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The height of a permutation and applications to distance between real line arrangements

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Abstract: We present a new notion of a distance between two real line arrangements. We define the height of a permutation and use this idea in our main theorem, which gives us a lower bound on the distance between the pair. We apply these techniques to the seven special cases of real arrangements with ten lines found in previous work by the authors.

Key words: Moduli space, automorphism group, line arrangements, distance invariant, degenerate arrangements

1. Introduction

A line arrangement $\mathcal{A} = \{L_1, \dots, L_n\}$ in $\mathbb{CP}^2$ is a finite collection of projective lines. The set $L(\mathcal{A}) = \{\bigcap_{i \in S} L_i | S \subseteq \{1, 2, \dots, n\}\}$ partially ordered by reverse inclusion is called the intersection lattice of $\mathcal{A}$. Two line arrangements $\mathcal{A}$ and $\mathcal{B}$ are lattice isomorphic, written $L(\mathcal{A}) \sim L(\mathcal{B})$, if up to a permutation on the labels of the lines their lattices are the same. In this case we say that the arrangements have the same combinatorics.

The moduli space of an arrangement $\mathcal{A}$ is

$$\mathcal{M}_{\mathcal{A}} = \{\mathcal{B} \in ((\mathbb{CP}^2)^*)^n | L(\mathcal{A}) = L(\mathcal{B})\} / PGL(3, \mathbb{C}),$$

where $L(\mathcal{A}) = L(\mathcal{B})$ if the permutation on the labels of the lines is the identity. Note that in other literature this may be called the ordered moduli space.

Our main goal is to learn how moduli spaces interact with each other geometrically, and for this we are interested in examples with disconnected moduli spaces.

A disconnected moduli space is a necessary condition for an important class of counterexamples called Zariski pairs. A Zariski pair of line arrangements is a pair of lattice isomorphic arrangements whose complements in $\mathbb{CP}^2$ have different embedding types. By Randell's isotopy theorem [8], the embedding types of arrangements in the same connected component are the same.

Rybnikov [9] found the first such pair of arrangements in 1998. Bartolo et al. [5] and then Guerville-Ballé [6] give other examples, with the latter example revisited by Bartolo et al. [4].

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The current state of affairs is that there is no Zariski pair of nine or fewer lines (see [7], [10]), but for ten this is still open. Eighteen potential pairs of ten lines are given by the authors with Ye in [3] and in [2].

Out of these eighteen arrangements, just seven can be realized in $\mathbb{RP}^2$ with 0-dimensional moduli spaces. The only smaller realizable arrangement with disconnected moduli space is the Falk–Sturmfels arrangement. We consider these seven arrangements below.

Three of these seven were proven not to be Zariski pairs in [1] using the automorphism group $Aut(\mathcal{A})$, the group of all lattice isomorphisms or symmetries of $L(\mathcal{A})$. We consider these three cases anyway so as to compare their geometry to the remaining four real cases.

In order to study these interesting moduli spaces, we propose an invariant with a real parametrization, enabling us to visualize the process. We now set up the details of this invariant.

We let A and B to be two real line arrangements that are the representative arrangements of two components of the moduli space $\mathcal{M}_{\mathcal{A}}$ of a fixed arrangement $\mathcal{A}$ given combinatorially. We have $\sigma \in Aut(\mathcal{A})$ taking the intersection lattice of one to that of the other.

We describe a path between A and B in the moduli space of all arrangements by some parameter $t \in I = [a, b]$ that changes the equations of the lines. We say that $\mathfrak{C}_t$ is a real parametrization of the path from A to B if for every $t \in I$ we have a real line arrangement, including both A and B.

Then our proposed invariant is the distance $d_t(\mathfrak{C}_a, \mathfrak{C}_b)$, the number of distinct combinatorial arrangements passed through along $\mathfrak{C}_t$ between A and B.

Let the distance $d_{min}(\mathfrak{C}_a, \mathfrak{C}_b)$ be the minimum value of $d_t(\mathfrak{C}_a, \mathfrak{C}_b)$ for all possible paths between A and B for some fixed equations for A and B.

We investigate $d_{min}(\mathfrak{C}_a, \mathfrak{C}_b)$ of the seven real examples out of the eighteen potential Zariski pairs of ten lines mentioned above and summarize the results in Table .

Table . A summary of the results from Sections 4 and 5.

Arrangement $\mathcal{A}$	Theorem	$Aut(\mathcal{A})$	$d_{min}(\mathfrak{C}_a, \mathfrak{C}_b)$
Case 1	Theorem 4.2	$\mathbb{Z}_2$	2
Case 2	Theorem 5.2	Id	≤ 9
Case 3	Theorem 5.3	Id	≤ 5
Case 4	Theorem 5.4	Id	≤ 8
Case 5	Theorem 5.5	$\mathbb{Z}_3$	≤ 4
Case 6	Theorem 4.3	$\mathbb{Z}_2$	2
Case 7	Theorem 4.4	$\mathbb{Z}_2$	2

We note here that Cases 1, 6, and 7 from [1] all have distance $d_{min}(\mathfrak{C}_a, \mathfrak{C}_b) = 2$, and we know these arrangements are not Zariski pairs. Beyond the scope of this paper, we question the relationship between these two properties, and we wonder whether this notion of distance could be used to measure the "strength" of a Zariski pair.

We can use this idea to define a stronger invariant $d(A, B)$ for future work, but it is complicated to compute. It is the minimum value of $d_t(\mathfrak{C}_a, \mathfrak{C}_b)$ for all possible equations of A and B and all possible paths between them. We have

$$d(A, B) \leq d_{min}(\mathfrak{C}_a, \mathfrak{C}_b) \leq d_t(\mathfrak{C}_a, \mathfrak{C}_b). \tag{1.1}$$

This paper is organized as follows.

In Section 2 we define the height of a permutation, the height between two pointed lines, and the height of an automorphism between a pair of arrangements.

In Section 3 we present the notion of a distance between two real line arrangements A and B. The main result of the paper is Theorem 3.9. We find a lower bound of the distance $d_{min}(\mathfrak{C}_a, \mathfrak{C}_b)$ between two real line arrangements.

In Section 4 we consider the three pairs of real line arrangements that appear in [1]. These three pairs have a $\mathbb{Z}_2$ symmetry.

In Section 5 we consider four pairs of real line arrangements. We compute upper bounds for $d_{min}(\mathfrak{C}_a, \mathfrak{C}_b)$ using the Main Theorem 3.9.

In the Appendix, we give code for the visualizations of the parametrizations via the Manipulate feature of `Mathematica`*. Except for Case 5, which includes a longer parametrization, these start with a real line arrangement A and end at a real line arrangement B.

2. Height of a permutation

2.1. Height of a permutation

Definition 2.1 *Let τ be a permutation of k symbols. Let L_1 (resp. L_2) be the line in $\mathbb{R}^2$ defined by $y = 0$ (resp. $y = 1$). Take k points $(x_1, 0), (x_2, 0), \ldots, (x_k, 0)$ on L_1, such that $x_1 < x_2 < \ldots < x_k$. Let $(x_{\tau(1)}, 1)$, $(x_{\tau(2)}, 1)$, $\ldots$, $(x_{\tau(k)}, 1)$ be k points on L_2.*

Let $S := \{(x, y) \in \mathbb{R}^2 | 0 \le y \le 1\}$ and let $\pi : S \to [0, 1]$ be $(x, y) \mapsto y$ a projection. Take continuous curves $C_i \subseteq S$ passing through $(x_i, 0)$ and $(x_{\tau(i)}, 1)$, here $1 \le i \le k$, such that every pair of curves intersects a finite number of times, π induces a homeomorphism between C_i and $[0, 1]$, and the projection from C_i to the x-axis is also a homeomorphism to its image.

Let

$$h(\tau; C_1, \ldots, C_k) = \#\{\pi(p) | p \in \cup_{i \neq j}(C_i \cap C_j)\}$$

be the number of the images of the intersections of the C_i under the projection π, and set

$$h(\tau) = \min_{C_1, \ldots, C_k} \{h(\tau; C_1, \ldots, C_k)\}$$

to be the minimum. The number $h(\tau)$ is called the height of the permutation τ.

Example 2.2 *Consider the permutation $\tau = (14365)(2)$ of six symbols, labeled as $A, \ldots, F$ on the left side of Figure 1.*

Observe that any curve passing through the points labeled A must intersect any three curves passing through the points labeled B, E, and D. Thus there are at least three intersection points with distinct heights by projection π, and hence the height is at least three. On the other hand, Figure 1 shows an example where the number of intersection point images under π is exactly three, and therefore the height $h(\tau) = 3$.

*Wolfram Research, Inc. 2010. Mathematica [online]. Website: http://www.wolfram.com/mathematica/?source=frontpage-quick-links [accessed 23 June 2013].

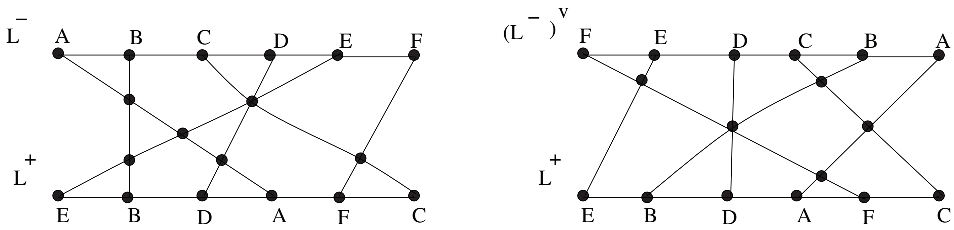

Figure 1. Calculating the height of a permutation for Example 2.2, the height between two pointed lines for Example 2.5, and the height of the identity automorphism between a pair of arrangements for Case 2 in Section 5.

2.2. Height between two pointed lines

Now we define the height between two pointed lines in a similar way.

Definition 2.3 *We say that* (L, P) *is a* k-pointed line *if* L *is an oriented real line and* $P = \{p^1, p^2, \ldots, p^k\}$ *is a set of* k *points on* L. *Sometimes we write* L *for short. We take* $(L^\vee, P)$, *or* $L^\vee$ *for short, to be the* k-*pointed line* (L, P) *with reverse orientation.*

Definition 2.4 *Let* (L_1, P) *and* (L_2, Q) *be two* k-*pointed lines in* $\mathbb{R}^2$, *where* $P = \{p^1, p^2, \ldots, p^k\}$ *and* $Q = \{q^1, q^2, \ldots, q^k\}$. *Choose two linear isomorphisms* f *and* g *of* $\mathbb{R}^2$, *such that* f *(resp.* g*) sends* L_1 *(resp.* L_2*) to the line* $y = 0$ *(resp.* $y = 1$*).*

Let τ *be the permutation of* k *symbols that identifies bijectively the points* P *on* L_1 *to the points* Q *on* L_2.

Let $\pi : S \to [0, 1]$, $(x, y) \mapsto y$ *be the projection as above. Take continuous curves* $C_i \subseteq S$ *passing through* $f(p^i)$ *and* $g(q^{\tau(i)})$, *here* $1 \le i \le k$, *such that there are finite intersection points for each pair of curves,* π *induces a homeomorphism between* C_i *and* $[0, 1]$, *and the projection from* C_i *to the* x-*axis is also a homeomorphism to its image.*

Let

$$h(f(L_1), g(L_2); C_1, \ldots, C_k) = \#\{\pi(p) | p \in \cup_{i \neq j}(C_i \cap C_j)\}$$

be the number of the images of the intersections of the C_i *under the projection* π, *and set*

$$h(f(L_1), g(L_2)) = \min_{C_1, \ldots, C_k}\{h(f(L_1), g(L_2); C_1, \ldots, C_k)\}$$

to be the minimum. Let $\vee : \mathbb{R}^2 \to \mathbb{R}^2$ *be the reflection defined by* $\vee(x, y) = (-x, y)$. *We denote the number* $\min\{h(f(L_1), g(L_2)), h(f(L_1), \vee \circ g(L_2))\}$ *as* $h(L_1, L_2)$ *and call it the* height between two pointed lines L_1 and L_2.

Note that $\vee$ will reverse the ordering of the points in Q and reverse the orientation of L_2. Since $h(f(L_1), g(L_2); C_1, \ldots, C_k)$ is an invariant under oriented linear isomorphisms of $\mathbb{R}^2$, we know that $h(L_1, L_2)$ is independent of the choice of f and g.

The height $h(L_1, L_2)$ is an invariant of a pair of pointed lines L_1 and L_2. Note that if $h(L_1, L_2) = 0$, then the points on L_1 have the same ordering as the points on either L_2 or $L_2^\vee$.

Example 2.5 *Consider the two pointed lines* L^- *and* L^+ *given in the left side of Figure 1, and consider* $\tau = (14365)(2)$ *as in Example 2.2. Then checking the case with* $(L^-)^\vee$*, as well, one can see that the intersections of the lines* C_i *can be arranged at three heights again, and so* $h(L^-, L^+) = 3$.

2.3. Height of an automorphism between a pair of arrangements

Let $\mathcal{A}$ be a complex projective line arrangement and $\mathcal{M}_\mathcal{A}$ be the moduli space of $\mathcal{A}$.

If $\mathcal{M}_\mathcal{A}$ is irreducible, then there is no Zariski pair in $\mathcal{M}_\mathcal{A}$ by Randell's isotopy theorem [8]. Thus we assume that $\mathcal{M}_\mathcal{A}$ is reducible.

Assumption. *In this work, we consider only the case where* $\mathcal{M}_\mathcal{A}$ *is of dimension 0 with exactly two components. Let* A *and* B *be the representative arrangements of the two components of* $\mathcal{M}_\mathcal{A}$*. Furthermore, we assume that* A *and* B *are real.*

Since $\mathcal{M}_\mathcal{A}$ is concerned with geometric realizations, we need to find geometric equations for our combinatorial lines. In order to determine $\mathcal{M}_\mathcal{A}$, one finds equations of lines using the combinatorial intersection data of $\mathcal{A}$ as in [10], [3] and [2]. By the assumption, we can write these as $L_i(s) : a_i(s)x + b_i(s)y + c_i(s)z = 0$, $i = 1, 2, \dots, n$, where s satisfies some quadratic equation $\alpha s^2 + \beta s + \gamma = 0$, where $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha \neq 0$ and $\beta^2 - 4\alpha\gamma > 0$. Thus we have

$$s^{\pm} = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}.$$

We let $L_i^+ = L_i(s^+)$ and $L_i^- = L_i(s^-)$ for $i = 1, 2, \dots, n$.

For example, the equation for Case 1 in Subsection 4.1 is $s^2 - s - 1 = 0$.

Definition 2.6 *Let* $\sigma \in Aut(\mathcal{A})$ *be an automorphism of the combinatorial arrangement* $\mathcal{A}$*. Let* $\mathcal{A}$ *have two representative arrangements* A *and* B *in the moduli space* $\mathcal{M}_\mathcal{A}$*. By the assumption, we may take* A *with lines* L_i^+ *and* B *with lines* L_i^-*. Let* $J(\sigma : A \to B)$ *be the set of indices* i *of the lines of these arrangements. We take care when mapping* $\mathbb{RP}^2$ *to* $\mathbb{R}^2$ *that no multiple point on* L_i^+ *or* L_i^- *gets sent to "infinity."*

Let $P_i = \{p_{i,1}, \dots, p_{i,k_i}\}$ *(resp.* $Q_i = \{q_{i,1}, \dots, q_{i,k_i}\}$*) be the set of all the multiple points of* L_i^+ *(resp.* L_i^-*), where* $q_{i,j} = (p_{i,\tau(j)})$ *for* $1 \le j \le k_i$*. Now we take* (L_i^+, P_i) *and* (L_i^-, Q_i) *to be* real k_i*-pointed lines in* $\mathbb{R}^2$.

Define $h(\sigma : A \to B)$ *to be the* height of the automorphism σ between A and B, *the maximum over all* $i \in J(\sigma : A \to B)$ *of* $h(L_i^+, L_i^-)$*, the height between two pointed lines* L_i^+ *and* L_i^-.

3. Distance invariant of line arrangements

3.1. Parametrization of a path

We first define a parametrization between two line arrangements and then use this to define three notions of a distance between two line arrangements.

Definition 3.1 *Let* $\mathcal{M}_\mathcal{A}^\mathbb{R}$ *be the* real moduli space of the arrangement $\mathcal{A}$, *where the solution space is over* $\mathbb{R}^2$ *instead of* $\mathbb{CP}^2$*. Let* $\mathcal{M}_n^\mathbb{R}$ *be the* moduli space of all real arrangements of n (or fewer) lines, *the union* $\bigcup \mathcal{M}_\mathcal{A}^\mathbb{R}$ *over all arrangements* $\mathcal{A}$ *of* n *or fewer lines. Observe that this notion is also useful for multiline arrangements, where two or more lines may coincide.*

By the assumption, we consider A and B to be two points in $\mathcal{M}_n^{\mathbb{R}}$. We want to find a path between A and B in $\mathcal{M}_n^{\mathbb{R}}$, that is, a path of real arrangements, but we need to find a parametrization of this path.

Definition 3.2 *Let $A = \mathfrak{C}_a$ and $B = \mathfrak{C}_b$ be two real line arrangements. We say that $\mathfrak{C}_t$ with $t \in I = [a, b] \subset \mathbb{R}$ is* a parametrization of the path from A to B *if we have a real line arrangement at every t in I and if the complements of these arrangements deform continuously. We refer to it in short as* a parametrization of A and B.

By choosing appropriate t_i, we can present this process as a sequence of line arrangements as follows:

$$A = \mathfrak{C}_a \to \mathfrak{C}_{t_1} \to \mathfrak{C}_{t_2} \to \ldots \to \mathfrak{C}_{t_k} \to \mathfrak{C}_b = B,$$

where $a = t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1} = b$ are sample points chosen from the interval I such that $\mathfrak{C}_{t_i} \not\sim \mathfrak{C}_{t_{i+1}}$ are distinct combinatorial arrangements. During this process, there are some special arrangements along the parametrization which occur when some lines coincide or when we get a higher multiplicity singularity. We call these degenerate arrangements.

Definition 3.3 *The arrangement $\mathfrak{C}_c = C$ is said to be* a degenerate arrangement *of the parametrization $\mathfrak{C}_t$ for some $c \in I$, if there is an $\varepsilon > 0$ such that the combinatorics of $C = \mathfrak{C}_c$ and $\mathfrak{C}_t$ are different for any $t \in (c - \varepsilon, c) \cup (c, c + \varepsilon)$.*

We denote the degenerate arrangements of the parametrization $\mathfrak{C}_t$ (that are not A or B) by $C_1, C_2, \ldots, C_m$, and we give a sequence which shows the order of the degenerate arrangements, according to their appearance along the parametrization.

$$A \rightsquigarrow C_1 \rightsquigarrow C_2 \rightsquigarrow \ldots \rightsquigarrow C_m \rightsquigarrow B. \tag{3.1}$$

We note that all of the arrangements $\mathfrak{C}_t$ between each two consecutive degenerate arrangements C_i and C_{i+1} have the same topology in their complements because they are in the same lattice in the moduli space $\mathcal{M}_n^{\mathbb{R}}$. When the lattice changes, the parametrization passes through a degenerate arrangement.

Now we define the distance $d_t(\mathfrak{C}_a, \mathfrak{C}_b)$ of the parametrization $\mathfrak{C}_t$ of the arrangements A and B using (3.1).

Definition 3.4 *We say the* distance between A and B along $\mathfrak{C}_t$, *written $d_t(\mathfrak{C}_a, \mathfrak{C}_b)$, is the number of combinatorial arrangements $\mathfrak{C}_i$ in $\mathfrak{C}_t$ of A and B such that $\mathfrak{C}_i \not\sim \mathfrak{C}_{i+1}$.*

3.2. The number of degenerate arrangements

In this subsection we define a parametrization between two line arrangements and we present our main theorem (Theorem 3.9). The motivation here is the study of the automorphism group $Aut(\mathcal{A})$ of the arrangement $\mathcal{A}$ that appears in [1].

Definition 3.5 *Let $\mathfrak{C}_t$ for $t \in I = [a, b]$ be a parametrization of A and B that is defined by the polynomial $L_1(t)L_2(t) \ldots L_n(t)$, where $L_i(t)$ is a linear polynomial whose coefficients are continuous functions of t. Assume that $L_i(a) = L_i^+$ for $i = 1, \ldots, n$. Let σ be an automorphism of $\mathcal{A}$.*

We say $\mathfrak{C}_t$ is a σ-parametrization of A and B *if $L_i(b) = \lambda_i L_i^-$ for some $\lambda_i \in \mathbb{R}$ and $i = 1, 2, \ldots, n$. In particular, when σ is the identity, $\mathfrak{C}_t$ is called an id*-parametrization. *When σ^2 is the identity but σ is not, $\mathfrak{C}_t$ is called a $\mathbb{Z}_2$*-parametrization.

Notation 3.6 *We denote by* $\delta_t(\mathfrak{C}_a, \mathfrak{C}_b)$ *the number of degenerate arrangements of the parametrization* $\mathfrak{C}_t$ *between* $\mathfrak{C}_a$ *and* $\mathfrak{C}_b$*.*

Proposition 3.7 *The distance* $d_t(\mathfrak{C}_a, \mathfrak{C}_b)$ *fulfills the inequality*

$$d_t(\mathfrak{C}_a, \mathfrak{C}_b) \leq \begin{cases} 2\delta_t(\mathfrak{C}_a, \mathfrak{C}_b) & \text{when neither } A \text{ nor } B \text{ is degenerate,} \\ 2\delta_t(\mathfrak{C}_a, \mathfrak{C}_b) + 1 & \text{when exactly one of } A \text{ and } B \text{ is degenerate, and} \\ 2\delta_t(\mathfrak{C}_a, \mathfrak{C}_b) + 2 & \text{when both } A \text{ and } B \text{ are degenerate.} \end{cases}$$

Proof Let $C_1, \dots, C_m$ be the degenerate arrangements between A and B, not including A and B as in (3.1), so that $\delta_t(\mathfrak{C}_a, \mathfrak{C}_b) = m$. As discussed just below (3.1), there is only one line arrangement along the parametrization between any two C_i and C_{i+1}. So in total there are m degenerate arrangements C_i and $m-1$ arrangements between each C_i and C_{i+1}; we also include the arrangement following C_m. If either of A and B is degenerate, they count, as well. Thus the distance $d_t(\mathfrak{C}_a, \mathfrak{C}_b)$ is at most $2\delta_t(\mathfrak{C}_a, \mathfrak{C}_b)$ plus the number of extremal degenerate arrangements. □

Before the main theorem, we introduce the following concept.

Definition 3.8 *Let* p *be a multiple point of the arrangement* A*. Let* $\mathfrak{C}_t$ *be a parametrization of* A *and* B*, where* $t \in [a, b]$*. When* t *runs from* a *to* b*, the multiple point* p *forms a curve, which we call the* locus curve of p.

We now have the notation and tools to prove our main theorem.

Theorem 3.9 (Main Theorem) *Let* $\mathfrak{C}_t$ *for* $t \in I = [a, b]$ *be a* σ*-parametrization of* A *and* B*. Then* $\delta_t(\mathfrak{C}_a, \mathfrak{C}_b) \geq h(\sigma : A \to B)$*, the height of the permutation.*

Proof We can consider $\mathfrak{C}_t$ to be a subset of $\mathbb{R}^2 \times I$ defined by $L_1(t)L_2(t)\dots L_n(t) = 0$. Let $\pi : \mathbb{R}^2 \times I \to I$ be the natural projection.

Choose a line L_i *not* at infinity. Let $P_i = \{p_i^1, \dots, p_i^k\}$ (resp. $Q_i = \{q_i^1, \dots, q_i^k\}$) be the set of all the multiple points of the line $L_i(a) = L_i^+$ (resp. $L_i(b) = L_i^-$). By Definition 2.4, we can identify p_i^j and $q_i^{\tau(j)}$ for $1 \leq j \leq k$.

Select some point $p_i^\ell \in P_i$. Since it is a multiple point, there is some other line L_j through it, and so we can write this point as $L_i(t) \cap L_{j_\ell}(t)$ for all t.

We now consider the locus curve of this point $C_i^\ell = \{L_i(t) \cap L_{j_\ell}(t) | t \in [a, b]\}$, passing through p_i^ℓ and $q_i^{\tau(\ell)}$. Let

$$h(L_i, \sigma : A \to B; C_1, \dots, C_k) = \#\{\pi(p) | p \in \cup_{\ell_1 \neq \ell_2}(C_i^{\ell_1} \cap C_i^{\ell_2})\}$$

be the number of the images of the intersections of the $C_i^{\ell_s}$ under the projection π. One sees that $\mathfrak{C}_{t_0}$ is a degenerate arrangement if $t_0 \in \{\pi(p) | p \in \cup_{\ell_1 \neq \ell_2}(C_i^{\ell_1} \cap C_i^{\ell_2})\}$. Thus since these locus curves behave well under projection we have

$$\delta_t(\mathfrak{C}_a, \mathfrak{C}_b) \geq \max_{1 \leq i \leq n} h(L_i, \sigma : A \to B; C_1, \dots, C_k).$$

By Definition 2.6 of $h(\sigma : A \to B)$, we know that $h(L_i, \sigma : A \to B; C_1, \dots, C_k) \geq h(\sigma : A \to B)$. □

4. $\mathbb{Z}_2$-parameterizations

In this section, we study Cases 1, 6 and 7 by $\mathbb{Z}_2$-parameterizations.

We need the following notation of $d_{min}(\mathfrak{C}_a, \mathfrak{C}_b)$:

Notation 4.1 *The minimum value of $d_t(\mathfrak{C}_a, \mathfrak{C}_b)$ for all possible paths between A and B for some fixed equations for A and B is $d_{min}(\mathfrak{C}_a, \mathfrak{C}_b)$.*

4.1. Case 1

We consider Equation (1) from [3], Theorem 4.4. Case 1 can be defined by the following equations:

$$L_1 : y = (1-s)z, \quad L_2 : y = z, \quad L_3 : y = 0, \quad L_4 : x = 0,$$
$$L_5 : x = z, \quad L_6 : x = sz, \quad L_7 : y = x + (1-s)z,$$
$$L_8 : y = x - sz, \quad L_9 : y = (2-s)x, \quad L_{10} : z = 0,$$

where $s^2 - s - 1 = 0$, i.e. $s^{\pm} = \frac{1 \pm \sqrt{5}}{2}$ and s^- for $1.A$ and s^+ for $1.B$.

Theorem 4.2 *There is a $\mathbb{Z}_2$-parametrization of the arrangements $1.A$ and $1.B$. It contains one degenerate arrangement $1.C_1$ of one line $y = x$. Moreover, $d_{min}(\mathfrak{C}_a, \mathfrak{C}_b) = 2$.*

Proof We give the following equations that define a parametrization of the arrangements $1.A$ and $1.B$:

$$L_1(t) : (1-t)y = tx + \frac{1+\sqrt{5}}{2}(1-2t)z, \quad L_2(t) : (1-t)y = tx + (1-2t)z,$$
$$L_3(t) : (t-1)y + tx = 0, \quad L_4(t) : ty = (1-t)x, \quad L_5(t) : ty = (1-t)x + (2t-1)z,$$
$$L_6(t) : ty = (1-t)x + \frac{\sqrt{5}-1}{2}(1-2t)z, \quad L_7(t) : y = x + \frac{\sqrt{5}+1}{2}(1-2t)z,$$
$$L_8(t) : y = x + \frac{\sqrt{5}-1}{2}(1-2t)z, \quad L_9(t) : y = \frac{(3+\sqrt{5})(1-t)+2t}{(3+\sqrt{5})t+2(1-t)}x, \quad L_{10}(t) : z = 0,$$

where $t = 0$ for $1.A$ and $t = 1$ for $1.B$. When $t = \frac{1}{2}$, we get the only degenerate arrangement $1.C_1$. The arrangement $1.C_1$ is just one line $y = x$.

When $t \neq \frac{1}{2}$, the combinatorics of all $\mathfrak{C}_t$'s are the same. Since the moduli space of this case has two irreducible components, any parametrization of $1.A$ and $1.B$ has at least one degenerate arrangement. Hence the distance $d_{min}(\mathfrak{C}_a, \mathfrak{C}_b)$ is 2. □

4.2. Case 6

Case 6 can be defined by the following equations:

$$L_1 : y = -(1+s)z, \quad L_2 : y = z, \quad L_3 : y = 0, \quad L_4 : x = 0,$$
$$L_5 : x = sz, \quad L_6 : x = z, \quad L_7 : y = (1 + \tfrac{2}{s})x - (1+s)z,$$
$$L_8 : y = -x + z, \quad L_9 : y = \tfrac{s+1}{s-1}(x - sz), \quad L_{10} : y = \tfrac{s+2}{s-1}x - \tfrac{1}{s-1},$$

where $s^2 + s - 1 = 0$, i.e. $s^{\pm} = \frac{-1 \pm \sqrt{5}}{2}$ and s^- for $6.A$ and s^+ for $6.B$.

Theorem 4.3 *There is a $\mathbb{Z}_2$-parametrization of the arrangements $6.A$ and $6.B$. It contains one degenerate arrangement $6.C_1$ of two lines $y = x$ and $y = -x + z$. Moreover, the distance $d_{min}(\mathfrak{C}_a, \mathfrak{C}_b)$ is 2.*

Proof We give the following equations that define a parametrization of the arrangements $6.A$ and $6.B$:

$$L_1(t) : (1-t)y = tx + \frac{\sqrt{5}-1}{2}(1-2t)z, \quad L_2(t) : (1-t)y = tx + (1-2t)z,$$

$$L_3(t) : (t-1)y + tx = 0, \quad L_4(t) : ty = (1-t)x,$$

$$L_5(t) : ty = (1-t)x + \frac{\sqrt{5}+1}{2}(1-2t)z, \quad L_6(t) : ty = (1-t)x + (2t-1)z,$$

$$L_7(t) : y = \frac{(2-\sqrt{5})(1-t)+t}{(2-\sqrt{5})t+1-t}x + \frac{(\sqrt{5}-1)(1-2t)}{2(1-\sqrt{5})t+2}z, \quad L_8(t) : y = -x + z,$$

$$L_9(t) : y = \frac{(\sqrt{5}-2)(1-t)+t}{(\sqrt{5}-2)t+1-t}x + \frac{(3-\sqrt{5})(1-2t)}{2(\sqrt{5}-3)t+2}z,$$

$$L_{10}(t) : y = \frac{(3\sqrt{5}-7)(1-t)+2t}{(3\sqrt{5}-7)t+2(1-t)}x + \frac{(3-\sqrt{5})(1-2t)}{3(\sqrt{5}-3)t+2}z,$$

where $t = 0$ for $6.A$ and $t = 1$ for $6.B$. When $t = \frac{1}{2}$, we get the only degenerate arrangement $6.C_1$. This arrangement $6.C_1$ is just two lines $y = x$ and $y = -x + z$.

When $t \neq \frac{1}{2}$, the combinatorics of all $\mathfrak{C}_t$'s are the same. Similarly to Case 1 above, the distance $d_{min}(\mathfrak{C}_a, \mathfrak{C}_b)$ is 2. □

4.3. Case 7

Case 7 can be defined by the following equations:

$$L_1 : y = z, \quad L_2 : y = (1-s)z, \quad L_3 : y = 0, \quad L_4 : x = 0,$$

$$L_5 : x = z, \quad L_6 : x = sz, \quad L_7 : y = -sx + z,$$

$$L_8 : y = \tfrac{s-1}{s}x + (1-s)z, \quad L_9 : y = \tfrac{1}{s-1}(x - z), \quad L_{10} : y = -x,$$

where $s^2 - s - 1 = 0$, i.e. $s^{\pm} = \frac{1 \pm \sqrt{5}}{2}$ and s^- for $7.A$ and s^+ for $7.B$.

Theorem 4.4 *There is a $\mathbb{Z}_2$-parametrization of the arrangements $7.A$ and $7.B$. It contains one degenerate arrangement $7.C_1$ of two lines $y = x$ and $y = -x$. Moreover, the distance $d_{min}(\mathfrak{C}_a, \mathfrak{C}_b)$ is 2.*

Proof We give the following equations that define a parametrization of the arrangements $7.A$ and $7.B$:

$$L_1(t) : (1-t)y = tx + (1-2t)z, \quad L_2(t) : (1-t)y = tx + \frac{1+\sqrt{5}}{2}(1-2t)z,$$

$$L_3(t) : (1-t)y = tx, \quad L_4(t) : ty = (1-t)x,$$

$$L_5(t) : ty = (1-t)x - (1-2t)z, \quad L_6(t) : ty = (1-t)x + \frac{1-\sqrt{5}}{2}(2t-1)z,$$

$$L_7(t) : y = \frac{(\sqrt{5}-1)(1-t)+2t}{(\sqrt{5}-1)t+2(1-t)}x + \frac{2(1-2t)}{(\sqrt{5}-3)t+2}z,$$

$$L_8(t) : y = \frac{(\sqrt{5}+3)(1-t)+2t}{(\sqrt{5}+3)t+2(1-t)}x + \frac{2(1-2t)}{\sqrt{5}-1+2t}z,$$

$$L_9(t) : y = \frac{(1-\sqrt{5})(1-t)+2t}{(1-\sqrt{5})t+2(1-t)}x + \frac{(\sqrt{5}-1)(1-2t)}{2-(\sqrt{5}+1)t}z, \quad L_{10}(t) : y = -x,$$

where $t = 0$ for $7.A$ and $t = 1$ for $7.B$. When $t = \frac{1}{2}$, we get the only degenerate arrangement $7.C_1$. The arrangement $7.C_1$ is just two lines $y = x$ and $y = -x$.

When $t \neq \frac{1}{2}$, the combinatorics of all $\mathfrak{C}_t$'s are the same. Similarly to Case 1 above, the distance $d_{min}(\mathfrak{C}_a, \mathfrak{C}_b)$ is 2. □

5. Id-parameterizations

In this section, we study Cases 2, 3, 4 and 5 by id-parameterizations. Recall Definition 3.8 of a locus curve of a point.

We need the following notation of $\delta_{min}(\mathfrak{C}_a, \mathfrak{C}_b)$:

Notation 5.1 *The minimum value of $\delta_t(\mathfrak{C}_a, \mathfrak{C}_b)$ for all possible paths between A and B for some fixed equations for A and B is denoted by $\delta_{min}(\mathfrak{C}_a, \mathfrak{C}_b)$.*

5.1. Arrangements with no symmetry

5.1.1. Case 2

We define two arrangements $2.A$ and $2.B$ by

$$\begin{aligned} F_t([x:y:z]) = \quad & xy(x-1)(y-1)(x-t)(y-\tfrac{t}{t-1})(y-\tfrac{1}{t-1}(x-1))(y-\tfrac{t}{t^2-1}x) \\ & \cdot(y-\tfrac{1}{1-t}(x-t))(y-\tfrac{1}{t-1}x-1), \end{aligned}$$

where $t=-\frac{\sqrt{2}}{2}$ for $2.A$ and $t=\frac{\sqrt{2}}{2}$ for $2.B$.

Theorem 5.2 *The id-parametrization of the arrangements $2.A$ and $2.B$ defined by F_t for $t \in [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$ contains four degenerate arrangements $2.C_1$, $2.C_2$, $2.C_3$ and $2.C_4$. The minimum number $\delta_{min}(\mathfrak{C}_a, \mathfrak{C}_b)$ of degenerate arrangements of all parameterizations of $2.A$ and $2.B$ satisfies $3 \le \delta_{min}(\mathfrak{C}_a, \mathfrak{C}_b) \le 4$. Moreover, we have $d_{min}(\mathfrak{C}_a, \mathfrak{C}_b) \le 9$.*

Proof The arrangements appear in Figure 2: the arrangement $2.A$ on the left-hand side of the figure, and the arrangement $2.B$ on the right-hand side.

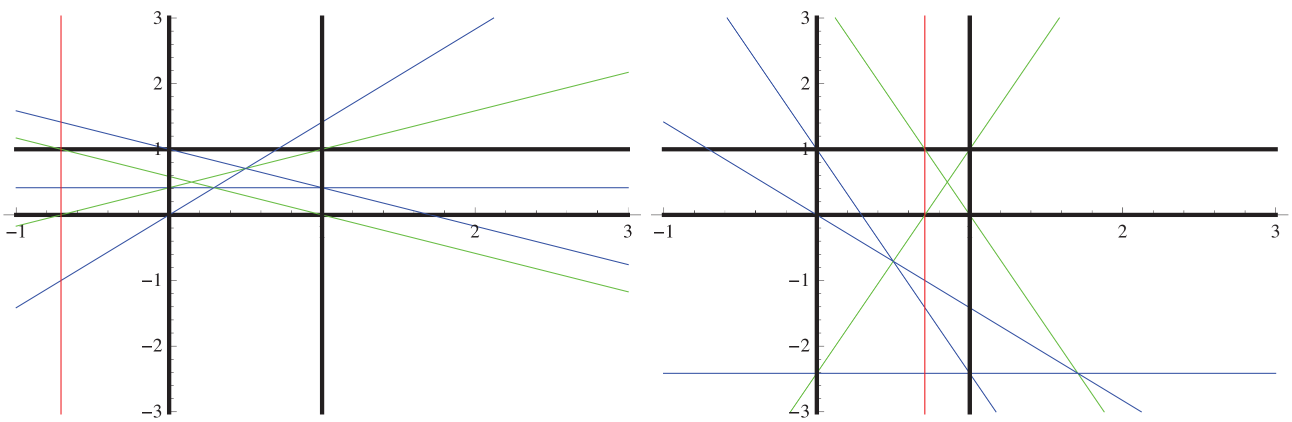

Figure 2. The arrangements 2.A and 2.B.

In each of the arrangements in Figure 2 we have ten lines with no line at infinity. There are four thick black lines that remain fixed under the parametrization. While the parameter t increases, the only red line in the arrangement moves to the right along the x-axis, intersecting it in the point $(t,0)$; the locus curve of the point $(t,0)$ is the x-axis.

In Figure 3, we depict degenerate arrangements $2.C_1$, $2.C_2$, $2.C_3$ and $2.C_4$ along the parametrization. The four values for the parameter t that give us these four degenerate arrangements are, respectively, $t=\frac{1-\sqrt{5}}{2}$, $t=0$, $t=\frac{1}{2}$ and $t=\frac{\sqrt{5}-1}{2}$. When $t=\frac{1-\sqrt{5}}{2}$, one blue line passes through the triple point $(1,1)$, and it becomes a quadruple point. When $t=0$, the red line coincides with the y-axis, two blue lines coincide with the x-axis, and one green line coincides with the third blue line. When $t=\frac{1}{2}$, the red line passes through the triple point $(\frac{1}{2},0)$, and it becomes a quadruple point. When $t=\frac{\sqrt{5}-1}{2}$, the red line passes through the intersection point of the two non-horizontal blue lines.

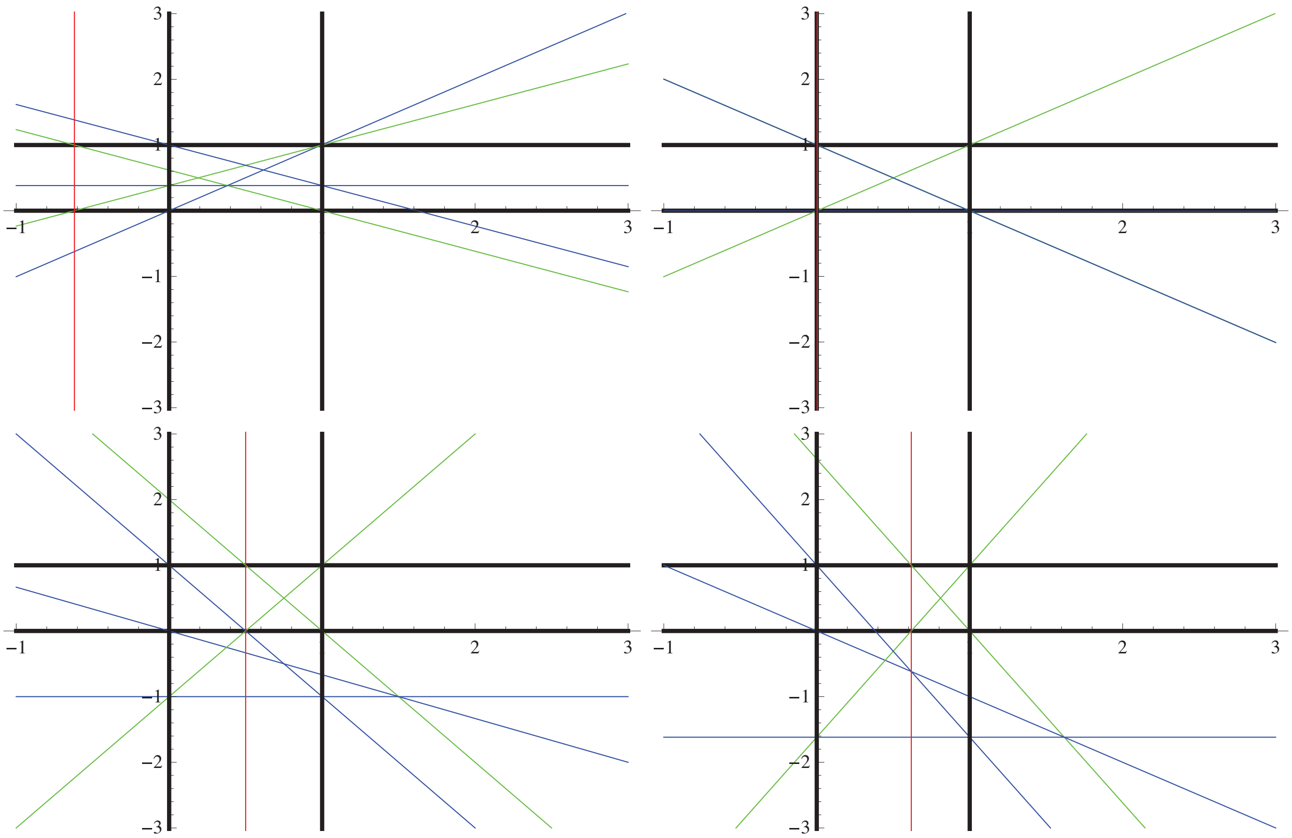

Figure 3. Degenerate arrangements of Case 2.

Observe from Figure 2 that there are just two lines with no multiple points at infinity, and so $J(id:A\to B)=\{\{y=\frac{1}{t-1}x+1\},\{y=\frac{t}{t^2-1}x\}\}$. Call the first blue line L in general with L^+ in A and L^- in B; we will not need to consider the second blue line.

There are three double and three triple points on L, and none of them is at infinity. We call them A,B,C,D,E,F. The ordering of them is in Figure 1, as appearing above in Example 2.2.

Figure 1 shows that $h(L^+,L^-)=3$; the second blue line in $J(id:A\to B)$ only gives height two, so we do not need to consider it. Hence $h(id:A\to B)\geq 3$ and by Theorem 3.9, the minimum number $\delta_t(\mathfrak{C}_a,\mathfrak{C}_b)$ of degenerate arrangements of all parameterizations of $2.A$ and $2.B$ is at least this height $h(id : A \to B)$. We have also given some parametrization containing four degenerate arrangements, and so $\delta(\mathcal{A}_t) \leq 4$; thus we have $3 \leq \delta(\mathcal{A}_t) \leq 4$.

Observe that among $2.A$ and $2.B$ just $2.A$ is a degenerate arrangement. By Proposition 3.7 and Definition 3.4, we have

$$d_{min}(\mathfrak{C}_a, \mathfrak{C}_b) \leq d_t(\mathfrak{C}_a, \mathfrak{C}_b) \leq 2\delta_t(\mathfrak{C}_a, \mathfrak{C}_b) + 1 \leq 2 * 4 + 1 = 9.$$

□

5.1.2. Case 3

We consider Equation (3) from [3], Theorem 4.4.

We define the two arrangements $3.A$ and $3.B$ by

$$\begin{aligned} F_t([x:y:z]) = \quad & xy(x-1)(x-t)(y-x)(y-2+x)(y+2x-2) \\ & \cdot(y - \frac{2t-1}{1-t}(x-1)-1)(y-\frac{2-t}{t-1}(x-1)) \\ & \cdot(y-\frac{t}{t-2}(x-2)), \end{aligned}$$

where $t = 3 - \sqrt{5}$ for $3.A$ and $t = 3 + \sqrt{5}$ for $3.B$.

Theorem 5.3 *The id-parametrization of the arrangements $3.A$ and $3.B$ defined by F_t for $t \in [3-\sqrt{5}, 3+\sqrt{5}]$ contains two degenerate arrangements $3.C_1$ and $3.C_2$. The minimum number $\delta_{min}(\mathfrak{C}_a, \mathfrak{C}_b)$ of degenerate arrangements of all parameterizations of $3.A$ and $3.B$ satisfies $\delta_{min}(\mathfrak{C}_a, \mathfrak{C}_b) = 2$. Moreover, we have $d_{min}(\mathfrak{C}_a, \mathfrak{C}_b) \leq 5$.*

Proof The arrangements appear in Figure 4: the arrangement $3.A$ on the left-hand side of the figure, and the arrangement $3.B$ on the right-hand side.

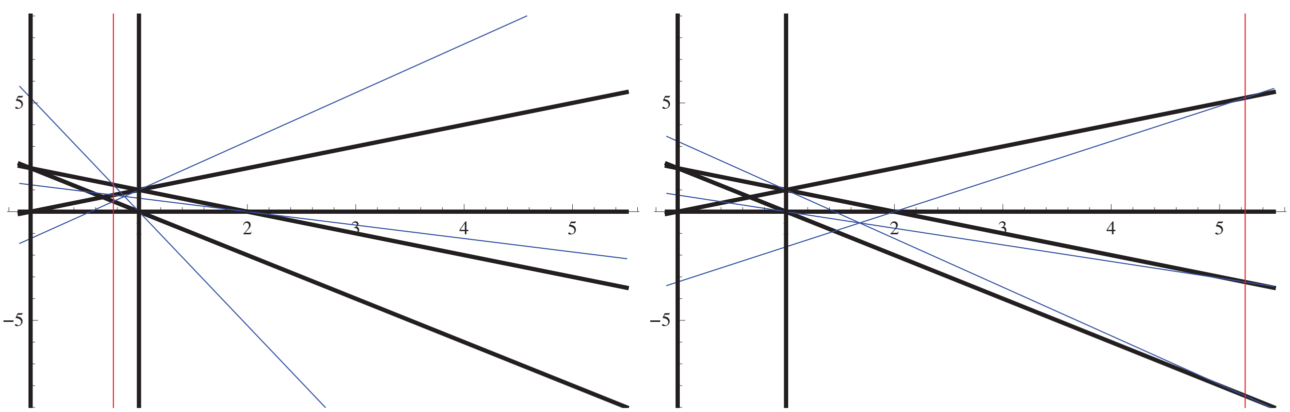

Figure 4. The arrangements $3.A$ and $3.B$.

In each of the arrangements of Figure 4 we have ten lines with no line at infinity. There are six thick black lines that remain fixed under the parametrization. While the parameter t increases, the only red line in the arrangement moves to the right along the x-axis, intersecting it in the point $(t, 0)$. Therefore, the locus curve of the point $(t, 0)$ in this case is just the x-axis.

In Figure 5, we depict degenerate arrangements $3.C_1$ and $3.C_2$. The two values for the parameter t that give us these two degenerate arrangements are, respectively, $t = 1$ and $t = 2$. When $t = 1$, one blue line coincides with one black line, and the red line coincides with the right black vertical line and with the other two blue lines, while for $t = 2$, one blue line coincides with the red line and another blue line coincides with the x-axis.

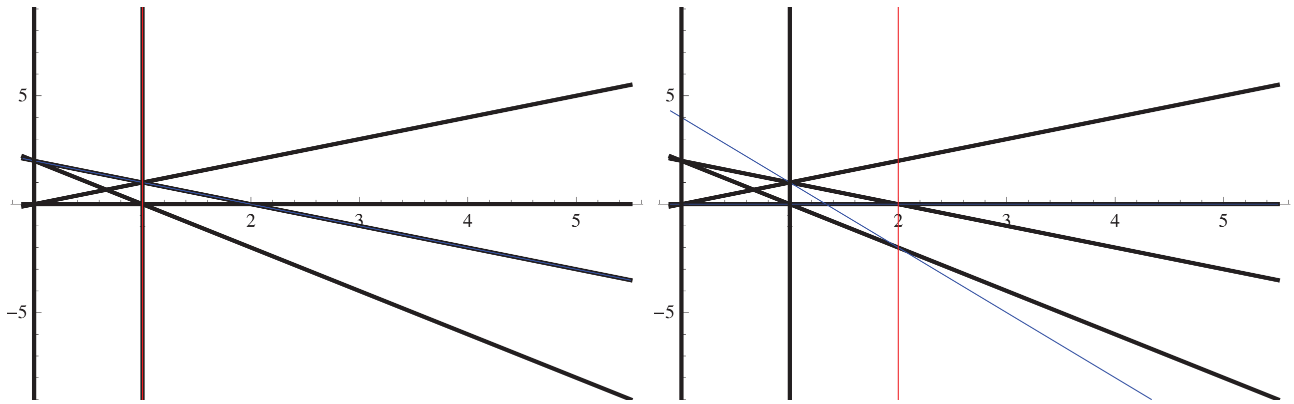

Figure 5. Degenerate arrangements of Case 3.

Observe from Figure 4 that there are seven lines with no multiple points at infinity, and so $|J(id : A \to B)| = 7$. We will only need to consider one of these lines: the thick black line L on the x-axis, $y = 0$; let us call it L^+ in A and L^- in B.

There are three double points, one triple point and one quadruple point on L, and none of them is at infinity. We call them A, B, C, D, E. The ordering of them is shown in Figure 6.

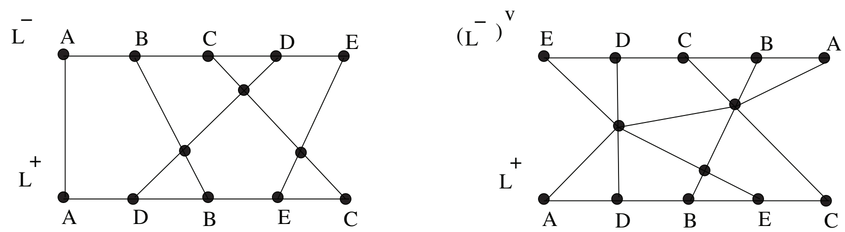

Figure 6. Calculating the height between two pointed lines for Case 3.

Figure 6 shows that $h(L^+, L^-) = 2$. Hence $h(id : A \to B) \geq 2$ and by Theorem 3.9, the minimum number $\delta_t(\mathfrak{C}_a, \mathfrak{C}_b)$ of degenerate arrangements of all parameterizations of $3.A$ and $3.B$ is at least this height $h(id : A \to B)$. We have also given some parametrization containing just two degenerate arrangements, and so $\delta(\mathcal{A}_t) \leq 2$; thus we have $\delta(\mathcal{A}_t) = 2$.

Observe that among $3.A$ and $3.B$ just $3.B$ is a degenerate arrangement. By Proposition 3.7 and Definition 3.4, we have

$$d_{min}(\mathfrak{C}_a, \mathfrak{C}_b) \leq d_t(\mathfrak{C}_a, \mathfrak{C}_b) \leq 2\delta_t(\mathfrak{C}_a, \mathfrak{C}_b) + 1 \leq 2 * 2 + 1 = 5.$$

□

5.1.3. Case 4

We consider Equation (4) from [3], Theorem 4.4. We define the two arrangements $4.A$ and $4.B$ by

$$\begin{aligned} F_t([x:y:z]) = \quad & xy(x-1)(x-2)(y-x)(y-2+x)(y+2x-2) \\ & \cdot(y-(t-1)(x-1)-1)(y-t(x-1)) \\ & \cdot(y-\frac{t-2}{2}(x-\frac{t+2}{t+1})-\frac{t}{t+1}), \end{aligned}$$

where $t = 2-2\sqrt{2}$ for $4.A$ and $t = 2+2\sqrt{2}$ for $4.B$.

Theorem 5.4 *The id-parametrization of the arrangements* $4.A$ *and* $4.B$ *defined by* F_t *for* $t \in [2-2\sqrt{2}, 2+2\sqrt{2}]$ *contains three degenerate arrangements* $4.C_1$, $4.C_2$ *and* $4.C_3$. *And the minimum number* $\delta_{min}(\mathfrak{C}_a, \mathfrak{C}_b)$ *of degenerate arrangements of all parameterizations of* $4.A$ *and* $4.B$ *satisfies* $\delta_{min}(\mathfrak{C}_a, \mathfrak{C}_b) = 3$. *Moreover, we have* $d_{min}(\mathfrak{C}_a, \mathfrak{C}_b) \leq 8$.

Proof The arrangements appear in Figure 7: the arrangement $4.A$ on the left-hand side of the figure, and the arrangement $4.B$ on the right-hand side.

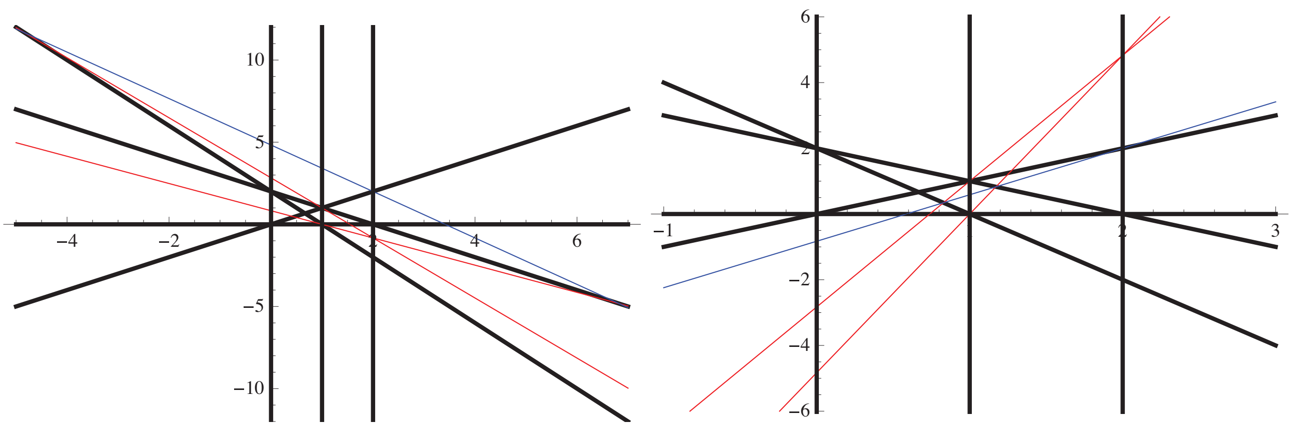

Figure 7. The arrangements $4.A$ and $4.B$.

In each of the arrangements of Figure 7 we have ten lines with no line at infinity. There are seven thick black lines that remain fixed. The intersection point of the blue line and two black lines, which is actually a triple, is fixed. While the parameter t increases, the intersection point of the two red lines moves upwards on the right black line $x = 2$. Thus this intersection point has the coordinates $(2, t)$ and the locus curve is $x = 2$.

Figure 8 depicts degenerate arrangements $4.C_1$ on the left, $4.C_2$ on the right and $4.C_3$ below. The three values for the parameter t that give us these three degenerate arrangements are respectively $t = 0$, $t = 2$ and $t = \sqrt{5}+1$. When $t = 0$ the blue line coincides with one black line and one red line, when $t = 2$ one of the red lines coincides with one black line, and when $t = \sqrt{5}+1$ the blue line passes through the origin.

Observe from Figure 7 that there are seven lines with no multiple points at infinity, and so $|J(id : A \to B)| = 7$. We will only need to consider one of these lines: the blue line $L : y = \frac{t-2}{2}(x - \frac{t+2}{t+1}) + \frac{t}{t+1}$ for $t = 2 \pm 2\sqrt{2}$; let us call it L^+ in A and L^- in B.

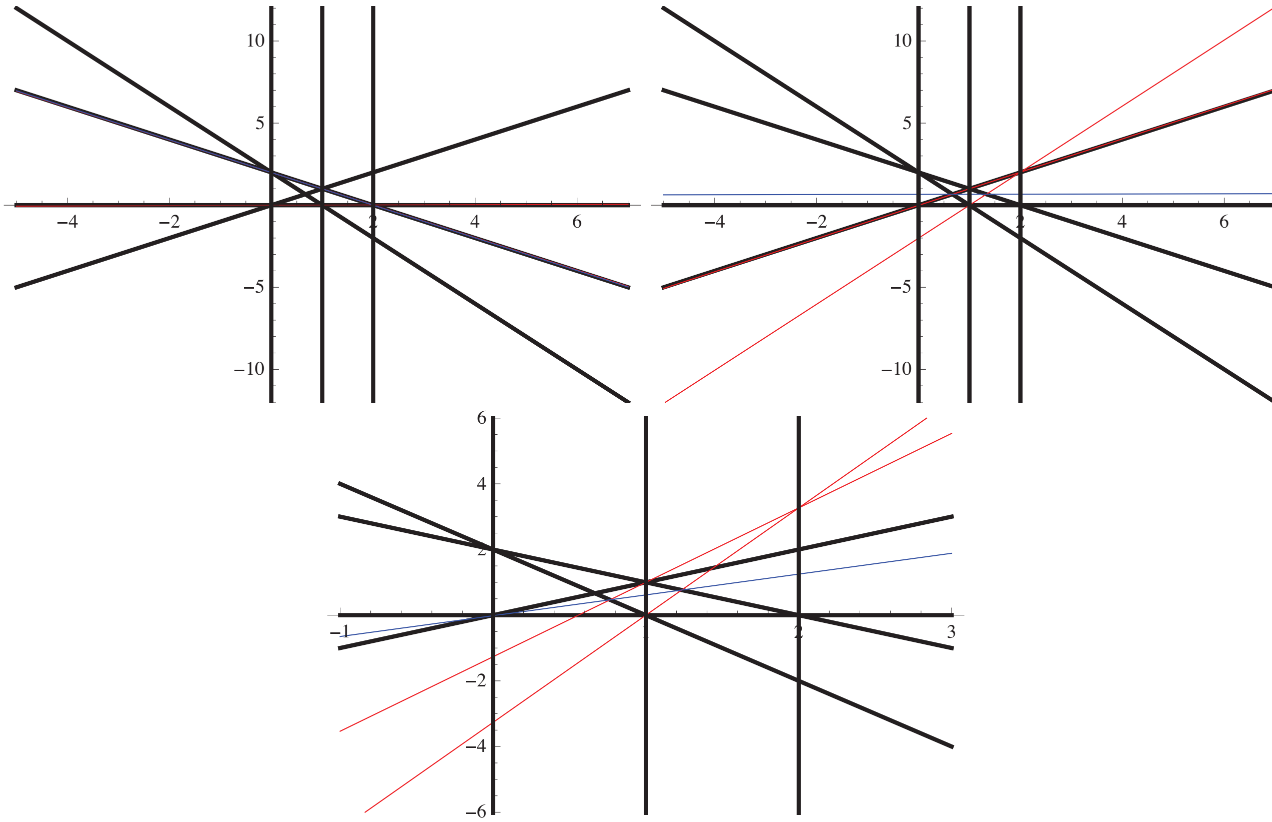

Figure 8. Degenerate arrangements of Case 4.

There are three double and three triple points on L, and none of them is at infinity. We call them A, B, C, D, E, F. The ordering of them is shown in Figure 9.

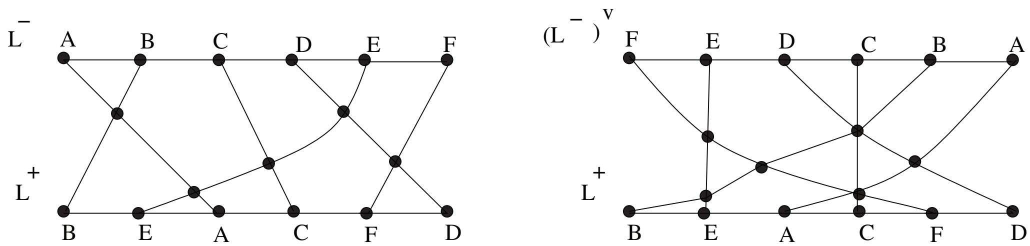

Figure 9. Calculating the height between two pointed lines for Case 4.

Figure 9 shows that $h(L^+, L^-) = 3$. Hence $h(id : A \to B) \geq 3$ and by Theorem 3.9, the minimum number $\delta_t(\mathfrak{C}_a, \mathfrak{C}_b)$ of degenerate arrangements of all parameterizations of $4.A$ and $4.B$ is at least this height $h(id : A \to B)$. We have also given some parametrization containing just three degenerate arrangements, and so $\delta(\mathcal{A}_t) \leq 3$; thus we have $\delta(\mathcal{A}_t) = 3$.

Observe that both $4.A$ and $4.B$ are themselves degenerate arrangements. By Proposition 3.7 and Definition 3.4, we have

$$d_{min}(\mathfrak{C}_a, \mathfrak{C}_b) \leq d_t(\mathfrak{C}_a, \mathfrak{C}_b) \leq 2\delta_t(\mathfrak{C}_a, \mathfrak{C}_b) + 2 \leq 2 * 3 + 2 = 8.$$

□

5.2. Arrangement with $\mathbb{Z}_3$ symmetry

In this subsection we present Case 5 whose automorphism group is $\mathbb{Z}_3$. However, we only consider the identity permutation.

5.2.1. Case 5

We define the two arrangements $5.A$ and $5.B$ by

$$\begin{aligned} F_t([x:y:z]) = \quad & yz(y-\sqrt{3}x)(y-4z+\sqrt{3}x)(y-z)(y+\sqrt{3}x-2z) \\ & \cdot(y-\sqrt{3}x+2z)(y-\tan(t)x)(y-\tan(t+\frac{2\pi}{3})(x-\frac{4}{\sqrt{3}})) \\ & \cdot(y-\tan(t+\frac{\pi}{3})(x-\frac{2}{\sqrt{3}})-2z), \end{aligned}$$

where $t = \arctan\sqrt{\frac{3}{5}}$ for $5.A$ and $t = \frac{\pi}{3} + \arctan\sqrt{\frac{3}{5}}$ for $5.B$.

Theorem 5.5 *The id-parametrization of the arrangements* $5.A$ *and* $5.B$ *defined by* F_t *for* $t \in [\arctan\sqrt{\frac{3}{5}}, \frac{\pi}{3} + \arctan\sqrt{\frac{3}{5}}]$ *contains one degenerate arrangement* $5.C_1$. *Moreover, we have* $d_{min}(\mathfrak{C}_a, \mathfrak{C}_b) \leq 4$.

Proof The arrangements appear in Figure 10: the arrangement $5.A$ on the left-hand side of the figure, and the arrangement $5.B$ on the right-hand side.

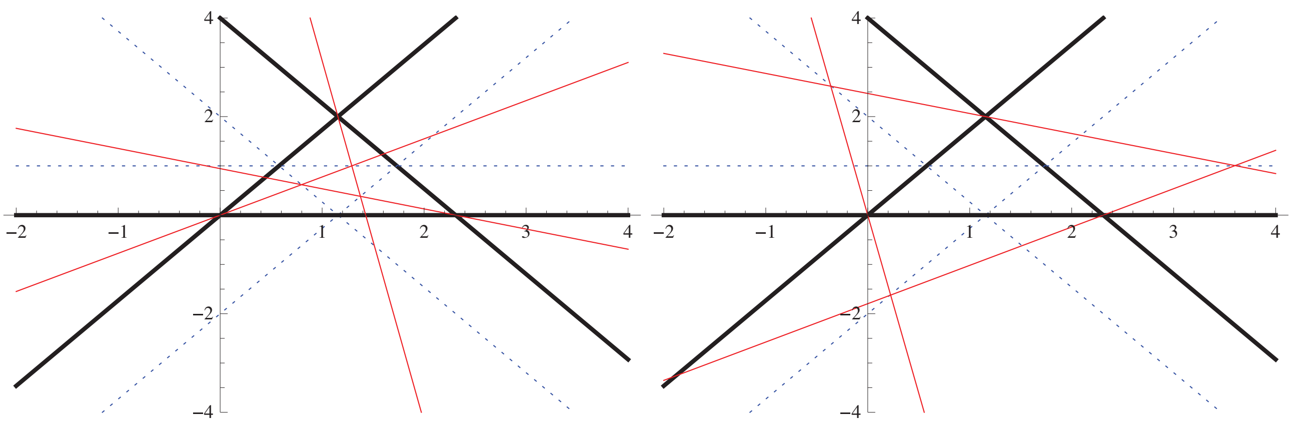

Figure 10. The arrangements $5.A$ and $5.B$.

In each of the arrangements in Figure 10 we have ten lines, where one line is fixed at infinity. First observe that the three thick black lines and the three blue dotted lines each form a triangle. These lines remain fixed. There is a red line passing through each vertex of the thick black triangle whose slope is parameterized by t_i for $i = 1,2,3$. We set $t = t_1 = t_2 = t_3$ to be the angle between each red-and-black line pair.

The three red lines form a triangle, as well. Choosing one vertex of the red triangle in arrangement $5.A$, we describe the locus curve of the vertex as the parameter t changes. For simplicity we only describe one of the three locus curves because the others are the same up to symmetry. The chosen vertex is an intersection of the red lines

$$y - \tan(t)x = 0 \quad \text{and} \quad y - \tan(t + \frac{2\pi}{3})(x - \frac{4}{\sqrt{3}}) = 0.$$

The coordinates of this intersection point are

$$\left(\frac{\frac{4}{\sqrt{3}}\tan(t+\frac{2\pi}{3})}{\tan(t+\frac{2\pi}{3}) - \tan(t)}, \frac{\frac{4}{\sqrt{3}}\tan(t)\tan(t+\frac{2\pi}{3})}{\tan(t+\frac{2\pi}{3}) - \tan(t)} \right).$$

These coordinates, parameterized by t give the locus curve while t changes.

The vertex proceeds on the locus curve until it meets a blue dotted line, where it forms a triple point. This occurs twice.

When $t = \frac{\pi}{3}$, the red and the black lines coincide, as in Figure 11. This is the degenerate arrangement of both arrangements $5.A$ and $5.B$. Note that in $5.A$ (respectively $5.B$), the red triangle is inside (respectively outside) the black triangle.

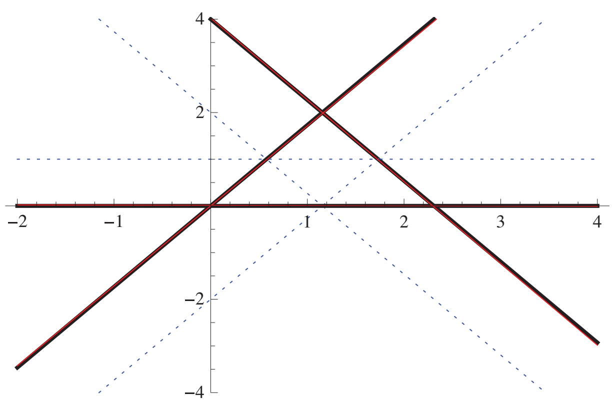

Figure 11. The degenerate arrangement of Case 5.

We have given some parametrization containing just one degenerate arrangement, and so $\delta_t(\mathfrak{C}_a, \mathfrak{C}_b) \leq 1$; because the moduli space has two components, there must be at least one degenerate arrangement, and thus we have $\delta_t(\mathfrak{C}_a, \mathfrak{C}_b) = 1$.

Observe that both $5.A$ and $5.B$ are themselves degenerate arrangements. By Proposition 3.7 and Definition 3.4, we have

$$d_{min}(\mathfrak{C}_a, \mathfrak{C}_b) \leq d_t(\mathfrak{C}_a, \mathfrak{C}_b) \leq 2\delta_t(\mathfrak{C}_a, \mathfrak{C}_b) + 2 \leq 2 * 1 + 2 = 4.$$

□

Acknowledgment

Thanks are given to Fei Ye for fruitful discussions and ideas.

6. Appendices

Here we give the codes for the seven visualizations in `Mathematica`. The reader can view them in order to follow the computations in the above sections.

Case 1:

```
L1[x_, t_] := -t/(t - 1) x
L2[x_, t_] := (t - 1)/-t x
L3[x_, t_] :=(1 - 2 t)/(1 - t) + t/(1 - t) x
L4[x_, t_] := (2 t - 1)/t + (1 - t)/t x
L5[x_, t_] := ((Sqrt[5] - 1) (1 - 2 t))/(2 t) + (1 - t)/t x
L6[x_, t_] := ((1 + Sqrt[5]) (1 - 2 t))/(2 (1 - t)) + t/(1 - t) x
L7[x_, t_] := x + (Sqrt[5] + 1)/2 (1 - 2 t)
L8[x_, t_] := x - (1 - Sqrt[5])/2 (1 - 2 t)
L9[x_, t_] := ((3 + Sqrt[5]) (1 - t) + 2 t)/((3 + Sqrt[5]) t + 2 (1- t)) x

Manipulate[Plot[{L1[x, t], L2[x, t], L3[x, t], L4[x, t], L5[x, t],
L6[x, t], L7[x, t], L8[x, t], L9[x, t]}, {x, -2, 2},
PlotRange -> 3, PlotStyle
 -> {{Black}, {Black}, {Black, Thick},
Black, Black, Red, {Black}, {Black}, {Black}}], {t, -1, 1.1}]
```

Case 2:

```
L1[x_, t_] := 0
L2[x_, t_] := 1
L3[x_, t_] := t/(t - 1)
L7[x_, t_] := 1/(t - 1) (x - 1)
L8[x_, t_] := t/(t^2 - 1) x
L9[x_, t_] := 1/(1 - t) (x - t)
L10[x_, t_] := 1/(t - 1) x + 1

Manipulate[Plot[{L1[x, t], L2[x, t], L3[x, t], L7[x, t], L8[x, t],
L9[x, t], L10[x, t]}, {x, -1, 3}, PlotRange -> 3,
PlotStyle -> {{Black, Thick}, {Black, Thick}, Blue, Green,
{Blue}, Green, {Blue}}, Epilog -> {{Thick, Line[{{0, -100},
{0, 100}}]}, {Thick, Line[{{1, -100}, {1, 100}}]},
{Red, Line[{{t, -100}, {t, 100}}]}}], {t, -(Sqrt[2]/2), Sqrt[2]/2}]
```

Case 3:

```
L1[x_, t_] := 0
L2[x_, t_] := x
L3[x_, t_] := 2 - x
L7[x_, t_] := 2 - 2 x
L8[x_, t_] := (2 t - 1)/(1 - t) (x - 1) + 1
L9[x_, t_] := (2 - t)/(t - 1) (x - 1)
L10[x_, t_] := t/(t - 2) (x - 2)

Manipulate[Plot[{L1[x, t], L2[x, t], L3[x, t], L7[x, t], L8[x, t],
```

```
L9[x, t], L10[x, t]}, {x, -0.1, 5.5}, PlotRange -> 9,
PlotStyle -> {{Black, Thick}, {Black, Thick}, {Black, Thick},
{Black, Thick}, {Blue}, {Blue}, {Blue}},
Epilog -> {{Thick, Line[{{0, -100}, {0, 100}}]}, {Thick,
Line[{{1, -100}, {1, 100}}]}, {Red, Line[{{t, -100}, {t, 100}}]}}],
{t, 3 - Sqrt[5], 3 + Sqrt[5]}]
```

Case 4:

```
L1[x_, t_] := 0
L2[x_, t_] := x
L3[x_, t_] := 2 - x
L7[x_, t_] := 2 - 2 x
L8[x_, t_] := (t - 1) (x - 1) + 1
L9[x_, t_] := t (x - 1)
L10[x_, t_] := (t - 2)/2 (x - (t + 2)/(t + 1)) + t/(t + 1)
```

```
Manipulate[Plot[{L1[x, t], L2[x, t], L3[x, t], L7[x, t], L8[x, t],
L9[x, t], L10[x, t]}, {x, -1, 3}, PlotRange -> 6, PlotStyle ->
{{Black, Thick}, {Black, Thick}, {Black, Thick}, {Black, Thick},
Red, Red, Blue}, Epilog -> {{Thick, Line[{{0, -100}, {0, 100}}]},
{Thick, Line[{{1, -100}, {1, 100}}]},
{Thick, Line[{{2, -100}, {2, 100}}]}}],
{t, 2 - 2 Sqrt[2], 2 + 2 Sqrt[2]}]
```

Case 5:

```
L1[x_, t_] := Sqrt[3] x
L2[x_, t_] := 4 - Sqrt[3] x
L3[x_, t_] := 0
L4[x_, t_] := 1
L6[x_, t_] := Tan[t + (2 \[Pi])/3] (x - 4/Sqrt[3])
L7[x_, t_] := Tan[t] x
L8[x_, t_] := Tan[t + \[Pi]/3] (x - 2/Sqrt[3]) + 2
L9[x_, t_] := 2 - Sqrt[3] x
L10[x_, t_] := Sqrt[3] x - 2
```

```
Manipulate[Plot[{L1[x, t], L2[x, t], L3[x, t], L4[x, t], L6[x, t],
L7[x, t], L8[x, t], L9[x, t], L10[x, t]}, {x, -1, 5}, PlotRange -> 3,
PlotStyle -> {Black, Black, Black, Black, Dashed, Dashed, Dashed,
Black, Black, Black}], {t, 0, \[Pi]}]
```

Case 6:

```
L1[x_, t_] := -(1 + t)
L2[x_, t_] := 1
```

```
L3[x_, t_] := 0
L5[x_, t_] := t
L7[x_, t_] := (1 + 2/t) x - 1 - t
L8[x_, t_] := - x + 1
L9[x_, t_] := (t + 1)/(t - 1) (x - t)
L10[x_, t_] := (t + 2)/(t - 1) x - 1/(t - 1)

Manipulate[Plot[{L1[x, t], L2[x, t], L3[x, t], L5[x, t], L7[x, t],
L8[x, t], L9[x, t], L10[x, t]}, {x, -4, 4}, PlotRange -> 6,
PlotStyle -> {{Black}, {Black}, {Black}, {Black}, {Black}, {Black},
{Black}, {Black}}, Epilog -> {{Thick, Line[{{0, -100}, {0, 100}}]},
{Thick, Line[{{1, -100}, {1, 100}}]} }], {t, (-1 - Sqrt[5])/2, (-1 +
Sqrt[5])/2}]
```

Case 7:

```
L1[x_, t_] := 1
L2[x_, t_] := 1 - t
L3[x_, t_] := 0
L7[x_, t_] := -t x + 1
L8[x_, t_] := (t - 1)/t x + 1 - t
L9[x_, t_] := 1/(t - 1) (x - 1)
L10[x_, t_] := -x
Manipulate[Plot[{L1[x, t], L2[x, t], L3[x, t],
L5[x, t], L7[x, t], L8[x, t], L9[x, t], L10[x, t]}, {x, -4, 4},
PlotRange -> 6, PlotStyle -> {{Black}, {Black}, {Black}, {Black},
{Black}, {Black}, {Black}, {Black}}, Epilog -> {{Thick, Line[{{0,
-100}, {0, 100}}]}, {Thick, Line[{{1, -100}, {1, 100}}]}, {Thick,
Line[{{t, -100}, {t, 100}}]} }], {t, (1 - Sqrt[5])/2, (1 +
Sqrt[5])/2}]
```

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