ring, and the skew Laurent series ring over R of endomorphism type, respectively. Also, $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the integers and the integers modulo n, respectively.

2. Basic results

In this section, we introduce a generalization of π-baer and generalized Baer rings. We discuss the notion of generalized right π-Baer rings. Examples and basic results for these rings are provided in this section. Moreover, the connections between the generalized π-Baer concept and related notions such as the π-Baer, generalized Baer, generalized quasi-Baer and generalized p.p. conditions are discussed. We begin with the following definition.

Definition 2.1 *We say a ring R (with unity) is generalized right projection invariant Baer (denoted generalized right π-Baer) if for each projection invariant left ideal Y (i.e. $Yf \subseteq Y$ for all idempotent $f \in R$), there exist a positive integer n and an idempotent $e \in R$ such that $r_R(Y^n) = eR$. Generalized left π-Baer rings are defined similarly. A ring R is called generalized π-Baer if it is both generalized right and left π-Baer.*

The following result will be used many times in the sequel.

Proposition 2.2 *Let R be a ring. Then the following are equivalent.*

(i) R is a generalized right (left) π-Baer ring;

(ii) For each projection invariant left (right) ideal Y, there are an integer $n \geq 1$ and an idempotent $e \in \boldsymbol{S}_\ell(R)$ ($e \in \boldsymbol{S}_r(R)$) such that $r_R(Y^n) = eR$ ($\ell_R(Y^n) = Re$).

Proof (i)⇒(ii) Let Y be a projection invariant left ideal of R. Then there is an idempotent $e \in R$ such that $r_R(Y^n) = eR$, for some positive integer n. We show that $e \in \mathbf{S}_\ell(R)$. By [15, Proposition 1], it is enough to show that $fe = efe$, for each idempotent $f \in R$. Let $f \in R$ be an idempotent. Since $Y^n f \subseteq Y^n$, $eR = r_R(Y^n) \subseteq r_R(Y^n f)$. Thus $Y^n fe = 0$ and so $fe \in r_R(Y^n)$. Hence $fe = efe$, and the result follows.

(ii)⇒(i) It is obvious. □

Proposition 2.3

(i) A ring R is a generalized right π-Baer ring if and only if whenever Y is a projection invariant left ideal of R there exist an integer $n \geq 1$ and an idempotent $e \in \boldsymbol{S}_r(R)$ such that $Y^n \subseteq Re$ and $r_R(Y^n) \cap Re = (1-e)Re$.

(ii) A ring R is a generalized left π-Baer ring if and only if whenever Y is a projection invariant right ideal of R there exist an integer $m \geq 1$ and an idempotent $f \in \boldsymbol{S}_\ell(R)$ such that $Y^m \subseteq fR$ and $\ell_R(Y^m) \cap fR = fR(1-f)$.

Proof We prove only part (i), part (ii) can be shown similarly. Suppose that R is a generalized right π-Baer ring. Let Y be a projection invariant left ideal of R. Then there exist an idempotent $f \in \mathbf{S}_\ell(R)$ and a positive integer n such that $r_R(Y^n) = fR$. So $Y^n \subseteq \ell_R(r_R(Y^n)) = R(1-f)$. Set $e = 1-f$. Then $e \in \mathbf{S}_r(R)$ and $r_R(Y^n) \cap Re = (1-e)R \cap Re = (1-e)Re$.

Conversely, let Y be a projection invariant left ideal of R. Choose an idempotent $e \in \mathbf{S}_r(R)$ and an integer $n \geq 1$ such that $Y^n \subseteq Re$ and $r_R(Y^n) \cap eR = (1-e)Re$. Let $a \in r_R(Y^n)$. Then $a = ea + (1-e)a$, and that $ae = eae + (1-e)ae$. Since $ae \in r_R(Y^n) \cap Re$, $ae = (1-e)ae$. Thus $eae = 0$. Since $e \in \mathbf{S}_r(R)$, $ea = eae = 0$ and so $a = (1-e)a \in (1-e)R$. Hence, $r_R(Y^n) \subseteq (1-e)R$. Therefore, R is generalized right π-baer. □

Observe that every π-Baer ring is generalized right and left π-Baer. We give an example to show that the converse does not hold true.

Example 2.4 *Let* $R = \begin{pmatrix} \mathbb{R} & V \\ 0 & \mathbb{R} \end{pmatrix}$, *where* V *is a vector space over* $\mathbb{R}$ *with dimension* 1*. Then all the projection invariant left ideals of* R *are*

$$0, R, \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbb{R} & V \\ 0 & 0 \end{pmatrix}, \text{and} \begin{pmatrix} 0 & V \\ 0 & \mathbb{R} \end{pmatrix}.$$

Since $r_R(\begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix})$ *is not generated by an idempotent of* R, R *is not a* π*-Baer ring. But, it is a generalized right* π*-Baer ring, since* $r_R(\begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}^2) = r_R(0) = R$.

Proposition 2.5 *Let* R *be a ring. Consider the following conditions.*

(i) R *is generalized right (left) Baer.*

(ii) R *is generalized right (left)* π*-Baer.*

(iii) R *is generalized right (left) quasi-Baer.*

Then (i)⇒(ii)⇒(iii).

Proof (i)⇒(ii) Let Y be a projection invariant left ideal of R. Then there are an integer $n \geq 1$ and an idempotent $e \in R$ such that $r_R(Y^n) = eR$. Thus, R is generalized right π-Baer.

(ii)⇒(iii) Let I be an ideal of R. Then it is also a projection invariant left ideal of R. Thus, there exist a positive integer n and an element $e = e^2 \in R$ such that $r_R(I^n) = eR$. Hence, R is a generalized right quasi-Baer ring. □

The following example shows that the converse of each of the implications in Proposition 2.5 does not hold true.

Example 2.6 *(i) Let* R *be the ring as in Example 2.4. Take* $v \in V \setminus \{0\}$*. Since* $\begin{pmatrix} 1 & v \\ 0 & 0 \end{pmatrix}^n = \begin{pmatrix} 1 & v \\ 0 & 0 \end{pmatrix}$ *for each positive integer* n*, and* $r_R(\begin{pmatrix} 1 & v \\ 0 & 0 \end{pmatrix})$ *cannot be generated by an idempotent,* R *is not a generalized right Baer ring.*

(ii) Let R be the ring as in (i). Then by Corollary 3.4 below, $T_2(R)$ is a generalized right π-Baer ring. We show that $T_2(R)$ is not a generalized right Baer ring. Consider

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in T_2(R).$$

Then $A^n = A$, for each positive integer n. Let

$$\begin{pmatrix} a_1 & v_1 & a_2 & v_2 \\ 0 & b_1 & 0 & b_2 \\ 0 & 0 & a_3 & v_3 \\ 0 & 0 & 0 & b_3 \end{pmatrix} \in r_{T_2(R)}(A).$$

By direct computation, we see that $a_1 = v_1 = b_1 = 0$, and $\begin{pmatrix} a_2 & v_2 \\ 0 & b_2 \end{pmatrix} = -\begin{pmatrix} a_3 & v_3 \\ 0 & b_3 \end{pmatrix}$. Thus

$$r_{T_2(R)}(A) = \left\{ \begin{pmatrix} 0 & 0 & a & v \\ 0 & 0 & 0 & b \\ 0 & 0 & -a & -v \\ 0 & 0 & 0 & -b \end{pmatrix} \in T_2(R) \,\middle|\, a, b \in \mathbb{R}, v \in V \right\}.$$

Now, we claim that $r_{T_2(R)}(A)$ does not contain nonzero idempotents. For this, suppose that

$$\begin{pmatrix} 0 & 0 & a & v \\ 0 & 0 & 0 & b \\ 0 & 0 & -a & -v \\ 0 & 0 & 0 & -b \end{pmatrix} \in r_{T_2(R)}(A)$$

be a nonzero idempotent. By direct computation, $a = -a^2$ and $b = -b^2$, a contradiction. Hence $T_2(R)$ is not generalized right Baer.

(iii) Let R be a prime ring with trivial idempotents which is not domain. For example, let $R = KG$, where K is a field of characteristic $p > 0$, and $G = C_p \wr A$ be the restricted wreath product of C_p, the cyclic group of order p, and an infinite elementary abelian p-group (see [12, Example 3.4]). Then R is a quasi-Baer ring which is not π-Baer. By Theorem 3.8 below, the ring $S_n(R)$ is not generalized right π-Baer ring, for each integer $n \geq 2$ (See Definition 3.5 below for the definition of $S_n(R)$). But [19, Theorem 3.2] implies that $S_n(R)$ is a generalized right quasi-Baer ring.

(iv) Let $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}) \middle| a - d \equiv b \equiv c \equiv 0 \ (mod\ 2) \right\}$. Since R is prime, it is quasi-Baer and hence generalized right quasi-Baer. Also, the idempotents of R are $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $R\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, the left ideal generated by $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, is a projection invariant left ideal of R. Now, for each integer $n \geq 1$, $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \in r_R((R\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix})^n)$. Therefore, R is not generalized right π-Baer.

Let M_R be a right R-module. A submodule N_R of M_R is called *essential* in M_R if for any $x \in M \setminus \{0\}$, there exists $r \in R$ such that $0 \neq xr \in N$. Also recall a *right essential overring* T of R is an overring of R such that R_R is essential in T_R.

Proposition 2.7 *Every prime ideal of a generalized right (left) π-Baer ring is either generated by an idempotent or it is a right (left) essential ideal.*

Proof Let P be a prime ideal of R not essential as a right ideal. Then there is a nonzero right ideal Y of R such that $P \cap Y = 0$. Since R is generalized right quasi-Baer by Proposition 2.5, there are a positive integer n and an idempotent $e \in R$ such that $r_R(Y^n) = eR$. It is obvious that $P \subseteq r_R(Y^n) = eR$. Let $r \in r_R(Y^n)$. So $Y^n(rR) = 0$. Since P is a prime ideal, $Y^n \subseteq P$ or $rR \subseteq P$. If $Y^n \subseteq P$ then $Y \subseteq P$. Thus $Y \cap P = Y = 0$, a contradiction. Therefore, $r \in P$ and that $P = eR$. □

Proposition 2.8 *Let R be a generalized right (left) π-Baer ring and let Y be a projection invariant left (right) ideal of R such that $r_R(Y)$ is essential in R. Then Y is nilpotent.*

Proof Since R is a generalized right π-Baer ring, there is an idempotent $e \in R$ such that $r_R(Y^n) = eR$ for some integer $n \geq 1$. We claim that $Y^n = 0$. Assume to the contrary that $Y^n \neq 0$, so $e \neq 1$. Then $r_R(Y) \cap (1-e)R \subseteq r_R(Y^n) \cap (1-e)R = eR \cap (1-e)R = 0$, a contradiction since $r_R(Y)$ is essential in R. Hence, $Y^n = 0$. □

Definition 2.9 *Let R be a ring and T be an overring of R. R is said to satisfy the power intersection of projection invariant right (left) ideals property (right (left) PII for short) related to T, if for every projection invariant right (left) ideal Y of T and every positive integer n, there exists $m \geq n$ such that $(Y \cap R)^m = Y^m \cap R$.*

Theorem 2.10 *Let R be a generalized right (left) π-Baer ring, and T be a right and left essential overring of R. If R has the left (right) PII property related to T, then T is a generalized right (left) π-Baer ring.*

Proof Let Y be a projection invariant left ideal of T and $X = Y \cap R$. It is easy to see that X is a projection invariant left ideal of R. So there are $n \in \mathbb{N}$ and $e = e^2 \in R$ such that $r_R(X^n) = eT$. We claim that $r_T(Y^n) = eT$. Let $a \in r_T(Y^n)$ and assume that $0 \neq (1-e)a$. Since $R_R \leq^{ess} T_R$, there exists $r \in R$ such that $0 \neq (1-e)ar \in R$. Then $0 \neq (1-e)ar \in r_R(Y^n) \subseteq r_R(X^n)$, a contradiction. Thus, $r_T(Y^n) \subseteq eT$. Now, suppose $eT \not\subseteq r_T(Y^n)$. Then there is a $y \in Y^n$ such that $0 \neq ye$. Since ${}_RR \leq^{ess} {}_RT$, there is $s \in R$ such that $0 \neq sye \in R$. Hence $sye \in Y^n \cap R = (Y \cap R)^n = X^n$. Then $sye = (sye)e \in X^n e = 0$, a contradiction. Therefore, $r_T(Y^n) = eT$, and so T is a generalized right π-Baer ring. □

Recall from [2] that, a ring R is said to satisfy the IFP *(insertion of factors property)* if $r_R(x)$ is an ideal of R for all $x \in R$. A ring R is called abelian if every idempotent in it is central. It is evident that any reduced ring satisfies IFP and any ring with IFP is abelian.

We include the following results to demonstrate the conditions in which the generalized right π-Baer, π-Baer ring, and generalized right quasi-Baer ring are coincide.

Proposition 2.11 *Let R be a ring, then:*

(i) A reduced ring R is generalized right (left) π-Baer if and only if R is π-Baer.

(ii) A semiprime ring R is generalized right (left) π-Baer if and only if it is π-Baer.

(iii) A ring R satisfying IFP is generalized right (left) π-Baer if and only if it is generalized right (left) quasi-Baer.

(iv) A ring R that is generated by its idempotents is generalized right (left) π-Baer if and only if it is generalized right (left) quasi-Baer.

(v) A semiprime generalized right (left) π-Baer ring R is quasi-Baer.

Proof (i) Let R be a reduced generalized right π-Baer ring and Y a projection invariant left ideal of R. Then $r_R(Y^n) = eR$, for some positive integer n and some idempotent $e \in R$. Since R is reduced, $r_R(Y) = r_R(Y^n) = eR$, which implies that R is a π-Baer ring. The converse is trivial.

(ii) Let R be a generalized right π-Baer ring and Y a projection invariant left ideal of R. Then $r_R(Y^n) = eR$, for some positive integer n, and some idempotent $e \in R$. By Proposition 2.2(ii) $e \in \mathbf{S}_\ell(R)$. So $Y^n e = (Ye)^n = 0$. Since R is semiprime, $Ye = 0$. Thus $eR \subseteq r_R(Y) \subseteq r_R(Y^n) = eR$. Hence $r_R(Y) = eR$ and that R is π-Baer. The converse is trivial.

(iii) Let Y be a projection invariant left ideal of R. Since R satisfying IFP, $r_R((YR)^n) = r_R(Y^n)$ for each positive integer n. Thus if R is generalized right quasi-Baer, then it is a generalized right π-Baer ring. The converse follows from Proposition 2.5.

(iv) Note that every projection invariant one-sided ideal of R is an ideal of R by [4, Corollary 2.2(iii)]. Now, Proposition 2.5 yields the result.

(v) The result follows from Proposition 2.5 and [19, Proposition 2.2]. □

Proposition 2.12 *Let R be a right (left) Noetherian ring with IFP. Then the following conditions are equivalents.*

(i) R is a generalized right (left) Baer ring;

(ii) R is a generalized right (left) π-Baer ring;

(iii) R is a generalized right (left) quasi-Baer ring;

(iv) R is a generalized right (left) p.q.-Baer ring;

(v) R is a generalized right (left) p.p. ring.

Proof (i)⇒(ii)⇒(iii)⇒(iv) These implications are clear.

(iv)⇔(v) Follows from [19, Proposition 2.2(ii)].

(v)⇒(i) Follows from [20, Proposition 2.5].

□

The next example shows the existence of a generalized p.p. ring, which is not generalized right π-Baer.

Example 2.13 *For a field F, take $F_n = F$ for $n = 1, 2, \ldots$. Let*

$$R = \begin{pmatrix} \prod_{n=1}^{\infty} F_n & \bigoplus_{n=1}^{\infty} F_n \\ \bigoplus_{n=1}^{\infty} F_n & \langle \bigoplus_{n=1}^{\infty} F_n, 1 \rangle \end{pmatrix},$$

which is considered as a subring of $\mathrm{M}_2(\prod_{n=1}^{\infty} F_n)$. *By [6, Example 1.6] R is a semiprime p.p. ring (and hence generalized p.p. ring) which is not p.q.-Baer. Thus by [19, Proposition 2.2(i)] R is not generalized right p.q.-Baer (and hence not generalized right* π*-Baer).*

Subrings of generalized right π-Baer rings need not be generalized right π-Baer as shown in the next example.

Example 2.14 *Let* p *be a prime number and* $R = \{(a,b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv b \ (mod\ p)\}$, *which is a subring of* $\mathbb{Z} \oplus \mathbb{Z}$. *Obviously,* $\mathbb{Z} \oplus \mathbb{Z}$ *is generalized right* π*-Baer. We notice that the only idempotents of* R *are* $(0,0)$ *and* $(1,1)$. *One can show that* $r_R((R(p,0))^n) = r_R(R(p,0)) = (0,p)R$, *for each positive integer* n. *Thus* $r_R(R(p,0))$ *does not contain a nonzero idempotent of* R. *Hence* R *is not generalized right* π*-Baer.*

Proposition 2.15 *Let* R *be a ring with* IFP. *Then the following conditions are equivalent.*

- *(i)* R *is generalized right* π*-Baer.*
- *(ii) For each nonempty set* S *such that* $Se \subseteq S$, *for all* $e = e^2 \in R$, *there exist* $c = c^2 \in R$ *and integer* $n \geq 1$ *such that* $r_R(S^n) = cR$.

Proof (i)⇒(ii) Let R be a generalized right π-Baer ring and S be a nonempty set such that $Se \subseteq S$, for all idempotent $e \in R$. Then RS is a left ideal of R and $RSe = R(Se) \subseteq RS$. Thus RS is a projection invariant left ideal of R. Hence $r_R(S^n) = r_R(R(S)^n) = r_R((RS)^n) = eR$ for some idempotent $e \in R$ and some positive integer n, and the result follows.

(ii)⇒(i) It is straightforward. □

Proposition 2.16 *Let* R *be a generalized right* π*-Baer ring. If every essential right ideal of* R *is an essential extension of an ideal of* R (*e.g., this is the case if* R *has essential right socle or if* R *is left perfect*), *then the right singular ideal* $\mathcal{Z}_r(R)$ *is nil.*

Proof Let $x \in \mathcal{Z}_r(R)$. Then $r_R(x)$ is right essential in R. So there exists an ideal I of R such that $I \subseteq r_R(x)$ and that is right essential in R. Since R is generalized right quasi-Baer by Proposition 2.5, there exists an idempotent $e \in R$ such that $r_R((xR)^n) = eR$ for some positive integer n. Since $0 = xI = xRI$, $(xR)^n I = 0$ and that $I \subseteq r_R((xR)^n) = eR$. We show that $x^n = 0$. Assume to the contrary that $x^n \neq 0$. So $e \neq 1$ and $I \cap (1-e)R \subseteq eR \cap (1-e)R = 0$. Thus $I \cap (1-e)R = 0$, a contradiction, since I is right essential. Therefore $x^n = 0$, and the result follows. □

Theorem 2.17 *Let* R *be a generalized right (left)* π*-Baer ring and* $e \in \boldsymbol{S}_\ell(R)$ $(e \in \boldsymbol{S}_r(R))$. *Then* eRe *is generalized right (left)* π*-Baer.*

Proof Assume Y is a projection invariant left ideal of eRe. Consider $X = R(1-e) \oplus Y$. We show that X is a projection invariant left ideal of R. It is easy to see that Y is a left ideal of R. Let $a(1-e) + y \in X$ and $f = f^2 \in R$. Then $(a(1-e)+y)f = a(1-e)f + yf = a(1-e)f + yfe + yf(1-e)$. Now $a(1-e)f = a(1-e)f(1-e) \in R(1-e)$ since $1-e \in \mathbf{S}_r(R)$. Also $fe = efe = (ef)^2 \in eRe$, so $yfe \in Y$. Thus $(a(1-e)+y)f \in X$, and that X is a projection invariant left ideal of R. Since R is generalized right π-Baer, there are an integer $n \geq 1$ and an idempotent $c \in \mathbf{S}_\ell(R)$ such that $r_R(X^n) = cR$. Then $er_R(X^n)e = ecRe$. We have $ece = cece = ecece = (ece)^2 \in eRe$. Then $ecRe = (ece)(cRe) = (ece)(ecRe) \subseteq (ece)(eRe)$. We claim that $(ece)(eRe) \subseteq ecRe$. Let $(ece)(ere) \in (ece)(eRe)$. Then

$$(ece)(ere) = (ec)(ere) = (cec)(ere) = (ecec)(ere) = (ece)(c(er)e)$$
$$= (ece)(e(c(er)e) \in (ece)(ecRe) = ecRe.$$

Hence $ecRe = (ece)(eRe)$ is a direct summand of eRe. We claim that $r_{eRe}(Y^n) = ecRe$. Since $Y^n \subsetneq X^n$, $(ecRe)Y^n = 0$, so $ecRe \subseteq r_{eRe}(Y^n)$. Next, assume that $a \in r_{eRe}(Y^n)$. Let $x = \sum_{i=1}^m x_i \in X^n$, where for each $i = 1, 2, \ldots, m$, x_i is a product of n elements of X. Then we have the following two cases.

(i) x_i is a product of n elements of Y. Then $x_i \in Y^n$, and that $x_i a = 0$.

(ii) x_i has at least one element of $R(1-e)$. Suppose that $x_i = y_1 \cdots y_t r(1-e) y_{t+1} \cdots y_{m-1}$. Since $1 - e \in \mathbf{S}_r(R)$, $x_i = y_1 \cdots y_t r(1-e) y_{t+1} \cdots y_{m-1}(1-e)$. Thus $x_i a = 0$, since $a \in eRe$.

Hence $xa = 0$ and so $X^n a = 0$. So $a \in eRe \cap r_R(X^n) = ecRe$. Thus $r_{eRe}(Y^n) = ecRe = (ece)(eRe)$, and that eRe is generalized right π-Baer. □

Lemma 2.18 *Let R be a ring, Y be a projection invariant left ideal of R, and $r_R(Y^n) = eR$ for some positive integer n and some idempotent $e \in B(R)$. Then $r_R(Y^n) = r_R(Y^m)$ for each integer $m \geq n$.*

Proof We show that $r_R(Y^n) = r_R(Y^{n+1})$. Let $r \in r_R(Y^{n+1})$. Then $Yr \subseteq r_R(Y^n) = eR$. Hence $Yr = Yre$ and so $Y^n r = Y^n re = Y^n er = 0$. Thus $r \in r_R(Y^n)$. This shows that $r_R(Y^{n+1}) \subseteq r_R(Y^n)$. The reverse inclusion is trivial. □

As a consequence of Lemma 2.18, we have the following.

Proposition 2.19 *Every generalized right π-Baer ring satisfies the* ACC *on the right annihilators of projection invariant left ideals.*

Corollary 2.20 *Let R be a right perfect generalized right π-Baer ring. Then Jacobson radical of R is nilpotent, and hence R is a semiprimary ring.*

Proof This follows from Proposition 2.19 and [18, Exercise 24.8]. □

Proposition 2.21 *Let Λ be a nonempty set and let R_λ be a ring for each $\lambda \in \Lambda$.*

(i) If $R = \prod_{\lambda \in \Lambda} R_\lambda$ is a generalized right π-Baer ring, then R_λ is a generalized right π-Baer ring for each $\lambda \in \Lambda$.

(ii) If $|\Lambda| < \infty$ and for each $\lambda \in \Lambda$, R_λ is a generalized right π-Baer ring with $\boldsymbol{S}_\ell(R_\lambda) = \boldsymbol{B}(R_\lambda)$, then $R = \prod_{\lambda \in \Lambda} R_\lambda$ is a generalized right π-Baer ring.

Proof (i) The proof follows immediately from Theorem 2.17.

(ii) It is enough to take $\Lambda = \{1, 2, \ldots, k\}$ for some $k \in \mathbb{N}$. Put $R = \prod_{i=1}^k R_i$. Assume that R_i be a generalized right π-Baer ring with $\mathbf{S}_\ell(R_i) = \mathbf{B}(R_i)$, for each $i = 1, \ldots, k$. Let Y be a projection invariant left ideal of R. It is easy to see that $Y = \prod_{i=1}^{k} Y_i$ for some projection invariant left ideal Y_i of R_i. Since R_i is generalized right π-Baer, $r_{R_i}(Y_i^{n_i}) = e_iR_i$, for some idempotents $e_i \in \mathbf{B}(R_i)$ and some positive integers n_i. Put $n = \max\{n_1, n_2, \ldots, n_k\}$. By Lemma 2.18, $r_{R_i}(Y_i^n) = e_iR_i$ for each i. Then $r_R(Y^n) = \prod_{i=1}^{k} e_iR_i$. Put $e = (e_1, \ldots, e_k) \in R$, it is clear that e is an idempotent of R. So $r_R(Y^n) = eR$. Therefore, R is a generalized right π-Baer ring. □

Proposition 2.22 *Let R be a generalized π-Baer ring, then $\mathrm{C}(R)$ is generalized Baer (and hence generalized π-Baer).*

Proof

Let S be a nonempty subset of $\mathrm{C}(R)$. Since R is a generalized π-Baer ring, Proposition 2.5 implies that $r_R(S^m) = r_R((RSR)^m) = eR$ and $\ell_R(S^n) = \ell_R((RSR)^n) = Rf$ for some idempotents $e, f \in R$ and some positive integers n, m. We may assume that m, n are the smallest such ones. We claim that $m = n$. First assume $m < n$, then $f \notin \ell_R(S^m)$. Since $S \subseteq \mathrm{C}(R)$ and $m < n$, $e \in \ell_R(S^n) = Rf$ and $fS^{n-m} \subseteq r_R(S^m) = eR$. Thus $ef = e$ and that $fS^{n-m} = efS^{n-m} = eS^{n-m}$. Now if $n \geq 2m$ then $n - m \geq m$ so $0 \neq fS^{n-m} = eS^{n-m} = S^m eS^{n-2m} = 0$, a contradiction. Thus $m < n < 2m$ so $0 \neq fS^m = fS^{n-m}S^{2n-m} = eS^{n-m}S^{2n-m} = eS^m = S^m e = 0$, a contradiction. It follows that $m \geq n$. Next suppose $m \geq n$. Then by similar arguments as in the preceding case we also get a contradiction. Hence $m = n$, and $eR = r_R(S^m) = \ell_R(S^m) = Rf$. Thus $e = ef = f \in B(R)$. Now, we have $r_{\mathrm{C}(R)}(S^n) = r_R(S^n) \cap \mathrm{C}(R) = eR \cap \mathrm{C}(R) = e\mathrm{C}(R)$. Hence $\mathrm{C}(R)$ is generalized Baer. □

There exists a ring R with generalized π-Baer center which is not generalized right π-Baer.

Example 2.23 *Let K be a field and $R = K[x, y, z]$, where x, y and z are indeterminants satisfying the relations $xy = xz = zx = yx = 0$ and $yz \neq zy$. Then R is reduced and $C(R) = K[x]$ is Baer and so generalized right π-Baer. But for each integer $n \geq 1$, $r_R((yR)^n) = r_R(yR) = xR$ does not have any nonzero idempotents. Thus R is not generalized right p.q.-Baer, and hence it is not generalized right π-Baer.*

3. Matrix extensions

In this section, we study the trivial extension, the full matrix extension, and certain triangular matrix extensions of generalized right π-Baer rings.

Proposition 3.1 *Let R be a generalized right (left) π-Baer ring. Then $\mathrm{M}_n(R)$ is a generalized right (left) π-Baer ring for each positive integer n.*

Proof Let R be a generalized right π-Baer ring. Then by Proposition 2.5, R is generalized right quasi-Baer. Now [19, Theorem 4.7] implies that $\mathrm{M}_n(R)$ is generalized right quasi-Baer. Since $\mathrm{M}_n(R)$ is generated by its idempotents, Proposition 2.11(iv) yields the result. □

We continue by describing a very useful ring-theoretic construction called the "trivial extension". Let R be a ring and M be a (R, R)-bimodule. We form $T := T(R, M)$, and define a multiplication on T by the rule

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

The ring T constructed in this way is called the "trivial extension" of M by R. It is sometimes useful to view T as the set of all matrices $\begin{pmatrix} a & m \\ 0 & a \end{pmatrix}$, where $a \in R$ and $m \in M$, using matrix multiplication.

The next Lemma follows from [4, Lemma 3.1].

Lemma 3.2 *Let T be the trivial extension of M by R. Let X, Y be additive subgroups of R and N be an additive subgroup of M. Then $\begin{pmatrix} X & N \\ 0 & Y \end{pmatrix}$ is a projection invariant left ideal of T if and only if the following conditions hold.*

(i) X and Y are projection invariant left ideals of R.

(ii) $RN + MY \subseteq N$.

(iii) N is a $(R, \boldsymbol{I}(R))$-bisubmodule of M.

(iv) $XM \subseteq N$.

Theorem 3.3 *Let T be the trivial extension of M by R. Then the following are equivalent.*

(i) T is a generalized right π-Baer ring;

(ii) (a) R is a generalized right π-Baer ring;

(b) If $\begin{pmatrix} X & N \\ 0 & Y \end{pmatrix}$ is a projection invariant left ideal of T, then there exist an integer $n \geq 1$ and an idempotent $e \in R$ such that $r_M(X^n) = (r_R(X^n))M$, and $r_R(Y^n) \cap r_R(X^{n-1}N) \cap r_R(X^{n-2}NY) \cap \ldots \cap r_R(NY^{n-1}) = eR$.

Proof (i)⇒(ii) (a) Since $\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbf{S}_\ell(T)$, Theorem 2.17 implies that $\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}$ is generalized right π-Baer. Thus R is generalized right π-Baer.

(b) Assume that $\begin{pmatrix} X & N \\ 0 & Y \end{pmatrix}$ is a projection invariant left ideal of T. Then there exist an idempotent $e \in \mathbf{S}_\ell(T)$ and a positive integer n such that $r_T(\begin{pmatrix} X & N \\ 0 & Y \end{pmatrix}^n) = eT$. By [9, Lemma 2.3], $e = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$ where $e_1, e_2 \in \mathbf{S}_\ell(R)$. So

$$r_T(\begin{pmatrix} X^n & \circledast \\ 0 & Y^n \end{pmatrix}) = \begin{pmatrix} r_R(X^n) & r_M(X^n) \\ 0 & r_R(Y^n) \cap r_R(\circledast) \end{pmatrix} = \begin{pmatrix} e_1 R & e_1 M \\ 0 & e_2 R \end{pmatrix},$$

where $\circledast = X^{n-1}N + X^{n-2}NY + \cdots + NY^{n-1}$. Thus $r_M(X^n) = e_1 M = (r_R(X^n))M$, and $r_R(Y^n) \cap r_R(X^{n-1}N) \cap r_R(X^{n-2}NY) \cap \ldots \cap r_R(NY^{n-1}) = r_R(Y^n) \cap r_R(\circledast) = e_2 R$.

(ii)⇒(i) Suppose that $\begin{pmatrix} X & N \\ 0 & Y \end{pmatrix}$ be a projection invariant left ideal of T. Then by Lemma 3.2, X, Y are projection invariant left ideals of R. So there are idempotents $e_1, e_2 \in R$ and a positive integer n such that $r_R(X^n) = e_1R$, $r_M(X^n) = e_1M$, and $r_R(Y^n) \cap r_R(X^{n-1}N + X^{n-2}NY + \cdots + NY^{n-1}) = e_2R$. Thus $r_T(\begin{pmatrix} X & N \\ 0 & Y \end{pmatrix}^n) = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} T$, and so T is generalized right π-Baer. □

Corollary 3.4 *Let R be a ring with $\boldsymbol{S}_\ell(R) = \boldsymbol{B}(R)$. Then R is generalized right π-Baer if and only if for each positive integer k, $T_{2^k}(R)$ is generalized right π-Baer.*

Proof We proceed by induction on k. Note that $T_{2^{k+1}}(R)$ is the trivial extension of $\mathrm{M}_{2^k}(R)$ by $T_{2^k}(R)$. Let $\begin{pmatrix} X & N \\ 0 & Y \end{pmatrix}$ be a projection invariant left ideal of $T_{2^{k+1}}(R)$. Since $T_{2^k}(R)$ is generalized right π-Baer, there exist idempotents $e_1, e_2 \in T_{2^k}(R)$ and positive integers n, m such that $r_{T_{2^k}(R)}(X^n) = e_1T_{2^k}(R)$ and $r_{T_{2^k}(R)}(Y^m) = e_2T_{2^k}(R)$. Put $t = max\{n, m\}$. Since $\mathbf{S}_\ell(R) = \mathbf{B}(R)$, [9, Lemma 2.3] implies that $e_1, e_2 \in \mathbf{B}(T_{2^k}(R))$. Now, by Lemma 2.18 $r_{T_{2^k}(R)}(X^{2t}) = e_1T_{2^k}(R)$ and $r_{T_{2^k}(R)}(Y^{2t}) = e_2T_{2^k}(R)$. Also $r_{\mathrm{M}_{2^k}(R)}(Y^{2t}) = e_2\mathrm{M}_{2^k}(R)$. Take $\divideontimes = X^{2t-1}N + X^{2t-2}NY + \cdots + NY^{2t-1}$. We show that $r_{T_{2^k}(R)}(Y^{2t}) \cap r_{T_{2^k}(R)}(\divideontimes) = e_1e_2T_{2^k}(R)$. Let $A \in r_{T_{2^k}(R)}(Y^{2t}) \cap r_{T_{2^k}(R)}(\divideontimes)$. Then $A \in e_2T_{2^k}(R)$. Since $X^{2t}\mathrm{M}_{2^k}(R) + \mathrm{M}_{2^k}(R)Y^{2t} \subseteq \divideontimes$, $(X^{2t}\mathrm{M}_{2^k}(R) + \mathrm{M}_{2^k}(R)Y^{2t})A = X^{2t}\mathrm{M}_{2^k}(R)A = 0$. Thus $A \in e_1T_{2^k}(R) \cap e_2T_{2^k}(R) = e_1e_2T_{2^k}(R)$. Hence the condition (b) of Theorem 3.3 holds. Thus, $T_{2^{k+1}}(R)$ is generalized π-Baer from Theorem 3.3. The converse is a consequence of Theorem 3.3.

□

In the following, we investigate the matrix algebras $S_n(R)$, $A_n(R)$, $B_n(R)$, $U_n(R)$ and $V_n(R)$, which also give a good supply of examples of rings which are generalized π-Baer rings.

Definition 3.5 ([1], Definition 3.1) *Let R be a ring with unity. Let $V_n = \sum_{i=1}^{n-1} E_{i,i+1}$, where $E_{i,j}$, $1 \leq i, j \leq n$, are the matrix units. For any integer $n \geq 2$, define*

$$A_n(R) = R\mathrm{I}_n + \sum_{\ell=2}^{[\frac{n}{2}]} RV_n^{\ell-1} + \sum_{i=1}^{[\frac{n+1}{2}]} \sum_{j=[\frac{n}{2}]+i}^{n} RE_{i,j},$$

and

$$B_n(R) = R\mathrm{I}_n + \sum_{\ell=3}^{[\frac{n}{2}]} RV_n^{\ell-2} + \sum_{i=1}^{[\frac{n+1}{2}]+1} \sum_{j=[\frac{n}{2}]+i-1}^{n} RE_{i,j}.$$

Then we have

$$A_n(R) = \left\{ \left(\begin{array}{cccccccc} a_1 & a_2 & \cdots & a_k & b_{1,k+1} & b_{1,k+2} & \cdots & b_{1,n} \\ 0 & a_1 & \cdots & a_{k-1} & a_k & b_{2,k+2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_1 & a_2 & \cdots & a_k & b_{\ell,n} \\ 0 & 0 & \cdots & 0 & a_1 & \cdots & a_{k-1} & a_k \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & a_1 & a_2 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_1 \end{array} \right) \middle| \begin{array}{c} k = [\frac{n}{2}] \\ \ell = [\frac{n+1}{2}] \\ a_t, b_{i,j} \in R \\ 1 \leq t \leq k \\ 1 \leq i \leq \ell \\ k+1 \leq j \leq n \end{array} \right\},$$

and

$$B_n(R) = \left\{ \left(\begin{array}{cccccccc} a_1 & a_2 & \cdots & a_{k-1} & b_{1,k} & b_{1,k+1} & \cdots & b_{1,n} \\ 0 & a_1 & \cdots & a_{k-2} & a_{k-1} & b_{2,k+1} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_1 & a_2 & \cdots & a_{k-1} & b_{\ell+1,n} \\ 0 & 0 & \cdots & 0 & a_1 & \cdots & a_{k-2} & a_{k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & a_1 & a_2 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_1 \end{array} \right) \middle| \begin{array}{c} k = [\frac{n}{2}] \\ \ell = [\frac{n+1}{2}] \\ a_t, b_{i,j} \in R \\ 1 \le t \le k-1 \\ 1 \le i \le \ell+1 \\ k \le j \le n \end{array} \right\}.$$

Furthermore, for an integer $n \geq 2$, *define*

$$U_n(R) = R\mathrm{I}_n + \sum_{i=1}^{[\frac{n-1}{2}]} \sum_{j=[\frac{n}{2}]+1}^{n} RE_{i,j} + \sum_{j=[\frac{n-1}{2}]+2}^{n} RE_{[\frac{n-1}{2}]+1,j}.$$

So for odd integers n, *we have*

$$U_n(R) = \left\{ \left(\begin{array}{ccccccccc} a & 0 & \cdots & 0 & 0 & b_{1,k+1} & b_{1,k+2} & \cdots & b_{1,n} \\ 0 & a & \cdots & 0 & 0 & b_{2,k+1} & b_{2,k+2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a & 0 & b_{\ell-1,k+1} & b_{\ell-1,k+2} & \cdots & b_{\ell-1,n} \\ 0 & 0 & \cdots & 0 & a & b_{\ell,k+1} & b_{\ell,k+2} & \cdots & b_{\ell,n} \\ 0 & 0 & \cdots & 0 & 0 & a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & a \end{array} \right) \middle| \begin{array}{c} k = [\frac{n}{2}] \\ \ell = [\frac{n-1}{2}] \\ a, b_{i,j} \in R \\ 1 \le i \le \ell \\ k+1 \le j \le n \end{array} \right\},$$

and for even integers, we have

$$U_n(R) = \left\{ \left(\begin{array}{ccccccccc} a & 0 & \cdots & 0 & 0 & b_{1,k+1} & b_{1,k+2} & \cdots & b_{1,n} \\ 0 & a & \cdots & 0 & 0 & b_{2,k+1} & b_{2,k+2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a & 0 & b_{\ell,k+1} & b_{\ell,k+2} & \cdots & b_{\ell,n} \\ 0 & 0 & \cdots & 0 & a & b_{\ell+1,k+1} & b_{\ell+1,k+2} & \cdots & b_{\ell+1,n} \\ 0 & 0 & \cdots & 0 & 0 & a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & a \end{array} \right) \middle| \begin{array}{c} k = [\frac{n}{2}] \\ \ell = [\frac{n-1}{2}] \\ a, b_{i,j} \in R \\ 1 \le i \le \ell+1 \\ k+1 \le j \le n \end{array} \right\}.$$

The ring $S_n(R)$ *is defined as a subring of* $T_n(R)$ *as follows:*

$$S_n(R) = R\mathrm{I}_n + \sum_{i<j} RE_{i,j}.$$

$$S_n(R)=\left\{\left.\begin{pmatrix} a & b_{1,2} & b_{1,3} & \cdots & b_{1,n-1} & b_{1,n}\\ 0 & a & b_{2,3} & \cdots & b_{2,n-1} & b_{2,n}\\ 0 & 0 & a & \cdots & b_{3,n-1} & b_{3,n}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & a & b_{n-1,n}\\ 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix}\right| \begin{array}{c} a, b_{i,j}\in R\\ 1\le i\le n-1\\ 2\le j\le n\end{array}\right\}.$$

Also, the ring $V_n(R)$ is defined as a subring of $S_n(R)$ as follows:

$$V_n(R)=R\mathrm{I}_n+\sum_{\ell=2}^{n} RV_n^{\ell-1}.$$

Then we have

$$V_n(R)=\left\{\left.\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n\\ 0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1}\\ 0 & 0 & a_1 & \cdots & a_{n-3} & a_{n-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & a_1 & a_2\\ 0 & 0 & 0 & \cdots & 0 & a_1 \end{pmatrix}\right| \begin{array}{l} a_i\in R\\ 1\le i\le n\end{array}\right\}.$$

Lemma 3.6 *Let R be a ring and $n\geqslant 2$ be an integer. Then the following conditions are equivalent.*

(i) $\boldsymbol{S}_\ell(R)=\boldsymbol{B}(R)$ *(resp.,* $\boldsymbol{S}_r(R)=\boldsymbol{B}(R)$*);*

(ii) $\boldsymbol{S}_\ell(S_n(R))=\boldsymbol{B}(S_n(R))$ *(resp.,* $\boldsymbol{S}_r(S_n(R))=\boldsymbol{B}(S_n(R))$*);*

(iii) $\boldsymbol{S}_\ell(A_n(R))=\boldsymbol{B}(A_n(R))$ *(resp.,* $\boldsymbol{S}_r(A_n(R))=\boldsymbol{B}(A_n(R))$*);*

(iv) $\boldsymbol{S}_\ell(B_n(R))=\boldsymbol{B}(B_n(R))$ *(resp.,* $\boldsymbol{S}_r(B_n(R))=\boldsymbol{B}(B_n(R))$*);*

(v) $\boldsymbol{S}_\ell(U_n(R))=\boldsymbol{B}(U_n(R))$ *(resp.,* $\boldsymbol{S}_r(U_n(R))=\boldsymbol{B}(U_n(R))$*);*

(vi) $\boldsymbol{S}_\ell(V_n(R))=\boldsymbol{B}(V_n(R))$ *(resp.,* $\boldsymbol{S}_r(V_n(R))=\boldsymbol{B}(V_n(R))$*).*

In particular every central idempotent in the ring $S_n(R)$ (resp., $A_n(R)$, $B_n(R)$, $U_n(R)$ or $V_n(R)$), is of the form $e\mathrm{I}_n$, where $e\in R$ is an idempotent.

Proof Note that the ring $S_n(R)$ (resp., $A_n(R)$, $B_n(R)$, $U_n(R)$ or $V_n(R)$) consisting of the elements such that all entries on their main diagonal are the same. Thus the proof follows by using a similar argument as in the proof of [16, Lemma 2]. □

Lemma 3.7 *Let $n\geq 2$, and I be a projection invariant left (right) ideal of the ring $S_n(R)$ (resp., $A_n(R)$, $B_n(R)$, $U_n(R)$ or $V_n(R)$). Then we have the following.*

(i) If I_1^1 denotes the set of matrices A in $S_{n-1}(R)$ (resp., $A_{n-1}(R)$, $B_{n-1}(R)$, $U_{n-1}(R)$ or $V_{n-1}(R)$) such that A is obtained by deleting the first row and the first column of a matrix in I, then I_1^1 is a projection invariant left (right) ideal.

(ii) *If I_n^n denotes the set of matrices A in $S_{n-1}(R)$ (resp., $A_{n-1}(R)$, $B_{n-1}(R)$, $U_{n-1}(R)$ or $V_{n-1}(R)$) such that A is obtained by deleting the n-th row and the n-th column of a matrix in I, then I_n^n is a projection invariant left (right) ideal.*

(iii) *If J is the set of entries of the main diagonal of the elements of I, then J is a projection invariant left (right) ideal of R.*

Proof We prove the case of $S_n(R)$. The other cases can be shown similarly.

(i) Let $A \in I_1^1$ be obtained by deleting the first row and the first column of $B = b\mathrm{I}_n + \sum_{1\le i<j\le n} b_{i,j}E_{i,j} \in I$. Then $A = b\mathrm{I}_{n-1} + \sum_{2\le i<j\le n} b_{i,j}E_{i-1,j-1}$. Now, let $E = e\mathrm{I}_{n-1} + \sum_{1\le i<j\le n-1} e_{i,j}E_{i,j}$ be an idempotent of $S_{n-1}(R)$. It is not hard to see that $E' = e\mathrm{I}_n + \sum_{1\le i<j\le n-1} e_{i,j}E_{i+1,j+1}$ is an idempotent of $S_n(R)$. Since I is a projection invariant left ideal of $S_n(R)$, $BE' \in I$. By computation we have

$$BE' = be\mathrm{I}_n + \sum_{j=2}^{n}(b_{1,2}e_{1,j-1} + b_{1,3}e_{2,j-1} + \cdots + b_{1,j-1}e_{j-2,j-1} + b_{1,j}e)E_{1,j}$$

$$+ \sum_{2\le i<j\le n}(be_{i-1,j-1} + b_{i,j-1}e_{i,j-1} + \cdots + b_{j-2,j-1}e_{j-2,j-1} + b_{i,j}e)E_{i,j}, \text{ and}$$

$$AE = be\mathrm{I}_{n-1} + \sum_{2\le i<j\le n}(be_{i-1,j-1} + b_{i,j-1}e_{i,j-1} + \cdots + b_{j-2,j-1}e_{j-2,j-1} + b_{i,j}e)E_{i-1,j-1}.$$

Thus AE is the matrix obtained by deleting the first row and the first column of BE', and so $AE \in I_1^1$. Therefore, I_1^1 is a projection invariant left ideal of $S_{n-1}(R)$.

(ii) follows from a similar argument as in the proof of (i).

(iii) Since $J = \underbrace{((I_1^1)_1^1)\cdots)_1^1}_{n-1}$, the proof follows by repeated use of (i). □

Theorem 3.8 *Let R be a ring with $\boldsymbol{S}_\ell(R) = \boldsymbol{B}(R)$ (resp., $\boldsymbol{S}_r(R) = \boldsymbol{B}(R)$), and $n \ge 2$. Then the following are equivalent.*

(i) *R is a generalized right (resp., left) π-Baer ring;*

(ii) *$S_n(R)$ is a generalized right (resp., left) π-Baer ring;*

(iii) *$A_n(R)$ is a generalized right (resp., left) π-Baer ring;*

(iv) *$B_n(R)$ is a generalized right (resp., left) π-Baer ring;*

(v) *$U_n(R)$ is a generalized right (resp., left) π-Baer ring;*

(vi) *$V_n(R)$ is a generalized right (resp., left) π-Baer ring.*

Proof We prove only the equivalence (i)⇔(ii), the other cases are similar.

(i)⇒(ii) We proceed by induction on n. Assume that R is a generalized right π-Baer ring. First, we claim that $S_2(R)$ is a generalized right π-Baer ring. Let I be a projection invariant left ideal of $S_2(R)$ and J be the set of entries of the main diagonal of the elements of I. Then by Lemma 3.7(iii) J is a projection invariant left ideal of R. Since R is generalized right π-Baer, $r_R(J^m) = eR$ for some idempotent $e \in R$ and some integer $m \geq 1$. By Proposition 2.2 and assumption, we may assume that $e \in \mathbf{B}(R)$. Thus $r_R(J^m) = r_R(J^{m+1}) = \cdots = r_R(J^{2m}) = eR$, by Lemma 2.18. For each $k \in \mathbb{N}$ and each $a_i\mathrm{I}_2 + b_iE_{1,2} \in I$, $1 \leq i \leq k$, we have

$$(a_1\mathrm{I}_2 + b_1E_{1,2}) \cdots (a_k\mathrm{I}_2 + b_kE_{1,2}) = a_1 \ldots a_k\mathrm{I}_2 + bE_{1,2},$$

where b is the sum of k terms in which each term is a product of $k-1$ elements of the set $\{a_1, a_2, \ldots, a_k\}$ and one element of the set $\{b_1, b_2, \ldots, b_k\}$. Put $f = e\mathrm{I}_2$, then $f \in \mathbf{B}(S_2(R))$. We claim that $r_{S_2(R)}(I^{2m}) = fS_2(R)$. Since $r_R(J^m) = eR$, $(a_1\mathrm{I}_2+b_1E_{1,2}) \cdots (a_{2m}\mathrm{I}_2+b_{2m}E_{1,2})(e\mathrm{I}_2) = 0$, for each $a_i\mathrm{I}_2+b_iE_{1,2} \in I$, $1 \leq i \leq 2m$. Thus $fS_2(R) \subseteq r_{S_2(R)}(I^{2m})$. Now, if $x\mathrm{I}_2 + yE_{1,2} \in r_{S_2(R)}(I^{2m})$, then $(a_1 \ldots a_{2m}\mathrm{I}_2 + bE_{1,2})(x\mathrm{I}_2 + yE_{1,2}) = 0$, for each $a_1 \ldots a_{2m}\mathrm{I}_2 + bE_{1,2} \in I^{2m}$. Then it follows that $a_1 \cdots a_{2m}x = 0$ and $a_1 \cdots a_{2m}y + bx = 0$. Consequently, $x \in eR$, and so $x = ex$. Therefore, $bx = bex = 0$, and that $a_1 \cdots a_{2m}y = 0$. Thus $y = ey$. So $x\mathrm{I}_2 + yE_{1,2} = (e\mathrm{I}_2)(x\mathrm{I}_2 + yE_{1,2})$. Hence $r_{S_2(R)}(I^{2m}) \subseteq fS_2(R)$, and so $S_2(R)$ is generalized right π-Baer.

Now, let $n \geq 3$ and I be a projection invariant left ideal of $S_n(R)$. Consider the sets I_1^1, I_n^n, and J as in Lemma 3.7. By Lemma 3.7(i) and (ii), I_1^1 and I_n^n are projection invariant left ideal of $S_{n-1}(R)$. Also, Lemma 3.7(iii) implies that J is a projection invariant left ideal of R. Then by the hypothesis of induction, Proposition 2.2, and Lemma 3.6, there exist central idempotents $e_1, e_2 \in R$, $f_1, f_2 \in S_{n-1}(R)$, and positive integers k_1, k_2 such that

$$r_{S_{n-1}(R)}((I_1^1)^{(n-1)k_1}) = f_1S_{n-1}(R),\ f_1 = e_1\mathrm{I}_{n-1},\ r_R(J^{k_1}) = e_1R,$$

$$r_{S_{n-1}(R)}((I_n^n)^{(n-1)k_2}) = f_2S_{n-1}(R),\ f_2 = e_2\mathrm{I}_{n-1},\ r_R(J^{k_2}) = e_2R.$$

Put $k = \max\{k_1, k_2\}$. Then $r_R(J^k) = r_R(J^{k_1}) = r_R(J^{k_2})$, by Lemma 2.18. Hence $e_1 = e_2$ and $f_1 = f_2$. Again by using Lemma 2.18, we have

$$r_{S_{n-1}(R)}((I_1^1)^{(n-1)k}) = r_{S_{n-1}(R)}((I_1^1)^{(n-1)k_1}) = r_{S_{n-1}(R)}((I_n^n)^{(n-1)k_2}) = r_{S_{n-1}(R)}((I_n^n)^{(n-1)k}).$$

Now, assume that

$$x\mathrm{I}_n + \sum_{i<j} x_{i,j}E_{i,j} \in r_{S_n(R)}(I^{nk}),\ a_1 \cdots a_{nk}\mathrm{I}_n + \sum_{i<j} y_{i,j}E_{i,j} \in I^{nk}.$$

Since $r_{S_{n-1}(R)}((I_1^1)^{(n-1)k}) = r_{S_{n-1}(R)}((I_n^n)^{(n-1)k}) = f_1S_{n-1}(R)$, x and $x_{i,j}$'s are in e_1R for each i and j except possibly $x_{1,n}$. We have $a_1 \cdots a_{nk}x_{1,n} + y_{1,2}x_{2,n} + \cdots + y_{1,n}x = 0$. Thus $a_1 \cdots a_{nk}x_{1,n} = 0$. Since $r_R(J^{nk}) = e_1R$, and $a_1, \ldots, a_{nk} \in I$ are arbitrary, it follows that $x_{1,n} \in e_1R$. Hence $r_{S_n(R)}(I^{nk}) \subseteq fS_n(R)$ where $f = e_1\mathrm{I}_n$. Note that f is a central idempotent of $S_n(R)$. Since $I^{nk}f = (I^kf)^n = 0$, $fS_n(R) \subseteq r_{S_n(R)}(I^{nk})$. Thus $r_{S_n(R)}(I^{nk}) = fS_n(R)$ and hence $S_n(R)$ is generalized right π-Baer.

(i)$\Rightarrow$(ii) Suppose that $S_n(R)$ is a generalized right π-Baer ring. We prove that R is generalized right π-Baer. Let J be a projection invariant left ideal of R. Put

$$I = \{a\mathrm{I}_n + \sum_{i<j} a_{i,j}E_{i,j} \in S_n(R) \mid a \in J\}.$$

It is easy to see that I is a projection invariant left ideal of $S_n(R)$. Since $S_n(R)$ is generalized right π-Baer, $r_{S_n(R)}(I^m) = fS_n(R)$ for some idempotent $f \in S_n(R)$ and some positive integer m. By Proposition 2.2 and assumption, we may assume that $e \in \mathbf{B}(R)$. Then Lemma 3.6 implies that $f = e\mathrm{I}_n$ where $e \in \mathbf{B}(R)$. Hence for each $a_i \in J$ with $0 \leq i \leq m$, we have $a_1 \cdots a_m e\mathrm{I}_n = 0$, since $I^m e = 0$. It follows that $a_1 \cdots a_m e = 0$. Thus $eR \subseteq r_R(I^m)$. For the reverse inclusion, let $b \in r_R(J^m)$. Then for any $a_i \in J$ we have $a_1 \cdots a_m b = 0$. Thus $b\mathrm{I}_n \in r_{S_n(R)}(I^m) = fS_n(R)$. It follows that $b \in eR$. Therefore, $r_R(J^m) = eR$, and so R is generalized right π-Baer. □

Corollary 3.9 *Let R be a reduced π-Baer ring. Then For every $n \geq 2$ the rings $S_n(R)$, $A_n(R)$, $B_n(R)$, $U_n(R)$, and $V_n(R)$ are generalized right (left) π-Baer rings which are not π-Baer rings.*

Proof Note that R is an abelian π-Baer ring by [4, Proposition 2.5]. Thus Theorem 3.8 implies that the rings $S_n(R)$, $A_n(R)$, $B_n(R)$, $U_n(R)$, and $V_n(R)$ are generalized right (left) π-Baer rings. By [4, Proposition 2.5] every abelian π-Baer ring is an abelian Baer ring and by [7, Proposition 1.5] every abelian Baer ring is reduced. So these rings are not π-Baer since these rings are not reduced. □

4. Polynomial extensions

In this section, we investigate the behavior of the generalized π-Baer condition with respect to polynomial extensions. The generalized Baer ring property may not transfer to polynomial rings or formal power series rings in general (e.g., see [20, Example 3.24]). However, the generalized π-Baer property transfers from a base ring to many of its polynomial extensions without additional requirements.

Lemma 4.1 ([14], Exercise 2R) *Let R be a ring and let R^{op} denote the opposite ring of R. Let α be an automorphism of R and δ be an α-derivation of R. Consider the map $\delta' : R^{op} \to R^{op}$ defined by $\delta'(a) := -\delta(\alpha^{-1}(a))$ for $a \in R$. Then*

(i) δ' *is an* α^{-1}*-derivation on* R^{op}*;*

(ii) $(R[x;\alpha,\delta])^{op} \cong R^{op}[x;\alpha^{-1},\delta']$.

Theorem 4.2 *Let R be a generalized left (right) π-Baer ring. Then the following polynomial extensions are generalized left (right) π-Baer rings, where X is an arbitrary nonempty set of not necessarily commuting indeterminates, α is a ring automorphism of R and δ is an α-derivation of R.*

(i) $R[X]$*;*

(ii) $R[[X]]$*;*

(iii) $R[x;\alpha,\delta]$*;*

(iv) $R[[x;\alpha]]$*;*

(v) $R[x;x^{-1};\alpha]$*;*

(vi) $R[[x;x^{-1};\alpha]]$.

Proof We will prove Part (iii), the other parts can be shown similarly. First, we prove the case when R is a generalized left π-Baer ring. Let Y be a nonzero projection invariant right ideal of $T := R[x;\alpha,\delta]$. Let Y_0 be the set of nonzero coefficients of the highest degree term of nonzero elements in Y together with 0. Then Y_0 is a nonzero projection invariant right ideal of R. Since R is generalized left π-Baer, there are an integer $n \geq 1$ and an idempotent $e \in \mathbf{S}_r(R)$ such that $\ell_R(Y_0^n) = Re$. We prove that $\ell_T(Y^n) = Te$. First, to see that $Te \subseteq \ell_T(Y^n)$, take $h(x) = a_0 + a_1x + \cdots + a_mx^m \in Y^n \setminus \{0\}$. If $a_m \neq 0$ then $a_m \in Y_0^n$. Thus $ea_m = 0$. Now $eh(x) = ea_0 + ea_1x + \cdots + ea_{m-1}x^{m-1} \in Y^n$. If $ea_{m-1} \neq 0$ then $ea_{m-1} \in Y_0^n$. But $ea_{m-1} = e(ea_{m-1}) = 0$, a contradiction. Hence $ea_{m-1} = 0$. Similarly, we get $ea_{m-2} = \cdots = ea_0 = 0$. So $eh(x) = 0$ and hence $e \in \ell_T(Y^n)$. Therefore, $Te \subseteq \ell_T(Y^n)$.

Next, we show that $\ell_T(Y^n) \subseteq Te$. We shall show that $g(x) = g(x)e$, for each $g(x) \in \ell_T(Y^n)$. The proof proceeds by induction on $k = deg(g(x))$, the degree of $g(x)$. Assume that $k = 0$. Take $y \in Y_0^n$, then there exists $f(x) = y_0 + y_1x + \cdots + yx^m \in Y^n$. Since $g(x)f(x) = 0$, $g(x)y = 0$ and so $g(x) \in \ell_R(Y_0^n) = Re$. Hence $g(x) = g(x)e$. Assume inductively that the assertion is true for all $g(x) \in \ell_T(Y^n)$ with $deg(g(x)) < k$. Let $g(x) = b_0 + b_1x + \cdots + b_kx^k \in \ell_T(Y^n)$. Since α is an automorphism, $b_k = \alpha^k(r)$ for some $r \in R$. Take $y \in Y_0^n$. There is $f(x) = y_0 + y_1x + \cdots + yx^m \in Y^n$ and $g(x)f(x) = 0$. Thus, $b_k\alpha^k(y) = \alpha^k(r)\alpha^k(y) = 0$ and that $ry = 0$. It follows that $r \in \ell_R(Y_0^n) = Re$, so $r = re$. We see that $b_k = \alpha^k(r) = \alpha^k(re) = \alpha^k(r)\alpha^k(e) = b_k\alpha^k(e)$. Consequently,

$$g(x) = b_0 + b_1x + \cdots + b_{k-1}x^{k-1} + b_k\alpha^k(e)x^k = b_0 + b_1x + \cdots + b_{k-1}x^{k-1} + b_kx^ne + h(x),$$

for some $h(x) \in T$ such that $deg(h(x)) \leq n-1$ or $h(x) = 0$. Thus,

$$0 = g(x)Y^n = (b_0 + b_1x + \cdots + b_{k-1}x^{k-1} + b_kx^ne + h(x))Y^n = (b_0 + b_1x + \cdots + b_{k-1}x^{k-1} + h(x))Y^n,$$

because $eY^n = 0$. Put $p(x) = b_0 + b_1x + \cdots + b_{k-1}x^{k-1} + h(x)$. Note that $g(x) = p(x) + b_kx^ne$. If $p(x) = 0$, then $g(x) = b_kx^ne$ and so $g(x) = g(x)e$. Next, assume that $p(x) \neq 0$. By the induction hypothesis, $p(x) = p(x)e$ as $p(x) \in \ell_T(Y^n)$. So $g(x) = p(x) + b_kx^ne = p(x)e + b_kx^ne = g(x)e$, hence $\ell_T(Y^n) \subseteq Te$. Therefore, $\ell_T(Y^n) = Te$.

For the case when R is a generalized right π-Baer ring, we notice that R is a generalized right π-Baer ring if and only if R^{op} is a generalized left π-Baer ring. Thus R^{op} is a generalized left π-Baer ring. By the case above, $R^{op}[x;\alpha^{-1},\delta']$ is a generalized left π-Baer ring. Since $(R[x;\alpha,\delta])^{op} \cong R^{op}[x;\alpha^{-1},\delta']$ by Lemma 4.1, $(R[x;\alpha,\delta])^{op}$ is a generalized left π-Baer ring. Therefore, $R[x;\alpha,\delta]$ is a generalized right π-Baer ring. □

Corollary 4.3 *Let R be a generalized right (left) π-Baer ring. Then the group ring $R\mathbb{Z}$ is a generalized right (left) π-Baer ring.*

Proof It is well known that $R\mathbb{Z} \cong R[x, x^{-1}]$. Now the statement follows from Theorem 4.2. □

Theorem 4.4 *Let R be a ring. Then the following statements are equivalent.*

(i) R is a generalized right (left) π-Baer ring;

(ii) $R[x]$ is a generalized right (left) π-Baer ring;

(iii) $R[[x]]$ is a generalized right (left) π-Baer ring.

Proof The implications (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii) follow immediately from Theorem 4.2. For (ii)$\Rightarrow$(i), let Y be a projection invariant left ideal of R. Then by [4, Lemma 4.1(iii)], $Y[x]$ is a projection invariant left ideal of $R[x]$. Since $R[x]$ is a generalized right π-Baer ring, there exists an idempotent $e(x) \in R[x]$ such that $r_{R[x]}((Y[x])^n) = e(x)R[x]$ for some positive integer n. Assume that e_0 be the coefficient of zero degree term of $e(x)$. We show that $r_R(Y^n) = e_0R$. Since $e(x)(Y[x])^n = e(x)Y^n[x] = 0$, $e_0Y^n = 0$. Thus, $e_0R \subseteq r_R(Y^n)$. Conversely, let $a \in r_R(Y^n)$, then $a \in r_{R[x]}(Y^n[x]) = r_{R[x]}((Y[x])^n) = e(x)R[x]$. Hence, $a = e(x)f(x)$. So $e(x)a = a$ and that $a = e_0a$. Hence $a \in e_0R$. Therefore, $r_R(Y^n) = e_0R$, and that R is a generalized right π-Baer ring. Similarly, it can be shown that (iii)$\Rightarrow$(i). □

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References

[1] Ahmadi M, Moussavi A, Golestani N. Generalized quasi-Baer $*$-rings and Banach $*$-algebras. Communications in Algebra 2020; 48 (5): 2207-2247. doi:10.1080/00927872.2019.1710841

[2] Bell HE. Near-Rings in which each element is a power of itself. Bulletin of the Australian Mathematical Society 1973; 2 (2): 363-368.

[3] Birkenmeier GF, Heatherly HE, Kim JY, Park JK. Triangular matrix representations of ring extensions. Journal of Algebra 2000; 230: 558-595.

[4] Birkenmeier GF, Kara Y, Tercan A. π-Baer rings. Journal of Algebra and Its Applications 2018; 16 (11): 1-19.

[5] Birkenmeier GF, Kim JY, Park JK. Polynomial extensions of Baer and quasi-Baer rings. Journal of Pure and Applied Algebra 2001; 159: 25-42.

[6] Birkenmeier GF, Kim JY, Park JK. Principally quasi-Baer rings. Communications in Algebra 2001; 29 (2): 639-660.

[7] Birkenmeier GF, Kim JY, Park JK. Quasi-Baer ring extensions and biregular rings. Bulletin of the Australian Mathematical Society 2000; 61: 39-52.

[8] Birkenmeier GF, Kim JY, Park JK. Right primary and nilary rings and ideals. Journal of Algebra 2013; 378: 133-152.

[9] Birkenmeier GF, Park JK, Rizvi ST. Generalized triangular matrix rings and the fully invariant extending property. Rocky Mountain Journal of Mathematics 2002; 32: 1299-1319.

[10] Birkenmeier GF, Tercan A, Yücel CC. The extending condition relative to sets of submodules. Communications in Algebra 2014; 42: 764-778.

[11] Birkenmeier GF, Tercan A, Yücel CC. Projection invariant extending rings. Journal of Algebra and Its Applications 2016; 15 (7): 11.

[12] Brown KA. The singular ideals of group rings. Quarterly Journal of Mathematics 1977; 28: 41-60.

[13] Clark WE. Twisted matrix units semigroup algebras. Duke Mathematical Journal 1997; 34: 417-424.

[14] Goodearl KR, Warfield RB. An Introduction to Noncommutative Noetherian Rings. Cambridge, UK: Cambridge University Press, 1989.

[15] Heatherly HE, Tucci RP. Central and semicentral idempotents. Kyungpook Mathematical Journal 2000; 40: 255-258.

[16] Huh C, Kim HK, Lee Y. p.p. rings and generalized p.p. rings. Journal of Pure and Applied Algebra 2002; 167: 37-52.

[17] Kaplansky I. Rings of operators. New York, NY, USA: W. A. Benjamin, 1968.

[18] Lam TY. A first course in noncommutative rings. Graduate Texts in Mathematics. New York, NY, USA: Springer, 2000.

[19] Moussavi A, Javadi HHS, Hashemi E. Generalized quasi-Baer rings. Communications in Algebra 2005; 33: 2115-2129.

[20] Paykan K, Moussavi A. A generalization of Baer rings. International Journal of Pure and Applied Mathematics 2015; 99 (3): 257-275.