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The relation between the existence of bounded global solutions of the differential equations and equations on time scales

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Abstract: This work is devoted to the study of existence of a bounded solution of the differential equation, defined on a family of time scales $\mathbb{T}_\lambda$, provided the graininess function μ_λ converges to zero as $\lambda \to 0$. We obtained the conditions, under which the existence of a bounded solution of differential equation implies the existence of a bounded solution of the corresponding equation, defined on time scales, and vice versa.

Key words: Time scale, differential equation, asymptotic stability, graininess function, bounded solution

1. Introduction

Differential (dynamic) equations on time scales were introduced by Hilger in [8] as an attempt to create a unified theory for both discrete and continuous dynamical systems. The theory of equations on time scales was further developed in [1, 2]. The behavior of the solutions of the dynamic equations, defined on a family of time scales $\mathbb{T}_\lambda$ when graininess function μ_λ goes to 0 as $\lambda \to 0$, is of particular interest to us. In this case intervals of the time scale $[t_0, t_1]_\lambda = [t_0, t_1] \cap \mathbb{T}_\lambda$ approach $[t_0, t_1]$ (e.g., in the Hausdorff metric). The natural question arises is whether the solutions of the equations on time scales and the corresponding differential equations share the same properties. While on the bounded time intervals it is not difficult to establish the convergence of solutions of dynamic time scale equations to the corresponding solutions of differential equations, for infinite intervals this problem is highly nontrivial.

In our work we consider the family of dynamical equations for sufficiently small μ_λ, which possesses a global bounded solution. Here the natural question arises whether the corresponding differential equation has a bounded solution as well, provided $\mathbb{T}_\lambda \to \mathbb{R}^1, \lambda \to 0$. Conversely, suppose we have a ordinary differential equation which has a bounded solution for $t \in \mathbb{R}$. Is it true that the corresponding equation on time scale $\mathbb{T}_\lambda$ will have a bounded solution as well, provide λ is sufficiently small.

It is worth mentioning that the question of existence of two-sided solutions for dynamic equations on time scales is not trivial by itself. In contrast with the classic theorem on the existence of solutions of the system of ordinary differential equations, where local double-sided existence with respect to the initial point holds, the

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situation for the equations on time scales is significantly more complicated. In particular, in order to extend the solution to the left, one needs to impose a very strong regression condition [5]. In our case, we got the existence of the two-sided global bounded solution without using this regression condition.

The proof of the main result requires a continuous dependence of the solutions on the initial data uniformly over all time scales. This question is nontrivial due to the topological complexity of the time scale. Our approach is different from the one considered in [9], where the analysis was done on the fixed scale.

The relation between the properties of solutions of the system of ordinary differential equation and the solutions of equations on Eulerian time scales was studied earlier. In particular, the paper [3] showed that the solutions of differential and the corresponding difference equations have the same oscillatory properties. The relation between the stability and the attractors of differential and difference equations was studied in [13]. The optimal control problems for the systems of ordinary differential equations and the corresponding dynamical equations on time scales were considered in [4, 6, 12]. The stability of perturbed dynamic system on time scales was addressed in [15] using the appropriate analogs of Lyapunov functions .

This work is devoted to the study of existence of a bounded solution of the the differential equation, defined on a family of time scales $\mathbb{T}_\lambda$, provided the graininess function μ_λ converges to zero as $\lambda \to 0$. Our work extends the results on the relation between the existence of bounded solutions of differential equations and the corresponding difference equations, obtained in [14], to the case of generic time scales. The main difficulty in our work is to compare the solutions of differential equations and equations on time scale for any $\mathbb{T}_\lambda$. This makes our analysis significantly different from [14], where only special case of time scale, namely $\mathbb{T} = \mathbb{Z}$, was considered.

This paper is organized as follows. In section 2.1 we provide some definitions and statements necessary for our research. In section 2.2 we state and prove the main results about the existence of a bounded solution of differential equation, defined on time scales. Examples of application of main results are provided in section 3.

2. Preliminaries and main results

2.1. Basic notions of time scale theory

For the convenience of the reader, we present the necessary concepts and notation, consistent with the ones introduced in the monograph [1].

- A *time scale* $\mathbb{T}$ is an arbitrary, nonempty, closed subset of the real axis.
- For every $A \subset \mathbb{R}$, denote $A_\mathbb{T} := A \cap \mathbb{T}$.
- A *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

- Similarly, a *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is

$$\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\}.$$

  Here we assume $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$).

- The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined as $\mu(t) := \sigma(t) - t$.

- A point $t \in \mathbb{T}$ is called *left-dense (LD), (left-scattered (LS), right-dense (RD)* or *right-scattered (RS))* if $\rho(t) = t$ ($\rho(t) < t$, $\sigma(t) = t$ or $\sigma(t) > t$) hold. If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$; otherwise, we set $\mathbb{T}^k = \mathbb{T}$.

- A function $f : \mathbb{T} \to \mathbb{R}^d$ is said to be Δ*-differentiable at* $t \in \mathbb{T}^k$ if the limit

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$$

exists in $\mathbb{R}^d$.

Let us recall the following classic results (see [1]):

a) if $t \in \mathbb{T}^k$ is right-dense point of $\mathbb{T}$, then f is Δ-differentiable at t if and only if the limit

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists in $\mathbb{R}^d$.

b) if $t \in \mathbb{T}^k$ is a right-scattered point of $\mathbb{T}$ and if f is continuous at t, then f is Δ differentiable at t and

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

2.2. Main results

Let $D \subset \mathbb{R}^d$ be a domain. Consider the system of differential equations:

$$\frac{dx}{dt} = X(t, x), \quad t \in \mathbb{R}, x \in D. \tag{2.1}$$

Let $\mathbb{T}_\lambda$ be a set of time scales and the system (2.1) defined on $\mathbb{T}_\lambda$ now reads as

$$x_\lambda^{\Delta}(t) = X(t, x_\lambda). \tag{2.2}$$

Here $t \in \mathbb{T}_\lambda$, $x_\lambda : \mathbb{T}_\lambda \to \mathbb{R}^d$, and $x_\lambda^{\Delta}(t)$ is delta-derivative of $x(t)$ on $\mathbb{T}_\lambda$. We assume that $\inf \mathbb{T}_\lambda = -\infty$, $\sup \mathbb{T}_\lambda = \infty$, $\lambda \in \Lambda \subset \mathbb{R}$, and $\lambda = 0$ is a limit point of Λ. We also assume that the function $X(t, x)$ is continuously differentiable and bounded together with its partial derivatives, i.e. $\exists C > 0$ such that

$$|X(t, x)| + \left| \frac{\partial X(t, x)}{\partial t} \right| + \left\| \frac{\partial X(t, x)}{\partial x} \right\| \le C \tag{2.3}$$

for $t \in \mathbb{R}, x \in D$, where $\frac{\partial X}{\partial x}$ is the corresponding Jacobi matrix.

Denote $\mu_\lambda := sup_{t \in \mathbb{T}_\lambda} \mu_\lambda(t)$, where $\mu_\lambda \colon \mathbb{T}_\lambda \to [0, \infty)$ is the graininess function. it is straightforward to see that if $\mu_\lambda(t) \to 0$ as $\lambda \to 0$, then $\mathbb{T}_\lambda$ coincides (e.g., in Hausdorff metric) with a continuous time scale $\mathbb{T}_0 = \mathbb{R}$ (see e.g., [7]). We start with the following lemma:

Lemma 2.1 *Let $t_0 \in \mathbb{T}_\lambda$, $t_0 + T \in \mathbb{T}_\lambda$, $x(t)$ and $x_\lambda(t)$ are the solutions of (2.1) and (2.2) on $[t_0, t_0 + T]$, and on $[t_0, t_0 + T]_{\mathbb{T}_\lambda}$ respectively. Then if $x(t_0) = x_\lambda(t_0) = x_0 \in D$ the following inequality holds*

$$|x(t) - x_\lambda(t)| \leq f(\lambda)K(T),$$

where K is constant, $f(\lambda) = sup_{t \in [t_0, t_0+T]_{\mathbb{T}_\lambda}} \mu_\lambda(t)$ for $t \in [t_0, t_0 + T]_{\mathbb{T}_\lambda}$.

Proof Without loss of generality, consider $t_0 = 0$, $T = 1$ and $x_\lambda(0) = x(0)$. We have

$$x(t) = x_0 + \int_0^t X(s, x(s))ds,$$

and

$$x_\lambda(t) = x_0 + \int_{[0,t]_{\mathbb{T}_\lambda}} X(s, x_\lambda(s))\Delta s. \tag{2.4}$$

Let us construct the function $\tilde{x}_\lambda(t)$, defined for $t \in [0, 1]$. Assume that on the time scale this function coincides with $x_\lambda(t)$, i.e.

$$\tilde{x}_\lambda(t) = x_\lambda(t), t \in \mathbb{T}_\lambda$$

On the remaining intervals this function is defined to be constant:

$$\tilde{x}_\lambda(t) = \begin{cases} x_\lambda(t), t \in [0, 1]_{\mathbb{T}_\lambda} \\ x_\lambda(r), t \in [r, \sigma(r)), \end{cases} \tag{2.5}$$

Here $r \in RS$, and $\sigma(r)$ is the forward jump operator. We have

$$|x(t) - x_\lambda(t)| \leq |x(t) - \tilde{x}_\lambda(t)| + |\tilde{x}_\lambda(t) - x_\lambda(t)|. \tag{2.6}$$

The second term in (2.6) is automatically zero for $t \in \mathbb{T}_\lambda$. Let us estimate first term in (2.6). By the analog of Gronwall's inequality on time scales [1] and Theorem 2 [11], for any $r > 0$ there exists constant $C(r) > 0$ that

$$\begin{aligned} |x_\lambda(t)| &\leq C(r), t \in [0, 1]_{\mathbb{T}_\lambda} \\ |x(t)| &\leq C(r), t \in [0, 1], |x_0| \leq r. \end{aligned} \tag{2.7}$$

Moreover first inequality holds uniformly in $t \in \mathbb{T}_\lambda$. Therefore there exists constant $C_1(r) > 0$ such that

$$\begin{aligned} |X(t, x_\lambda(t))| &\leq C_1(r), \\ |X(t, x(t))| &\leq C_1(r), \\ \|X_x(t, x_\lambda(t))\| &\leq C_1(r), \forall t \in [0, 1]_{\mathbb{T}_\lambda}, \forall \lambda, \\ \|X_x(t, x(t))\| &\leq C_1(r), \forall t \in [0, 1]. \end{aligned} \tag{2.8}$$

We now rewrite (2.4) as

$$x_\lambda(t) = x_0 + \int_{[0,t]_{\mathbb{T}_\lambda} \setminus RS} X(s, x_\lambda(s))ds + \sum_{r \in RS} X(r, x_\lambda(r))\mu(r). \tag{2.9}$$

By (2.8) the sum (2.9) is bounded from above by convergent series $C_1 \sum_{r\in RS} \mu_\lambda(r)$. Then

$$\sum_{r\in RS} X(r_k, x_\lambda(r_k))\mu(r_k) = \sum_{k=1}^{N} X(r_k, x_\lambda(r_k))\mu_\lambda(r_k) + \sum_{k\geq N+1} X(r_k, x_\lambda(r_k))\mu_\lambda(r_k).$$

For every λ choose $N(\lambda, T) \geq 1$ such that

$$\sum_{k\geq N+1} \mu_\lambda(r_k) \leq \frac{\mu_\lambda}{2}. \tag{2.10}$$

Now let us remove the right-scattered points from the time scale in (2.10). Let $A = \cup_r (r, \sigma(r))$, where the union is taken over all right scattered r. Without loss of generality, we assume that the time scale $\mathbb{T}_\lambda$ has the following structure:

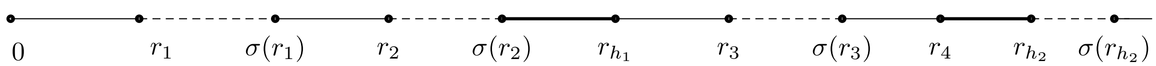

Figure . ???

Here the solid lines stand for the line segments which consist of the limit points, and the dashed lines illustrate the line segments $[r_i, \sigma(r_i))$, with the right-scattered points r_i included. Finally, the boldface solid lines denote the set A. The proof for others time scale structures is analogues.

In order to proceed, we will carefully consider all possible scenarios.

1) On $[0, r_1]$ we have $\tilde{x}_\lambda(t) = x_\lambda(t) = x(t)$.

2) On $[r_1, \sigma(r_1)]$, we have $\tilde{x}_\lambda(t) = x_\lambda(r_1) = x(r_1)$ but $x(t) = x(r_1) + \int_{r_1}^{t} X(s, x(s))ds$. Therefore for $t \in [r_1, \sigma(r_1)]$ $x(t)$ is a smooth function. By Taylor's theorem with the remainder in Lagrange form, we get

$$x(t) = x(r_1) + X(r_1, x(r_1))(t - r_1) + X'_x(s_1, x(s_1)) \cdot X(s_1, x(s_1))\frac{(t - r_1)^2}{2}, \tag{2.11}$$

where s_1 is in $[r_1, \sigma(r_1)]$, and X'_x is Jacobi matrix. By (2.8), we have

$$max_{t\in[0,1]}|X'_x(t, x(t))X(t, x(t))| \leq C_1^2. \tag{2.12}$$

Therefore, if $t \in (r_1, \sigma(r_1))$, we get

$$|x(t) - \tilde{x}_\lambda(t)| \leq \int_{r_1}^{\sigma(r_1)} |X(t, x(t))|dt \leq C_1\mu(r_1). \tag{2.13}$$

At the point $\sigma(r_1)$ we have

$$\tilde{x}_\lambda(\sigma(r_1)) = x_\lambda(r_1) + X(r_1, x_\lambda(r_1))\mu(r_1) = x(r_1) + X(r_1, x(r_1))\mu(r_1).$$

Thus, by (2.11) and (2.12) we get

$$|x(\sigma(r_1)) - \tilde{x}_\lambda(\sigma(r_1))| \leq C_1^2 \frac{\mu_\lambda^2(r_1)}{2}. \tag{2.14}$$

3) For $t \in [\sigma(r_1), r_2]$, we have

$$\tilde{x}_\lambda(t) = x_\lambda(t) = x_\lambda(\sigma(r_1)) + \int_{\sigma(r_1)}^{t} X(s, x_\lambda(s))ds,$$

$$x(t) = x(\sigma(r_1)) + \int_{\sigma(r_1)}^{t} X(s, x(s))ds,$$

hence

$$|x(t) - \tilde{x}_\lambda(t)| \leq |x_\lambda(\sigma(r_1)) - x(\sigma(r_1))| + C\int_{\sigma(r_1)}^{t} |\tilde{x}_\lambda(s) - x(s)|ds.$$

By (2.14) and Gronwall's inequality we have

$$|\tilde{x}_\lambda - x(t)| \leq \frac{\mu_\lambda^2(r_1)}{2} C_1^2 e^{C(r_2 - \sigma(r_1))}. \tag{2.15}$$

4) On $[r_2, \sigma(r_2)]$ we have

$$\tilde{x}_\lambda(t) = \tilde{x}_\lambda(r_2), \quad x(t) = x(r_2) + \int_{r_2}^{t} X(s, x(s))ds.$$

Thus

$$|x(t) - \tilde{x}_\lambda| \leq \frac{\mu_\lambda^2(r_1)}{2} C_1^2 e^{C(r_2 - \sigma(r_1)) + \mu_\lambda(r_2)C_1}. \tag{2.16}$$

At $t = \sigma(r_2)$ we get

$$x(\sigma(r_2)) = x(r_2) + X(r_2, x(r_2))\mu_\lambda(r_2) + \frac{1}{2}X'_x(s_2, x(s_2)) \cdot X(s_2, x(s_2))\frac{(t - r_2)^2}{2}$$

for some $s_2 \in [r_2, \sigma(r_2)]$, and $\tilde{x}_\lambda(\sigma(r_2)) = \tilde{x}_\lambda(r_2) + X(r_2, \tilde{x}_\lambda(r_2))\mu_\lambda(r_2)$. Therefore

$$|x(\sigma(r_2) - \tilde{x}_\lambda(\sigma(r_2))| \leq |x(r_2) - \tilde{x}_\lambda(r_2)| + \mu_\lambda(r_2)C|x(r_2) - \tilde{x}_\lambda(r_2)| + \frac{\mu_\lambda^2(r_2)}{2}C_1^2 =$$

$$= (1 + C\mu_\lambda(r_2))|x(r_2) - \tilde{x}_\lambda(r_2)| + \frac{\mu_\lambda^2(r_2)}{2}C_1^2 \leq (1 + C\mu_\lambda(r_2))\frac{\mu_\lambda^2(r_1)}{2}C_1^2 e^{C(r_2 - \sigma(r_1))^2} + \frac{\mu_\lambda^2(r_2)}{2}C_1^2.$$

5) Next consider the line segment $[\sigma(r_2), r_{h_1}]$. We have

$$|\tilde{x}(r_{h_1}) - \tilde{x}(\sigma(r_2))| \leq C_1(r_{h_1} - r_2) = C_1\mu_1,$$

$$|x(r_{h_1}) - x(\sigma(r_2))| \leq C_1\mu_1.$$

Consequently

$$|\tilde{x}_\lambda(r_{h_1}) - x(r_{h_1})| \leq 2C_1\mu_1 + (1 + C\mu_\lambda(r_2))\frac{\mu_\lambda^2(r_1)}{2}C_1^2 e^{C(r_2 - \sigma(r_1))} + \frac{\mu_\lambda^2(r_2)}{2}C_1^2. \tag{2.17}$$

6) The interval $[r_{h_1}, r_3)$ consists entirely of limit points, hence we have

$$|\tilde{x}_\lambda(t) - x(t)| \leq e^{C(r_3 - r_{h_1})}[2C_1\mu_1 +$$

$$+(1 + C\mu_\lambda(r_2))\frac{\mu_\lambda^2(r_1)}{2}C_1^2 e^{C(r_2-\sigma(r_1))} + \frac{\mu_\lambda^2(r_2)}{2}C_1^2]. \tag{2.18}$$

7) The estimate on the interval $[r_3, \sigma(r_3))$ is similar to the one on $[r_2, \sigma(r_2))$:

$$|\tilde{x}_\lambda(t) - x(t)| \leq (2C_1\mu_1 + \frac{\mu_\lambda^2(r_1)}{2}C_1^2 e^{C(r_2-\sigma(r_1))}(1 + C\mu_\lambda(r_2)) + +\frac{\mu_\lambda^2(r_2)}{2}C_1^2)e^{K(r_3-r_{h_1})} + \mu_\lambda(r_3)C_1,$$

hence

$$|\tilde{x}_\lambda(\sigma(r_3)) - x(\sigma(r_3))| \leq |\tilde{x}_\lambda(r_3) - x(r_3)|(1 + C\mu_\lambda(r_3)) + \frac{\mu_\lambda^2(r_3)}{2}C_1^2.$$

In view of (2.18), we get

$$|\tilde{x}_\lambda(\sigma(r_3)) - x(\sigma(r_3))| \leq (2C_1\mu_1 + \frac{\mu_\lambda^2(r_1)}{2}C_1^2 e^{C(r_2-\sigma(r_1))}(1 + C\mu_\lambda(r_2)) +$$

$$+\frac{\mu_\lambda^2(r_2)}{2}C_1^2)e^{C(r_3-r_{h_1})}(1 + C\mu_\lambda(r_3)) + \frac{\mu_\lambda^2(r_3)}{2}C_1^2.$$

8) In a similar way, on the segment $[\sigma(r_3), r_4]$ we have

$$|\tilde{x}_\lambda(t) - x(t)| \leq [(2C_1\mu_1 + \frac{\mu_\lambda^2(r_1)}{2}C_1^2 e^{C(r_2-\sigma(r_1))}(1 + C\mu_\lambda(r_2)) +$$

$$+\frac{\mu_\lambda^2(r_2)}{2}C_1^2)e^{C(r_3-r_{h_1})}(1 + C\mu_\lambda(r_3)) + \frac{\mu_\lambda^2(r_3)}{2}C_1^2]e^{C(r_4-\sigma(r_3))} :\ = \delta_*.$$

9) The interval (r_4, r_{h_2}), whose length is at most μ_2, is removed from the time scale. Therefore,

$$|\tilde{x}_\lambda(r_{h_2}) - x(r_{h_2})| \leq \delta_* + 2C_1\mu_2.$$

10) On the interval $[r_{h_2}, \sigma(r_{h_2}))$ we have

$$|\tilde{x}_\lambda(t) - x(t)| \leq \delta_* + 2C_1\mu_2 + \mu_\lambda(r_{h_2})C_1. \tag{2.19}$$

Hence

$$|\tilde{x}_\lambda(\sigma(r_{h_2})) - x(\sigma(r_{h_2}))| \leq |\tilde{x}_\lambda(r_{h_2}) - x(r_{h_2})|(1 + C\mu_\lambda(r_{h_2})) + \frac{\mu_\lambda^2(r_{h_2})}{2}C_1^2 \leq$$

$$\leq [\delta_* + 2C_1\mu_2](1 + C\mu_\lambda(r_{h_2})) + \frac{\mu_\lambda^2(r_{h_2})}{2}C_1^2 :\ = \delta_{**}$$

11) Then on the segment $[\sigma(r_{h_2}), r_5]$ we have:

$$|\tilde{x}_\lambda(t) - x(t)| \leq \delta_{**}e^{C(r_5-\sigma(r_{h_2}))} \leq$$

$$\leq (1+C\mu_\lambda(r_{h_2}))(1+C\mu_\lambda(r_3))2C_1\mu_1 e^{C((r_4-\sigma(r_3))+(r_3-r_{h_1}))}+$$

$$+\frac{1}{2}(1+C\mu_\lambda(r_{h_2}))e^{C((r_4-\sigma(r_3))}\mu_\lambda^2(r_1)C_1^2 e^{C((r_2-\sigma(r_1))+(r_3-r_{h_1}))}(1+C\mu_\lambda(r_2))(1+C\mu_\lambda(r_3))+$$

$$+\frac{1}{2}(1+C\mu_\lambda(r_{h_2}))e^{C((r_4-\sigma(r_3))}\mu_\lambda^2(r_2)C_1^2 e^{C(r_3-\sigma(r_{h_1}))}(1+C\mu_\lambda(r_3))+$$

$$+\frac{1}{2}(1+C\mu_\lambda(r_{h_2}))e^{C((r_4-\sigma(r_3))}\mu_\lambda^2(r_3)C_1^2+2C_1\mu_2(1+C\mu_\lambda(r_{h_2}))+\frac{1}{2}\mu_\lambda^2(r_{h_2})C_1^2)e^{C(r_5-\sigma(r_{h_2}))}. \tag{2.20}$$

Denote

$$\Pi=(1+C\mu_\lambda(r_1))(1+C\mu_\lambda(r_2))(1+C\mu_\lambda(r_3))(1+C\mu_\lambda(r_{h_2}))\dots(1+C\mu_\lambda(r_N)).$$

where the product is taken over all right-scattered points on Figure. This way

$$\ln\Pi\leq C(\mu_\lambda(r_1)+\mu_\lambda(r_2)+\mu_\lambda(r_3)\dots)\leq C.$$

Note that the sum of all powers of exponents appearing in (2.20) does not exceed C, since these exponents are essentially the lengths of disjoint subintervals of $[0,1]$. Consequently, for $t\notin[r_k,\sigma(r_k))$ we obtain

$$|\tilde{x}_\lambda(t)-x(t)|\leq \Pi e^C 2C_1\sum_i \mu_i+\frac{1}{2}\Pi C_1^2 e^C\sum_i \mu^2(r_i)\leq$$

$$\leq \Pi e^C C_1\mu_\lambda+\frac{1}{4}\mu_\lambda\Pi C_1^2 e^C\sum_i\mu(r_i)\leq \mu_\lambda(\Pi e^C C_1+\frac{1}{4}\Pi C_1^2 e^C):=L_1\to 0,\lambda\to 0. \tag{2.21}$$

Similarly to (2.19) and (2.21) , if $t\in[r_k,\sigma(r_k))$, we have

$$|\tilde{x}_\lambda(t)-x(t)|\leq \mu_\lambda\Pi e^C(C_1+\frac{C_1^2}{4})+2C_1\sum_i\mu_i+\mu_\lambda C_1\leq \mu_\lambda(\Pi e^C(C_1+\frac{C_1^2}{4})+3C_1):=L_2\to 0,\lambda\to 0. \tag{2.22}$$

Therefore $|x(t)-\tilde{x}_\lambda(t)|\leq f(\lambda)K(T)$, where $K(T)=max_{[L_1,L_2]}$, and the proof of Lemma 2.1 follows from (2.6):

$$|x(t)-x_\lambda(t)|\leq f(\lambda)K(T), f(\lambda)\to 0,\lambda\to 0.$$

□

Under the condition (2.3), the following lemma holds.

Lemma 2.2 *The solution of system (2.2) $x_\lambda^\Delta(t)$ continuously depends on the initial data until the moment it leaves the region D.*

Proof Let x_λ and y_λ be the solutions of (2.2) with $x_\lambda(t_0)=x_0$, $y_\lambda(t_0)=y_0$, respectively. In other words

$$x_\lambda=x_0+\int_{[t_0,t]_{\mathbb{T}_\lambda}}X(s,x_\lambda(s))\Delta s \text{ and } y_\lambda=y_0+\int_{[t_0,t]_{\mathbb{T}_\lambda}}X(s,y_\lambda(s))\Delta s.$$

By the analog of Gronwall's inequality [1, p.257] for $s\in[t_0,t]_{\mathbb{T}_\lambda}$, we have

$$|x_\lambda(t)-y_\lambda(t)|\leq|x_0-y_0|e_C(t,t_0),$$

where $e_K(t,t_0)$ is an exponential function on time scale [1]. By Lemma 3.1 [12] the exponential function $e_C(t,t_0)$ is bounded on $[t_0,t]_{\mathbb{T}_\lambda}$ uniformly for all time scales, i.e:

$$|x_\lambda(s) - y_\lambda(s)| \le C_1|x_0 - y_0|$$

for any $s \in [t_0,t]_{\mathbb{T}_\lambda}$, $x, y \in \mathbb{R}^d$, and the constant $C_1 > 0$ is independent of t, x and y. This completes the proof of Lemma 2.2. □

Theorem 2.3 *Suppose the system (2.1) has a bounded on $\mathbb{R}$, asymptotically stable uniformly in $t_0 \in \mathbb{R}$ solution $x(t)$, which belongs to the ρ - neighborhood of the domain D, for some $\rho > 0$. Then there exists $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$ the system (2.2) has a bounded on $\mathbb{T}_\lambda$ solution $x_\lambda(t)$.*

Proof Since $x(t)$ is asymptotically stable, for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}$ there exist $\delta > 0$ and $T > 0$ such that for any other solution $y(t)$ of (2.1) satisfying

$$|x(t_0) - y(t_0)| \le \delta$$

we have

$$|x(t) - y(t)| < \varepsilon \quad \text{if} \quad t \ge t_0,$$

$$|x(t) - y(t)| \le \frac{\delta}{2} \quad \text{if} \quad t \ge t_0 + T,$$

where δ and T are independent of t_0. Let $x_\lambda(t)$ be a solution of (2.2) such that $x_\lambda(t_0) = y(t_0)$ at the initial point $t_0(\lambda) \in \mathbb{T}_\lambda$. By lemma 2.1

$$|y(t) - x_\lambda(t)| \le f(\lambda)K(T), \tag{2.23}$$

for some $K > 0$ and $f(\lambda) = sup_{t\in\mathbb{T}_\lambda}\mu_\lambda(t)$, $t_0(\lambda) \in \mathbb{T}_\lambda$, $t_0 + T \in \mathbb{T}_\lambda$. The stability implies that $y(t)$ as well as its neighborhood belongs to the domain D for $t > t_0$. Then, it follows from (2.23) that for sufficiently small λ, $x_\lambda(t)$ is in D for $t \in [t_0, t_0 + T]_{\mathbb{T}_\lambda}$. Consider the interval $[t_0, t_0 + T]$. In case $t_0 + T \notin \mathbb{T}_\lambda$, take the first point from the time scale $t_1 \in \mathbb{T}_\lambda$: $t_0 + T \le t_1 \le t_0 + T + 1$. The condition $\mu_\lambda(t) \to 0$, $\lambda \to 0$ makes such choice possible. Next we choose λ such that $f(\lambda)K(t_1) \le \frac{\delta}{2}$. Altogether we have

$$|y(t) - x_\lambda(t)| \le \frac{\delta}{2}, t \in [t_0, t_1]_{\mathbb{T}_\lambda}.$$

Therefore, for any solution $y(t)$ of (2.1) with $x_\lambda(t_0) = y(t_0)$ we have

$$|x(t) - x_\lambda(t)| \le |x(t) - y(t)| + |y(t) - x_\lambda(t)| < 2\varepsilon,$$

$$|x(t_1) - x_\lambda(t_1)| \le \delta$$

for $t \in [t_0, t_1]_{\mathbb{T}_\lambda}$. Now consider the interval $[t_1, t_1 + T]$, $t_1 + T \in \mathbb{T}_\lambda$. In case $t_1 + T \notin \mathbb{T}_\lambda$, as earlier, we choose the first point from the time scale $t_2 \in \mathbb{T}_\lambda$: $t_1 + T \le t_2 \le t_1 + T + 1$. Let $\tilde{y}(t)$ be a solution of (2.1) with $x_\lambda(t_1) = \tilde{y}(t_1)$. Then

$$|x(t_1) - \tilde{y}(t_1)| \le \delta,$$

$$|x(t) - \tilde{y}(t)| \le \varepsilon, t \in [t_1, t_2],$$

and we have

$$|x(t) - x_\lambda(t)| < 2\varepsilon$$

$$|x(t_2) - x_\lambda(t_2)| < \delta.$$

Continuing this process, we construct the solution $x_\lambda(t)$ of (2.2), which is in 2ε-neighborhood of the bounded solution (2.1) $x(t)$ for $t \geq t_0$. Thus $x_\lambda(t)$ is itself bounded. In other words, for any $t_0 \in \mathbb{T}_\lambda$, and for sufficiently small $\lambda \in \Lambda$ we constructed a solution $x_\lambda(t)$ of (2.2), which is bounded for $t \geq t_0$.

Let us now construct solution $x_\lambda(t)$ of (2.2), which is a bounded on the entire axis. To this end, note that all solutions of (2.2) that start in the δ-neighborhood of the solution $x(t)$ at $t_0 = t_0(\lambda) \in \mathbb{T}_\lambda$ stay in its 2ε-neighborhood. Furthermore, due to asymptotic stability, for $t \geq t_0 + t_1(\lambda)$, with $t_1 \in [T, T+1]$, $x_\lambda(t)$ is in δ-neighborhood of $x(t)$ again. Therefore, there exists $\lambda_0 \in \Lambda$ such that if $\lambda \leq \lambda_0$ for every time scale $\mathbb{T}_\lambda$ exists a sequence $t_n(\lambda) \in \mathbb{T}_\lambda$ with the following properties:

1) $t_n \to -\infty$;

2) $t_{n-1} - t_n \in [T, T+1]$;

3) if $|x_\lambda(t_n) - x(t_n)| < \delta$, then $x_\lambda(t)$ for $t \geq t_n$ does not leave the 2ε-neighborhood of $x(t)$ and at the moment t_{n-1} gets back into the δ-neighborhood of the bounded solution $x(t)$.

Let S_n be the set of values of the solutions (2.2) at the point t_1, which are within δ-neighborhood of $x(t_n)$ at t_n. Clearly, S_n is nonempty and $S_n \subset S_{n-1}$. The mapping that generates S_n is continuous by Lemma 2.2, therefore S_n is closed. Denote $z_0 := \bigcap_n S_n$, and consider the solution $x_\lambda(t)$ of (2.2) with $x_\lambda(t_1) = z_0$. Since $z_0 \in S_n$ for all n, by construction of S_n, this solution may be extended to the left oft_n for all n, and belongs to 2ε-neighborhood of the bounded solution (2.1) $x(t)$. Thus the solution of (2.2) can be extended to the left without any further conditions. Hence it is bounded on both semiaxis, which completes the proof. □

The next theorem provides the conditions for the existence of a bounded solution of (2.1) assuming (2.2) has such a solution.

Theorem 2.4 *Assume there exists $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$ the system (2.2) has a uniformly in $t_0 \in \mathbb{T}_\lambda$ asymptotically stable solution $x_\lambda(t)$, bounded on the entire time scale. Also assume that the ρ-neighborhood of this solution is in D, for some $\rho > 0$. Then the system (2.1) has a bounded on axis solution.*

Proof Using the asymptotic stability, for any $\varepsilon > 0$ there exist $\delta > 0$ and $T > 0$, which do not depend on t_0 and λ, such that

$$|x_\lambda(t_0) - y_\lambda(t_0)| \leq \delta \tag{2.24}$$

$$|x_\lambda(t) - y_\lambda(t)| < \varepsilon, \quad \text{with} \quad t \geq t_0 \tag{2.25}$$

$$|x_\lambda(t) - y_\lambda(t)| \leq \frac{\delta}{2}, \quad \text{with} \quad t \in [t_0 + T, \infty)_{\mathbb{T}_\lambda}. \tag{2.26}$$

Since the system (2.2) has a bounded solution $x_\lambda(t)$ for each $\lambda < \lambda_0$, then there exists $C_1(\lambda) > 0$such that

$$|x_\lambda(t)| \leq C_1(\lambda), \forall t \in \mathbb{T}_\lambda.$$

Let $x(t)$ be the solution of (2.1) with $x(t_0) = y_\lambda(t_0)$, where $y_\lambda(t)$ is the solution of (2.2) satisfying (2.24)-(2.26). By Lemma 2.1

$$|x(t) - y_\lambda(t)| \leq f(\lambda)K(T),\ t \in [t_0, t_0 + T]_{\mathbb{T}_\lambda}$$

where $f(\lambda) = \sup_{t\in\mathbb{T}_\lambda} \mu_\lambda(t)$, and the constant K depends only on T.

Consider the interval $[t_0, t_0 + T]$. If $t_0 + T \notin \mathbb{T}_\lambda$, we take $t_1 \in \mathbb{T}_\lambda$ such that $t_0 + T \le t_1 \le t_0 + T + 1$. As before, since $\mu_\lambda(t) \to 0,\ \lambda \to 0$, such choice is possible. Next, choose λsuch that

$$f(\lambda)K(T+1) \le \frac{\delta}{2}.$$

We have

$$|x(t) - y_\lambda(t)| \le \frac{\delta}{2}, \quad t \in [t_0, t_1]_{\mathbb{T}_\lambda}.$$

It follows from the integral representation of the solution $x(t)$ of (2.1) that

$$|x(t)| \le |x(t_0)| + \int_{t_0}^{t_1} |X(s, x(s))|ds, \quad t \in [t_0, t_1].$$

Hence

$$|x(t)| \le C_1(\lambda) + \delta + C(T+1). \tag{2.27}$$

Thus, any solution $x(t)$ of (2.1), which starts in the δ-neighborhood of $x_\lambda(t_0)$, is also at δ -neighborhood of $x_\lambda(t_1)$ at t_1, and satisfies the inequality (2.27), provided $x(t)$ is defined on $[t_0, t_1]$. Let us show that $x(t)$ is indeed defined on this interval provided $\mu_\lambda(t) \to 0$ as $\lambda \to 0$. Choose $\varepsilon > 0$ such that the ε-neighborhood of the bounded solution $x_\lambda(t)$ of (2.2) is $\frac{\rho}{2}$ distance away from the boundary of D (note that the conditions of this theorem make such choice possible.) Therefore, the solutions of (2.1), which start in such $\frac{\rho}{2}$-neighborhood, may be extended both to the left and to the right on an interval of length at least $\frac{\rho}{2C}$. Let $t_* \in [t_0, t_1]_{\mathbb{T}_\lambda}$ be such that the conditions of Lemma 2.1 hold, and fix $t_*(\lambda) \le \frac{\rho}{2C}$. This way, the solution $x(t)$ starts in the δ-neighborhood of $x_\lambda(t_0)$ and is extended onto interval $[0, \frac{\rho}{2C}]$, so that

$$|x(t_*) - y_\lambda(t_*)| \le \frac{\delta}{2},$$

where $y_\lambda(t_*)$ is the solution of the system (2.2) mentioned above. Hence, the $\frac{\rho}{2}$-neighborhood of $x(t_*)$ is in D, and the solution $x(t)$ is extended into the interval $[0, \frac{\rho}{2C}]$ and is also $\frac{\rho}{2}$ distance away from the boundary of D. Continuing this process, we get that $x(t)$ is defined on the interval $[t_0, t_1]$.

We now take a look at the interval $[t_1, t_1 + T]$ if $t_1 + T \in \mathbb{T}_\lambda$. In case $t_1 + T \notin \mathbb{T}_\lambda$, take the first point from the time scale $t_2 \in \mathbb{T}_\lambda$: $t_1 + T \le t_2 \le t_1 + T + 1$. Arguing as before, for any $t_0(\lambda) \in \mathbb{R}$ we construct the solution $x(t)$ of (2.1) which stays in the δ-neighborhood of the bounded solution $x_\lambda(t)$ of (2.2), and hence this solution is bounded for $t \ge t_0$. Let is describe the construction of a bounded solution $x(t)$ on the whole axis. To this end, all solutions of (2.1), which begin in δ-neighborhood of the $x_\lambda(t_0)$, provided the inequality (2.27) holds for $t = t_1$, are in its δ-neighborhood. In particular, a bounded solutions $x(t)$, which at $t = -t_1$ starts in the δ-neighborhood of the $x_\lambda(-t_1)$, also remains in the δ-neighborhood of $x_\lambda(t_0)$. Let S_p be a set of values of the solutions of (2.1) at $t = t_0$, which start in the δ-neighborhood of $x_\lambda(-t_p)$. This set is nonempty for any natural p, and the inclusion $S_p \subset S_{p-1}$ holds. By construction, sets S_p consist of images of solutions (2.1), which begin at points $t = -t_p(\lambda)$. Since the solutions continuously depend on the initial data, the mapping that generates S_p is continuous, and S_p is closed. Since these sets are also bounded as they are contained in the

δ-neighborhood of $x_\lambda(t_0)$, they are compact. Consequently, their intersection is nonempty. Choose $z \in \cap_{p\geq 1} S_p$, and consider the solution of differential equation $x(t)$ with $x(t_0) = z$. By construction, this solution can be extended to the left, and at $t = -t_p(\lambda)$ is in the δ-neighborhood of $x_\lambda(-t_p)$ for all$p \in \mathbb{N}$. Thus, $x(t)$ can be is extended to the left without any additional assumptions, and it also satisfies the inequality (2.27), hence is bounded. Finally, this solution may be extended to the right so that is it also bounded for $t \geq 0$. This completes the proof of the Theorem.

□

3. Example

The following example illustrates the application of Theorem 2.4.

Example 3.1 . *Consider in* $\mathbb{R}^2$ *the following system of type (2.2) when* $\mathbb{T}_\lambda = \lambda\mathbf{Z}$ *is Eulerian time scale:*

$$\begin{cases} x_\lambda^\Delta(t) = \alpha x_\lambda(t) + f_1(t, x_\lambda(t), y_\lambda(t)), \\ y_\lambda^\Delta(t) = \beta y_\lambda(t) + f_2(t, x_\lambda(t), y_\lambda(t)), \end{cases} \tag{3.1}$$

where $\lambda \geq 0$, $\alpha \in [-1; 0)$ *and* $\beta \in [-1; 0)$. *We assume that the functions* f_1 *and* f_2 *are defined on* $\mathbb{R}^3$, *these functions are in* $C^1(\mathbb{R}^3)$ *and the functions as well all of their partial derivatives are uniformly bounded by some constant* $C > 0$. *Let* $x_\lambda(k\lambda) = x_k^\lambda$. *Then the system, which corresponds to (3.1), has the form*

$$\begin{cases} x_{k+1}^\lambda = x_k^\lambda + \alpha\lambda x_k^\lambda + \lambda f_1(k\lambda, x_k^\lambda, y_k^\lambda), \\ y_{k+1}^\lambda = y_k^\lambda + \beta\lambda y_k^h + \lambda f_2(k\lambda, x_k^\lambda, y_k^\lambda), \end{cases} \tag{3.2}$$

where $\mu_\lambda(t) = \lambda \to 0$. *Then the system*

$$\begin{cases} x_{k+1}^\lambda = (1 + \alpha\lambda)x_k^\lambda, \\ y_{k+1}^\lambda = (1 + \beta\lambda)y_k^\lambda \end{cases} \tag{3.3}$$

is a linear system, which corresponds to (3.2). Its fundamental matrix has the form

$$X(n, k) = X(n - k) = \begin{pmatrix} (1 + \alpha\lambda)^{n-k} & 0 \\ 0 & (1 + \beta\lambda)^{n-k} \end{pmatrix},$$

$n \geq k$, *and the general solution is*

$$\begin{pmatrix} x_n^\lambda \\ y_n^\lambda \end{pmatrix} = X(n - t_0) \begin{pmatrix} x_{t_0}^\lambda \\ y_{t_0}^\lambda \end{pmatrix}. \tag{3.4}$$

Denote $z = \begin{pmatrix} x \\ y \end{pmatrix}$, *and* $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$. *Clearly, the fundamental matrix may be estimated as*

$$\|X(n - t)\| \leq \sqrt{2}(1 + \gamma\lambda)^{n-t}, \tag{3.5}$$

where $\gamma = \max\{\alpha; \beta\}$. *It is not difficult to see that then the system (3.2) has a bounded solution* $\eta_\lambda(t)$, *which satisfies the equation*

$$\eta_k^\lambda = \sum_{n=-\infty}^{k-1} X(k - 1 - n) f(\lambda n, \eta_n^\lambda). \tag{3.6}$$

For example, this fact can be proved using the method of successive approximations. From (3.6) we have that

$$|\eta_k^\lambda| \le \lambda \sum_{n=-\infty}^{k-1} \|X(k-1-n)\| C = -\frac{C\sqrt{2}}{\gamma} = C_0. \tag{3.7}$$

Note that C_0 *does not depend on* μ_λ. *Let us verify the asymptotical stability of the bounded solution* $\eta_\lambda(t)$. *A direct calculation shows that any solution* $z_\lambda(t)$ *of (3.2) satisfies*

$$z_k^\lambda = X(k-k_0) z_{k_0}^\lambda + \lambda \sum_{n=k_0}^{k-1} X(k-1-n) f(n\lambda, z_n^\lambda). \tag{3.8}$$

Hence,for the difference of two solutions, we have

$$|z_k^\lambda - \eta_k^\lambda| \le \sqrt{2}(1+\gamma\lambda)^{k-k_0} |z_{k_0}^\lambda - \eta_{k_0}^\lambda| + C\lambda \sum_{n=k_0}^{k-1} \sqrt{2}(1+\gamma\lambda)^{k-1-n} |z_n^\lambda - \eta_n^\lambda|$$

or

$$(1+\gamma\lambda)^{-k} |z_k^\lambda - \eta_k^\lambda| \le \sqrt{2}(1+\gamma\lambda)^{-k_0} |z_{k_0}^\lambda - \eta_{k_0}^\lambda| + \frac{C\lambda\sqrt{2}}{1+\gamma\lambda} \sum_{n=k_0}^{k-1} (1+\gamma\lambda)^{-n} |z_n^\lambda - \eta_n^\lambda|.$$

Using the analog of the Gronwall's inequality, we obtain

$$|z_k^\lambda - \eta_k^\lambda| \le \sqrt{2}(1+\gamma\lambda)^{k-k_0} \left(1 + \frac{C\lambda\sqrt{2}}{(1+\gamma\lambda)}\right)^{k-k_0} |z_{k_0}^\lambda - \eta_{k_0}^\lambda|$$

or

$$|z_k^\lambda - \eta_k^\lambda| \le \sqrt{2}(1+\gamma\lambda + C\lambda\sqrt{2})^{k-k_0} |z_{k_0}^\lambda - \eta_{k_0}^\lambda|. \tag{3.9}$$

Now let $\gamma + C\sqrt{2} < 0$ *and let choose* λ *from the condition* $0 < 1 + \gamma\lambda + C\lambda\sqrt{2}$.
Observe that η_k^λ *is exponentially stable. Therefore, the system of differential equations, corresponding to (3.1):*

$$\begin{cases} \dfrac{dx}{dt} = \alpha x + f_1(t,x,y), \\[2mm] \dfrac{dy}{dt} = \beta y + f_2(t,x,y) \end{cases}$$

has a bounded solution, defined on $\mathbb{R}$.

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