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Recent developments in δ-Casorati curvature invariants

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Abstract: The theory of δ-invariants, initiated by the author in the early 1990s, is a challenging topic in modern differential geometry, having a lot of applications. In the spirit of δ-invariants, Decu et al. (2007) initiated the study of δ-Casorati curvatures. Since then there are many interesting results on δ-Casorati curvatures obtained by many authors. In this article we provide a comprehensive survey on recent developments in δ-Casorati curvatures done during the last decade.

Key words: δ-invariant, Chen invariant, δ-Casorati curvatures, Chen ideal submanifold, Casorati ideal submanifold, optimal inequality

1. Introduction

One of the most fundamental problems in submanifold theory is the immersibility of a Riemannian manifold in a Euclidean space. In 1956, Nash proved the following well-known theorem.

Theorem 1.1 *[92] Every n-dimensional Riemannian manifold can be isometrically embedded in a Euclidean m-space with $m = \frac{n}{2}(n+1)(3n+11)$.*

Nash's embedding theorem was aimed for in the hope that if a Riemannian manifold could be regarded as isometrically embedded submanifold, this would then yield the opportunity to use help from extrinsic geometry. But this hope was not materialized according to Gromov's article [69] published in 1985.

There were several reasons why it is so difficult to apply Nash's theorem. One reason is that it requires in general very large codimension for a Riemannian manifold to admit an isometric embedding in Euclidean spaces. On the other hand, submanifolds of higher codimension are very difficult to be understood. Another reason is that at that time there do not exist general optimal relationships between the known intrinsic invariants and the main extrinsic invariants for arbitrary submanifolds of Euclidean spaces except the three fundamental equations (cf. [33]). This leads to another fundamental problem in submanifold theory (cf. [24]).

Problem 1.2 *Find simple relationship between the main extrinsic invariants and the main intrinsic invariants of a submanifold.*

In order to provide some answers to this fundamental problem, the author introduced in the early 1990's new types of Riemannian invariants [23, 26, 27], known as δ-invariants or Chen invariants, which are very

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different in nature from the "classical" Ricci and scalar curvatures. In contrast to Ricci and scalar curvatures, the main feature of δ-invariants is to remove portion of sectional curvatures from the scalar curvature [see (3.1) and (3.2)]. Via δ-invariants, the author was able to provide solutions to Problem 1.2 by establishing optimal relationships between δ-invariants and the main extrinsic invariant; namely, the squared mean curvature H^2. Applying these results, the author was also able to introduce and to study the notion of ideal immersions. The δ-invariants and the related inequalities have many applications (see, e.g., [32]). Since then these results have been extended to many geometrical inequalities of similar nature in other ambient spaces such as complex, Sasakian, cosymplectic, Kenmotsu or quaternionic space forms as well as in locally product Riemannian manifolds and statistical manifolds.

For surfaces in a Euclidean 3-space $\mathbb{E}^3$, Casorati [19] introduced in 1890 what is today called the Casorati curvature. This curvature was preferred by Casorati over Gauss curvature because Gauss curvature may vanish for surfaces that look intuitively curved, while the Casorati curvature only vanishes at planar points. Note that in computer vision Casorati curvature represents the bending energy of surfaces in $\mathbb{E}^3$ (cf. [73]). The Casorati curvature have been extended to arbitrary submanifolds in Riemannian geometry (cf. e.g., [32, 54]). In general, the Casorati curvature $\mathcal{C}$ of a submanifold in a Riemannian manifold is defined to be the normalized squared norm of the second fundamental form. In particular, for hypersurfaces of a Riemannian manifold $\widetilde{M}^{n+1}$, the Casorati curvature is given by

$$\mathcal{C} = \frac{1}{n}(\kappa_1^2 + \cdots + \kappa_n^2),$$

where $\kappa_1, \ldots, \kappa_n$ denote the principal curvatures of the hypersurfaces. For the importance of the Casorati curvature for submanifolds in the view of geometry as the science of human vision, we refer to [114].

In the spirit of δ-invariants, Decu et al. introduced the normalized Casorati curvatures $\delta_C(n-1)$ and $\hat{\delta}_C(n-1)$ in 2007 (see [54]). In 2008, they extended normalized Casorati curvatures to *generalized normalized δ-Casorati curvatures* $\delta_C(r; n-1)$ and $\hat{\delta}_C(r; n-1)$ in [55]. At the same time, they were able to establish optimal inequalities involving the (intrinsic) scalar curvature and the (extrinsic) δ-Casorati curvatures. Consequently, they were able to provide further solutions to Problem 1.2. Since then the study of δ-Casorati curvatures becomes a very active research topic in modern differential geometry. Many interesting results on δ-Casorati curvatures were obtained during the last decade.

The main purpose of this article is to present a comprehensive survey on recent developments in δ-Casorati curvatures. It is the author's intention that this survey article will provide a useful reference for graduate students and researchers working on this interesting subject.

2. Preliminaries

For the basic knowledge on Riemannian manifolds and submanifolds, we refer to [36, 94, 101].

2.1. Sectional, scalar and normalized scalar curvature

Let M^n be an n-dimensional submanifold of a Riemannian m-manifold $\widetilde{M}^m$. We choose a local field of orthonormal frame $e_1, \ldots, e_n, \xi_{n+1}, \ldots, \xi_m$ in $\widetilde{M}^m$ such that, restricted to M^n, the vectors $e_1, \ldots, e_n$ are tangent to M^n and hence $\xi_{n+1}, \ldots, \xi_m$ are normal to M^n. Let $K(e_i \wedge e_j)$ and $\widetilde{K}(e_i \wedge e_j)$ denote respectively the sectional curvatures of M^n and $\widetilde{M}^m$ of the plane section spanned by e_i and e_j.

The scalar curvature τ of M^n at p is defined by

$$\tau(p) = \sum_{1\le i<j\le n} K(e_i \wedge e_j). \tag{2.1}$$

Similarly, if L is ℓ-dimensional linear subspace of T_pM^n, then scalar curvature $\tau(L)$ of L is defined by

$$\tau(L) = \sum_{1\le i<j\le \ell} K(e_i \wedge e_j), \tag{2.2}$$

where $e_1, \ldots, e_\ell$ is an orthonormal basis of L. The normalized scalar curvature ρ of M^n is given by

$$\rho = \frac{2}{n(n-1)}\tau. \tag{2.3}$$

2.2. Basic formulas and fundamental equations

Let M^n be an n-dimensional submanifold in a Riemannian m-manifold $\widetilde{M}^m$. We denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of M^n and $\widetilde{M}^m$, respectively. The Gauss and Weingarten formulas are then given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.4}$$

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.5}$$

for vector fields X, Y tangent to M^n and ξ normal to M^n, where h denotes the second fundamental form, D the normal connection, and A the shape operator of the submanifold.

Let $\{h^r_{ij}\}$, $i,j = 1,\ldots,n$; $r = n+1,\ldots,m$, be the coefficients of the second fundamental form h with respect to $e_1,\ldots,e_n,\xi_{n+1},\ldots,\xi_m$. Then $h^r_{ij} = \langle h(e_i,e_j),\xi_r\rangle = \langle A_r e_i, e_j\rangle$, where $A_r = A_{\xi_r}$ and $\langle\ ,\ \rangle$ denotes the inner product. The mean curvature vector $\overrightarrow{H}$ is defined by

$$\overrightarrow{H} = \frac{1}{n}\operatorname{trace} h = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i).$$

The squared mean curvature is given by $H^2 = \langle \overrightarrow{H}, \overrightarrow{H}\rangle$. A submanifold M^n is said to be minimal (resp., totally geodesic) if its mean curvature vector (resp., its second fundamental form) vanishes identically.

Let R and $\widetilde{R}$ be the Riemann curvature tensors of M^n and $\widetilde{M}^m$, respectively. If $\widetilde{M}^m$ is of constant curvature c, then the three fundamental equations of Gauss, Codazzi and Ricci are given respectively by

$$\langle R(X,Y)Z,W\rangle = \left\langle A_{h(Y,Z)}X,W\right\rangle - \left\langle A_{h(X,Z)}Y,W\right\rangle + c\left(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\right), \tag{2.6}$$

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z), \tag{2.7}$$

$$\left\langle R^\perp(X,Y)\xi,\eta\right\rangle = \langle [A_\xi, A_\eta]X, Y\rangle \tag{2.8}$$

for X,Y,Z,W tangent to M^n and ξ,η normal to M^n, where $R^\perp$ denotes the normal curvature tensor associated with D and $\overline{\nabla}h$ is given by

$$(\overline{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.9}$$

2.3. Casorati curvature and principal Casorati directions

For a submanifold M^n in a real space form $R^m(c)$ of constant sectional curvature c, the Casorati curvature $\mathcal{C}$ of M^m is defined by

$$\mathcal{C} = \frac{1}{n} \sum_{r=n+1}^{m} \left(\sum_{i,j=1}^{n} (h_{ij}^r)^2 \right). \tag{2.10}$$

The Casorati curvature of an ℓ-dimensional linear subspace $L \subset T_pM^n$, spanned by $\{e_1, \ldots, e_\ell\}$, is given by

$$\mathcal{C}(L) = \frac{1}{\ell} \sum_{r=n+1}^{m} \left(\sum_{i,j=1}^{\ell} (h_{ij}^r)^2 \right). \tag{2.11}$$

By contraction, it follows from Equation (2.6) of Gauss that

$$Ric(Y, Z) = (n-1)\, c\, g(Y, Z) + ng(A_{\overrightarrow{H}}(Y), Z) - g(A^C(Y), Z), \tag{2.12}$$

where $A^C = \sum_{r=n+1}^{m} A_r^2$ is the Casorati operator. In terms of Casorati operator, (2.12) can be expressed as

$$S(X) = c(n-1)X + nA_H X - A^C(X), \tag{2.13}$$

where S is the Ricci operator defined by $\langle S(X), Y \rangle = Ric(X, Y)$.

The eigenvectors and eigenvalues of A^C are called the principal Casorati directions and principal Casorati curvatures, respectively, so that the principal Casorati curvatures $\mathcal{C}_1, \ldots, \mathcal{C}_n$ satisfy $\mathcal{C} = \mathcal{C}_1 + \cdots + \mathcal{C}_n$.

Remark 2.1 *Let M^n be a hypersurface of a Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$ with $n \geq 3$ and let $\kappa_1, \ldots, \kappa_n$ be the principal curvatures at a point $p \in M^n$ with principal directions $e_1, \ldots, e_n$. Assume L_i^{n-1} is the hyperplane of T_pM^n spanned by $e_1, \ldots, e_{i-1}$, $e_{i+1}, \ldots, e_n$. It was shown by Brubaker and Suceavă [15] that if Casorati curvature satisfies $(n-1)\sqrt{\mathcal{C}_{n-1}(L_i^{n-1})} \leq nH(p)$ with $H = \frac{1}{n}(\kappa_1 + \ldots + \kappa_n)$ for every $p \in M^n$ and for each $i \in \{1, \ldots, n\}$, then M^n is a convex hypersurface.*

3. The Chen δ-invariants

Curvature invariants are known to be the $N^o\,1$ Riemannian invariants and the most natural ones. Since Casorati's δ-curvatures were motivated after δ-invariants $\delta(n-1)$ and $\hat{\delta}(n-1)$ on Riemannian n-manifolds, we recall in this section basic definitions and fundamental results on Chen's δ-curvature invariants.

3.1. Definition of δ-invariants

Let n be a positive integer ≥ 3. For a positive integer $k \leq \frac{n}{2}$, let $\mathcal{S}(n, k)$ denote the set consisting of k-tuples $(n_1, \ldots, n_k)$ of integers ≥ 2 such that $n_1 < n$ *and* $n_1 + \cdots + n_k \leq n$. Put $\mathcal{S}(n) = \cup_{k \geq 1} \mathcal{S}(n, k)$.

For a given point p in a Riemannian n-manifold M^n and each $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, the author introduced in [23, 26, 27] the following *δ-invariants*:

$$\delta(n_1, \ldots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \tag{3.1}$$

$$\hat{\delta}(n_1, \ldots, n_k)(p) = \tau(p) - \sup\{\tau(L_1) + \cdots + \tau(L_k)\}, \tag{3.2}$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_pM^n such that $\dim L_j = n_j$, $j = 1, \ldots, k$. In particular, wee see from (3.1) and (3.2) that

(a) $\delta(\emptyset) = \tau \quad (k = 0$, the trivial δ-invariant),

(b) $\delta(2) = \tau - \inf K$ and $\hat{\delta}(2) = \tau - \sup K$, where K is the sectional curvature,

(c) $\delta(n-1)(p) = \max Ric(p)$ and $\hat{\delta}(n-1)(p) = \min Ric(p)$.

3.2. Universal inequalities involving δ-invariants

For each $(n_1, \ldots, n_k) \in \mathcal{S}(n,k)$, we put (cf. [23, 26, 27])

$$a(n_1, \ldots, n_k) = \frac{1}{2}n(n-1) - \frac{1}{2}\sum_{j=1}^{k} n_j(n_j - 1), \quad b(n_1, \ldots, n_k) = \frac{n^2(n+k-1-\sum_j n_j)}{2(n+k-\sum_j n_j)}. \tag{3.3}$$

For δ-invariants we have the following optimal universal inequalities.

Theorem 3.1 [30] *Let $f : M^n \to \widetilde{M}^m$ be an isometric immersion of a Riemannian n-manifold into a Riemannian m-manifold. Then, for each $p \in M^n$ and each k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, we have*

$$\delta(n_1, \ldots, n_k)(p) \le b(n_1, \ldots, n_k)H^2(p) + a(n_1, \ldots, n_k) \max \tilde{K}(p), \tag{3.4}$$

where $\max \widetilde{K}(p)$ denotes the maximum of the sectional curvature function of $\widetilde{M}^m$ restricted to 2-plane sections of the tangent space T_pM^n of M^n at p.

The equality case of inequality (3.4) holds at $p \in M$ if and only if the following conditions hold:

(a) There is an orthonormal basis $e_1, \ldots, e_n, \xi_{n+1}, \ldots, \xi_m$ at p such that the shape operators of M^n in $\tilde{M}^m$ at p take the following form:

$$A_{e_r} = \begin{pmatrix} A_1^r & \ldots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \ldots & A_k^r & \\ & 0 & & \mu_r I \end{pmatrix}, \quad r = n+1, \ldots, m, \tag{3.5}$$

where I is an identity matrix and A_j^r is a symmetric $n_j \times n_j$ submatrix such that

$$\text{trace}\,(A_1^r) = \cdots = \text{trace}\,(A_k^r) = \mu_r. \tag{3.6}$$

(b) For mutual orthogonal subspaces $L_1, \ldots, L_k \subset T_pM^n$ satisfying $\delta(n_1, \ldots, n_k) = \tau - \sum_{j=1}^{k} \tau(L_j)$ at p, we have $\tilde{K}(e_{\alpha_i}, e_{\alpha_j}) = \max \tilde{K}(p)$ for $\alpha_i \in \Gamma_i, \alpha_j \in \Gamma_j$, $0 \le i \ne j \le k$, where

$$\Gamma_0 = \{1, \ldots, n_1\}, \ \ldots, \ \Gamma_{k-1} = \{n_1 + \cdots + n_{k-1} + 1, \ldots, n_1 + \cdots + n_k\},$$
$$\Gamma_k = \{n_1 + \cdots + n_k + 1, \ldots, n\}.$$

An important case of Theorem 3.1 is the following.

Theorem 3.2 [26, 27] *For an isometric immersion $f : M^n \to R^m(c)$ from a Riemannian n-manifold M^n into a real space form $R^m(c)$ of constant curvature c and for an k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, we have*

$$\delta(n_1, \ldots, n_k) \le b(n_1, \ldots, n_k)H^2 + a(n_1, \ldots, n_k)c, \tag{3.7}$$

where H^2 is the squared mean curvature of M^n in $R^m(c)$.

The equality case of inequality (3.7) holds at a point $p \in M^n$ if and only if there is an orthonormal basis $e_1, \ldots, e_n, \xi_{n+1}, \ldots, \xi_m$ at p such that the shape operators at p take the forms (3.5) and (3.6).

Definition 3.3 *A submanifold M^n of a real space form $R^m(c)$ is called $(n_1, \ldots, n_k)$-*ideal *if it satisfies the equality sign of (3.7) identically.*

Roughly speaking, an ideal submanifold in $R^m(c)$ is a submanifold which receives the least amount of tension from its ambient space at each point (cf. [26, 32, 33]). Ideal submanifolds associated with Chen's δ-invariants are also known in some literatures as Chen ideal submanifolds.

Remark 3.4 *Inequalities analogous to (3.7) for submanifolds in various space forms (in particular, for the special case with $k = 1, n_1 = 2$) have been studied by many authors (cf. [32]).*

The δ-invariants and their associated inequalities have many applications to several areas in mathematics (cf. e.g., [26, 27, 32, 33]). For instance, it have been applied in [35] to show that if $\pi : M \to N$ is a covering map between two compact irreducible homogeneous spaces and if the first eigenvalues of the Laplacian of M and N satisfy $\lambda_1(M) \ne \lambda_1(N)$, then N doesn't admit an ideal embedding into any Euclidean space regardless of codimension, although M may could.

3.3. Algebraic Chen δ-invariants and inequalities

A $(0, 4)$-tensor field T on a Riemannian manifold is said to be *curvature-like* if it has all the formal properties of the Riemannian curvature tensor so that it satisfies the following properties:

$$T(X, Y, Z, W) = -T(Y, X, Z, W), \tag{3.8}$$

$$T(X, Y, Z, W) = T(Z, W, X, Y), \tag{3.9}$$

$$T(X, Y, Z, W) + T(Y, Z, X, W) + T(Z, X, Y, W) = 0. \tag{3.10}$$

In [41], Chen et al. introduced the notion of δ-invariant for curvature-like tensor fields and they established optimal general inequalities in case the curvature-like tensor field satisfies algebraic Gauss equation as follows. Let (M^n, g) be a Riemannian n-manifold and let T be a curvature-like $(0, 4)$-tensor field on M^n. Then one may define the T*-sectional curvature* $K_T(\pi)$ associated with a 2-plane $\pi \subset T_pM^n$, $p \in M^n$ as usual. For an ℓ-dimensional linear subspace $L \subset T_pM^n$ with $\ell \ge 2$ and with an orthonormal basis $\{e_1, \ldots, e_\ell\}$ of L, the T-scalar curvature $\tau_T(L)$ of L is defined by $\tau_T(L) = \sum_{i<j} K_T(e_i \wedge e_j)$, $1 \le i, j \le \ell$. In particular, we have $\tau_T(p) = \tau_T(T_pM^n)$.

For any $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, we define the algebraic δ_T-invariant $\delta_T(n_1, \ldots, n_k)$ as we did in (3.1) by

$$\delta_T(n_1, \ldots, n_k)(p) = \tau_T(p) - \inf\{\tau_T(L_1) + \cdots + \tau_T(L_k)\}, \tag{3.11}$$

where $L_1, \dots, L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1, \dots, k$.

Let (M^n, g) be a Riemannian n-manifold and (B, g) a Riemannian vector bundle over M^n. Let σ be a B-valued symmetric $(1,2)$-tensor field. If T is a $(0,4)$-tensor field on M^n such that

$$T(X,Y,Z,W) = g(\sigma(Y,Z), \sigma(X,W)) - g(\sigma(X,Z), \sigma(Y,W)) \tag{3.12}$$

for vector vector fields X, Y, Z, W tangent to M^n, then T is curvature-like. Equation (3.12) is called an algebraic Gauss equation.

Typical examples of settings in which (3.12) occurs is for a submanifold of Euclidean space, B being the normal bundle, σ the second fundamental form, T the curvature tensor. For such submanifolds in [23, 27, 30] optimal inequalities involving the δ-invariant have been established. The proofs can be immediately generalized to prove the following general inequality.

Theorem 3.5 *[41] Let (M^n, g) be a Riemannian n-manifold and let T be a curvature-like $(0,4)$-tensor field on M^n. If (B, g) is a Riemannian vector bundle over M^n and σ is a B-valued symmetric $(1,2)$-tensor field which satisfy the algebraic Gauss equation, then for each k-tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$ we have*

$$\delta(n_1, \dots, n_k) \le \frac{n^2 \left(n + k - 1 - \sum_{j=1}^k n_j\right)}{2\left(n + k - \sum_{j=1}^k n_j\right)} g(\operatorname{trace}\sigma, \operatorname{trace}\sigma), \quad \operatorname{trace}\sigma = \sum_{i=1}^n \sigma(e_i, e_i). \tag{3.13}$$

The equality case of inequality (3.13) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \dots, e_n\}$ at p such that with respect this basis every linear map σ_ξ, $\xi \in B_p$ of the tangent space T_pM^n, defined by $g(\sigma_\xi X, Y) = g(\sigma(X,Y), \xi)$ for all $X, Y \in T_pM^n$ takes the following form:

$$\sigma_\xi = \begin{pmatrix} A_1^\xi & & & 0 \\ & \ddots & & \\ & & A_k^\xi & \\ 0 & & & \eta_\xi I \end{pmatrix},$$

where $\{A_j^\xi\}_{j=1}^k$ are symmetric $n_j \times n_j$ submatrices satisfying $\operatorname{trace}(A_1^\xi) = \cdots = \operatorname{trace}(A_k^\xi) = \lambda_\xi$ for some λ_ξ.

Remark 3.6 *Let M^n be a convex hypersurface in $\mathbb{E}^{n+1}$. For natural numbers n_1, n_2 with $n = n_1 + n_2$. Suceavă and Vajiac [108] proved that the mean curvature H, the Casorati curvature $\mathcal{C}$, and Chen's $\hat{\delta}(n_1, n_2)$-invariant satisfy the inequality $H \ge \frac{4\,\hat{\delta}(n_1,n_2)}{n^2\sqrt{\mathcal{C}}}$, with the equality holding if and only if p is an umbilical point. For further results in this respect, see [108].*

4. δ-invariants for Lagrangian submanifolds in complex space forms

The next result follows immediately from the fact that the Gauss equation of a Lagrangian submanifold in a complex space form has the same expression as the one for a submanifold in a real space form (see [42]).

Theorem 4.1 *Let M^n be a Lagrangian submanifold in a complex space form $\widetilde{M}^n(4c)$ with constant holomorphic sectional curvature $4c$. Then inequality (3.6) holds for each k-tuple $(n_1, \dots, n_k)$ which satisfies $2 \le n_1, \dots, n_k < n$, $\sum_{i=1}^k n_i < n$.*

The next theorem extends a result on $\delta(2)$ from [42].

Theorem 4.2 *[28] A Lagrangian submanifold of a complex space form $\widetilde{M}^n(4c)$ is minimal at a point p if it satisfies the equality case of inequality (3.7) at p.*

The reason behind the above theorem is that (3.7) is not an optimal equality in the Lagrangian setting.

4.1. Optimal inequalities for $\delta(n_1,\dots,n_k)$ with $\sum_{i=1}^k n_i < n$

The following optimal inequalities for Lagrangian submanifold in the case $\sum_{i=1}^k n_i < n$ was proved by Chen et al. in [38, 39].

Theorem 4.3 *Let M^n be a Lagrangian submanifold of $\widetilde{M}^n(4c)$. Then, for any k-tuple $(n_1,\dots,n_k)$ satisfying $\sum_{i=1}^k n_i < n$, we have*

$$\delta(n_1,\dots,n_k) \le \frac{n^2\Big\{\Big(n-\sum_{i=1}^k n_i+3k-1\Big)-6\sum_{i=1}^k(2+n_i)^{-1}\Big\}}{2\Big\{\Big(n-\sum_{i=1}^k n_i+3k+2\Big)-6\sum_{i=1}^k(2+n_i)^{-1}\Big\}}H^2+\frac{1}{2}\Big\{n(n-1)-\sum_{i=1}^k n_i(n_i-1)\Big\}c. \tag{4.1}$$

The equality sign holds at a point $p\in M^n$ if and only if there is an orthonormal basis $\{e_1,\dots,e_n\}$ at p such that the second fundamental form h satisfies

$$\begin{aligned}
&h(e_{\alpha_i},e_{\beta_i})=\sum_{\gamma_i}h^{\gamma_i}_{\alpha_i\beta_i}Je_{\gamma_i}+\frac{3\delta_{\alpha_i\beta_i}}{2+n_i}\lambda Je_{N+1},\ \ \sum_{\alpha_i=1}^{n_i}h^{\gamma_i}_{\alpha_i\alpha_i}=0,\\
&h(e_{\alpha_i},e_{\alpha_j})=0,\ i\ne j;\ h(e_{\alpha_i},e_{N+1})=\frac{3\lambda}{2+n_i}Je_{\alpha_i},\ h(e_{\alpha_i},e_u)=0,\\
&h(e_{N+1},e_{N+1})=3\lambda Je_{N+1},\ h(e_{N+1},e_u)=\lambda Je_u,\ N=n_1+\cdots+n_k,\\
&h(e_u,e_v)=\lambda\delta_{uv}Je_{N+1},\ i,j=1,\dots,k;\ u,v=N+2,\dots,n.
\end{aligned} \tag{4.2}$$

Note that inequalities (3.7) and (4.1) coincide for minimal immersions.

4.2. Optimal inequalities for $\delta(n_1,\dots,n_k)$ with $\sum_{i=1}^k n_i = n$

For the case $\sum_{i=1}^k n_i = n$, we also have the following theorem from [39].

Theorem 4.4 *Let M^n be a Lagrangian submanifold of a complex space form $\widetilde{M}^n(4c)$. Then for each $(n_1,\dots,n_k)\in\mathcal{S}(n)$ with $\sum_{i=1}^k n_i = n$ we have*

$$\delta(n_1,\dots,n_k)\le\frac{n^2\{k-1-2\sum_{i=2}^k(2+n_i)^{-1}\}}{2\{k-2\sum_{i=2}^k(2+n_i)^{-1}\}}H^2+\frac{1}{2}\Big\{n(n-1)-\sum_{i=1}^k n_i(n_i-1)\Big\}c, \tag{4.3}$$

where we assume that $n_1=\min_{i=1}^n\{n_i\}$. If the equality sign of (4.3) holds at $p\in M^n$, then the components of the second fundamental form with respect to some suitable orthonormal basis $\{e_1,\dots,e_n\}$ for T_pM^n satisfy the following conditions:

(a) $h^A_{\alpha_i\alpha_j} = 0$ *for* $i \neq j$ *and* $A \neq \alpha_i, \alpha_j$*;*

(b) if $n_j \neq \min\{n_1, \dots, n_k\}$*,* $h^{\beta_j}_{\alpha_i\alpha_i} = 0$ *if* $i \neq j$ *and* $\sum_{\alpha_j \in \Delta_j} h^{\beta_j}_{\alpha_j\alpha_j} = 0$*,*

(c) if $n_j = \min\{n_1, \dots, n_k\}$*,* $\sum_{\alpha_j \in \Delta_j} h^{\beta_j}_{\alpha_j\alpha_j} = (n_i + 2)h^{\beta_j}_{\alpha_i\alpha_i}$ *for any* $i \neq j$*, and any* $\alpha_i \in \Delta_i$*.*

The next result from [39] shows that inequality (4.3) is sharp.

Theorem 4.5 *For each* $(n_1 \dots, n_k) \in \mathcal{S}(n)$ *with* $\sum_{i=1}^k n_i = n$*, there exists a Lagrangian submanifold in* $\widetilde{M}^n(4c)$ *which satisfies the equality of the improved inequality* (4.3) *identically.*

Remark 4.6 *Theorem 4.5 implies that inequality* (4.3) *cannot be improved further.*

Remark 4.7 *Lagrangian submanifolds of complex space forms which satisfy some special cases of inequalities* (4.1) *and* (4.3) *have been classified in [14, 32, 38–40, 43, 45, 46].*

5. The First two articles on δ-Casorati curvatures

5.1. δ-Casorati curvatures $\delta_{\mathcal{C}}(n-1)$ and $\hat{\delta}_{\mathcal{C}}(n-1)$

In the spirit of δ-invariants, Decu et al. introduced in [54] the notion of normalized Casorati δ-curvatures $\delta_{\mathcal{C}}(n-1)$ and $\hat{\delta}_{\mathcal{C}}(n-1)$ as follows:

$$\big[\delta_{\mathcal{C}}(n-1)\big]_p = \frac{1}{2}\mathcal{C}_p + \frac{(n+1)}{2n(n-1)}\inf\{\mathcal{C}(L) \mid L \text{ a hyperplane of } T_pM\}, \tag{5.1}$$

$$\big[\hat{\delta}_{\mathcal{C}}(n-1)\big]_p = 2\mathcal{C}_p - \frac{(2n-1)}{2n}\sup\{\mathcal{C}(L) \mid L \text{ a hyperplane of } T_pM\}. \tag{5.2}$$

Note that in contrast to author's δ-invariants which are intrinsic, the δ-Casorati curvatures are extrinsic.

Decu et al. proved the following result for normalized Casorati δ-curvatures.

Theorem 5.1 *[54] Let* M^n *be a Riemannian submanifold of a real space form* $R^m(c)$*. Then we have:*

$$\rho \leq \delta_{\mathcal{C}}(n-1) + c, \quad \big(\textit{resp.}, \ \rho \leq \hat{\delta}_{\mathcal{C}}(n-1) + c\big), \tag{5.3}$$

where ρ *is the normalized scalar curvature defined by* (2.3)*.*

In addition, the equality sign of (5.3) *holds identically if and only if there exists an orthonormal frame* $e_1, \dots, e_n, \xi_{n+1}, \dots, \xi_m$ *such that with respect to this frame the shape operator satisfies*

$$A_{n+1} = \begin{pmatrix} \lambda I_{n-1} & 0 \\ 0 & 2\lambda \end{pmatrix} \ \left(\textit{resp.}, \ A_{n+1} = \begin{pmatrix} \lambda I_{n-1} & 0 \\ 0 & \frac{1}{2}\lambda \end{pmatrix}\right) \ \textit{and} \ A_{n+2} = \ldots = A_m = 0, \tag{5.4}$$

for some function λ*, where* I_{n-1} *denotes the* $(n-1) \times (n-1)$*-identity submatrix.*

Remark 5.2 *An alternate proof of Theorem 5.1 for* $\hat{\delta}_{\mathcal{C}}(n-1)$ *was given by Zhang and Zhang [121].*

5.2. Generalized δ-Casorati curvatures $\delta_{\mathcal{C}}(r, n-1)$ and $\hat{\delta}_{\mathcal{C}}(r, n-1)$

For any positive real number $r \neq n(n-1)$, put

$$a(r) = \frac{(n-1)(r+n)(n^2-n-r)}{nr}. \tag{5.5}$$

Let M^n be an n-dimensional submanifold of a real space form $R^m(c)$. The generalized δ-Casorati curvatures $\delta_{\mathcal{C}}(r; n-1)$ and $\hat{\delta}_{\mathcal{C}}(r; n-1)$ were defined by Decu et al. in [55] as

$$\delta_{\mathcal{C}}(r; n-1)\,|_p = \; r\,\mathcal{C}\,|_p + a(r)\cdot \inf\{\mathcal{C}(L) \mid L \text{ a hyperplane of } T_pM\} \quad if \;\; 0 < r < n(n-1), \tag{5.6}$$

$$\hat{\delta}_{\mathcal{C}}(r; n-1)\,|_p = \; r\,\mathcal{C}\,|_p + a(r)\cdot \sup\{\mathcal{C}(L) \mid L \text{ a hyperplane of } T_pM\} \quad if \; r > n(n-1). \tag{5.7}$$

Decu et al. proved the following theorem for generalized δ-Casorati curvatures.

Theorem 5.3 *[55] Let M^n be n-dimensional submanifold of a real space form $R^m(c)$. For a real number r such that $0 < r < n(n-1)$ we have*

$$2\tau \leq \delta_{\mathcal{C}}(r; n-1) + n(n-1)c; \tag{5.8}$$

and for a real number $r > n(n-1)$ we have

$$2\tau \leq \hat{\delta}_{\mathcal{C}}(r; n-1) + n(n-1)c. \tag{5.9}$$

Equality holds in the inequalities (5.8) and (5.9) if and only if

$$h^r_{ij} = 0 \quad for \; i \neq j \in \{1, \ldots, n\}, \;\; and \tag{5.10}$$

$$h^r_{11} = \ldots = h^r_{n-1,n-1} = \frac{r}{n(n-1)} h^r_{nn} \quad for \;\; r \in \{n+1, \ldots, m\} \tag{5.11}$$

holds, respectively.

Condition (5.10) means that the *normal connection* $\nabla^\perp$ is flat. And, condition (5.11) means that there exist m mutually orthogonal unit normal vector fields $\xi_1, \ldots, \xi_m$ such that the shape operators with respect to all directions ξ_α have an eigenvalue of multiplicity $n-1$ and that for each ξ_α the distinguished eigendirection is the same (namely e_n), that is, that the submanifold is *invariantly quasi-umbilical*. Thus, we have the following.

Corollary 5.4 *[55] Let M^n be a Riemannian submanifold of a real space form $R^m(c)$. Equality holds in (5.8) and (5.9) if and only if M^n is invariantly quasi-umbilical with trivial normal connection in $\widetilde{M}^m(c)$ and with respect to suitable tangent and normal orthonormal frames, the shape operators are given by*

$$A_{n+1} = \begin{pmatrix} \lambda I_{n-1} & 0 \\ 0 & \frac{n(n-1)}{r}\lambda \end{pmatrix}, \quad A_{n+2} = \ldots = A_m = 0. \tag{5.12}$$

Remark 5.5 *The techniques used for proving Theorem 5.1 and Theorem 5.3 were based on the Oprea's optimization procedure given in [93] by showing that a quadratic polynomial in the components of the second fundamental form is parabolic, different from the proofs of Chen type inequalities between δ-invariants and the squared mean curvature of submanifolds.*

Remark 5.6 *These two papers [54, 55] by Decu et al. were the starting point of the investigations of the δ-Casorati curvatures.*

5.3. Casorati ideal submanifolds

Analogous to Chen ideal submanifolds, a submanifold which realizes the equality case of an inequality involving a δ-Casorati curvature is called a Casorati ideal submanifold.

Now, I would like to present the following new result which provides a simple link between Casorati and Chen ideal submanifolds.

Proposition 5.7 *Let M^n be a Casorati $\delta_C(r;n-1)$-ideal submanifold of a real space form $R^m(c)$ with $n \geq 3$. Then M^n is a Chen $\delta(k)$-ideal submanifold with $\frac{n}{2} < k \leq n-1$ if and only if either $r = \frac{n(n-1)}{2k+1-n}$ or M^n is totally geodesic. In particular, a Casorati $\delta_C(r;n-1)$-ideal submanifold M^n in $R^m(c)$ is Chen $\delta(n-1)$-ideal if and only if either $r = n$ or M^n is totally geodesic.*

Proof Since $\delta_C(r;n-1)$ is defined only for r satisfying $0 < r < n(n-1)$, we have $\frac{n(n-1)}{r} > 1$. Now, let us assume that M^n is a Casorati $\delta_C(r;n-1)$-ideal submanifold of a real space form $R^m(c)$ with $n \geq 3$. Then, according to Corollary 5.4, there exists suitable tangent and normal orthonormal frames such that with respect these frames the shape operator of M^n satisfy

$$A_{n+1} = \begin{pmatrix} \lambda I_{n-1} & 0 \\ 0 & \frac{n(n-1)}{r}\lambda \end{pmatrix}, \quad A_{n+2} = \ldots = A_m = 0. \tag{5.13}$$

If $\lambda = 0$, then (5.13) implies that M^n is totally geodesic in $R^m(c)$.

Now, we assume $\lambda \neq 0$. If M^n is a Chen $\delta(k)$-ideal submanifold with $n-1 \geq k > \frac{n}{2}$. Then it follows from Theorem 3.1 and (5.13) that the shape operator A_{n+1} must satisfies $k\lambda = \frac{n(n-1)}{r}\lambda + (n-k-1)\lambda$. Because λ is assumed to be nonzero, we must have $r = \frac{n(n-1)}{2k+1-n}$.

Conversely, if $r = \frac{n(n-1)}{2k+1-n}$ holds, then it follows from (5.13) and Theorem 3.1 that M^n is $\delta(k)$-ideal. □

Remark 5.8 *Proposition 5.7 holds true for Casorati $\delta_C(r;n-1)$-ideal submanifolds in many other space forms.*

The next result on Casorati ideal submanifolds follows from (5.4) and [36, Proposition 1.1, p. 88].

Corollary 5.9 *[54] The Casorati ideal submanifolds of dimension ≥ 4 for (5.3) are conformally flat submanifolds with flat normal connection.*

An *obstruction* for a manifold to be conformally flat in terms of Chen's δ-curvatures was given in [31].

A rotation hypersurface of a real space form $\widetilde{M}^{n+1}(c)$ of constant curvature c is generated by moving an $(n-1)$-dimensional totally umbilical submanifold along a curve in $\widetilde{M}^{n+1}(c)$ [61]. If M^n is a Casorati ideal hypersurface in $\widetilde{M}^{n+1}(c)$, then it follows from [60, 61] that M^n is a rotation hypersurface whose profile curve is the graph of a function f of one real variable which satisfies the differential equation

$$f(f'' + \tilde{c} f) + \frac{n(n-1)}{r}(\varepsilon - \tilde{c} f^2 - f'^2) = 0. \tag{5.14}$$

where $\varepsilon = 1$ if $c \geq 0$; and if $c < 0$, then $\varepsilon = 0$, 1 or -1 depends on the rotation hypersurface M^n is parabolical, spherical or hyperbolical, respectively.

Corollary 5.10 *[55] The Casorati ideal hypersurfaces of real space forms are rotation hypersurfaces whose profile curves are given by the solutions of (5.14).*

6. Modified δ-Casorati curvature $\delta_{\mathcal{C}}(n-1)$

Lee and Vîlcu [82] pointed out that it is better to replace the coefficient "$\frac{n+1}{2n(n-1)}$" in (5.1) by "$\frac{n+1}{2n}$", since the normalized δ-Casorati curvature $\delta_{\mathcal{C}}(n-1)$ should able be recovered from the generalized normalized δ-Casorati curvature $\delta_{\mathcal{C}}(r;n-1)$ with $r=\frac{1}{2}$. For this reason, the modified δ-Casorati curvature $\tilde{\delta}_{\mathcal{C}}(n-1)$ was defined in [82] as

$$\tilde{\delta}_{\mathcal{C}}(n-1)\mid_p=\frac{1}{2}\mathcal{C}\mid_p+\frac{n+1}{2n}\inf\{\mathcal{C}(L)\mid L \text{ a hyperplane of } T_pM^n\}, \tag{6.1}$$

It is direct to verify the following two relations:

$$\delta_C\left(\frac{n(n-1)}{2};n-1\right)=n(n-1)\tilde{\delta}_C(n-1),\quad \hat{\delta}_C\left(2n(n-1);n-1\right)=n(n-1)\hat{\delta}_C(n-1). \tag{6.2}$$

6.1. Inequality involving the modified δ-Casorati curvature

Zhang and Zhang proved in [121] that Theorem 5.1 remains true if the Casorati δ-curvature $\delta_{\mathcal{C}}(n-1)$ were replaced by the modified Casorati δ-curvature $\tilde{\delta}_{\mathcal{C}}(n-1)$. More precisely, Zhang and Zhang proved the next result for the modified Casorati δ-curvature $\tilde{\delta}_{\mathcal{C}}(n-1)$.

Theorem 6.1 *[121] Let M^n be a Riemannian submanifold of a real space form $R^m(c)$. Then*

$$\rho\le\tilde{\delta}_{\mathcal{C}}(n-1)+c. \tag{6.3}$$

The equality sign holds identically if and only if there is an orthonormal frame $e_1,\dots,e_n,\ \xi_{n+1},\dots,\xi_m$ such that with respect to this frame the shape operator satisfies $A_{n+1}=\begin{pmatrix}\lambda I_{n-1} & 0\\ 0 & 2\lambda\end{pmatrix}$ and $A_{n+2}=\ldots=A_m=0$.

6.2. Algebraic δ-Casorati curvatures

Let (M^n,g) be a Riemannian n-manifold, (B,g_B) a Riemannian vector bundle over M^n, σ a B-valued symmetric $(1,2)$-tensor field on M^n, and T a curvature-like tensor field which satisfies the algebraic Gauss equation. Tripathi [111] extended δ-Casorati curvatures $\delta_{\mathcal{C}}(n-1)$, $\hat{\delta}_{\mathcal{C}}(n-1)$, $\delta_{\mathcal{C}}(r;n-1)$ and $\hat{\delta}_{\mathcal{C}}(r;n-1)$ to algebraic δ-Casorati curvatures $\delta_{\mathcal{C}^{T,\sigma}}(n-1)$, $\hat{\delta}_{\mathcal{C}^{T,\sigma}}(n-1)$, $\delta_{\mathcal{C}^{T,\sigma}}(r;n-1)$ and $\hat{\delta}_{\mathcal{C}^{T,\sigma}}(r;n-1)$ as follows:

$$[\delta_{\mathcal{C}^{T,\sigma}}(n-1)]_p=\frac{1}{2}\,\mathcal{C}_p^{T,\sigma}+\frac{n+1}{2n}\inf\left\{\mathcal{C}^{T,\sigma}(L)\mid L \text{ is a hyperplane of } T_pM^n\right\};$$

$$[\hat{\delta}_{\mathcal{C}^{T,\sigma}}(n-1)]_p=2\,\mathcal{C}_p^{T,\sigma}-\frac{2n-1}{2n}\sup\{\mathcal{C}^{T,\sigma}(L)\mid L \text{ is a hyperplane of } T_pM^n\};$$

$$[\delta_{\mathcal{C}^{T,\sigma}}(r;n-1)]_p=r\,\mathcal{C}_p^{T,\sigma}+\mathrm{a}(r)\inf\left\{\mathcal{C}^{T,\sigma}(L)\mid L \text{ is a hyperplane of } T_pM^n\right\}\quad \textit{if } 0<r<n(n-1);$$

$$[\hat{\delta}_{\mathcal{C}^{T,\sigma}}(r;n-1)]_p=r\,\mathcal{C}_p^{T,\sigma}+\mathrm{a}(r)\sup\left\{\mathcal{C}^{T,\sigma}(L)\mid L \text{ is a hyperplane of } T_pM^n\right\}\quad \textit{if } n(n-1)<r,$$

where $a(r)$ is given by (5.5), $\mathcal{C}^{T,\sigma}(p)=\mathcal{C}^{T,\sigma}(T_pM^n)$ and $\mathcal{C}^{T,\sigma}(L_p)=\frac{1}{\ell}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{\ell}\left(\sigma_{ij}^{\alpha}\right)^2$, where L_p is an ℓ-dimensional subspace of T_pM^n, with $\ell\geq 2$, spanned by an orthonormal basis $\{e_1,\dots,e_\ell\}$ of L_p.

In [111], Tripathi also established the corresponding inequalities for algebraic δ-Casorati curvatures.

6.3. Casorati ideal submanifolds

Now, we present some results on Casorati ideal submanifolds.

Proposition 6.2 *[121] The Casorati ideal submanifolds for (5.3) and (6.3) are Einstein if and only if they are totally geodesic submanifolds.*

An isometric immersion of a Riemannian manifold M^n into a Euclidean m-space $\mathbb{E}^m$ is called *rigid* if the isometric immersion of M^n is unique up to isometries of $\mathbb{E}^m$.

Similar to [34, Theorem 4.2], Zhang and Zhang proved the following.

Theorem 6.3 *[121] The Casorati ideal hypersurface M^3 for (6.3) in $\mathbb{E}^4$ is rigid.*

Casorati ideal hypersurfaces M^3 for (6.3) in $\mathbb{E}^4$ were also classified in [121].

Theorem 6.4 *The Casorati ideal hypersurface M^3 for (6.3) in $\mathbb{E}^4$ is congruent to*

$$\left(\frac{1}{a}\mathrm{sd}\left(at,\frac{1}{\sqrt{2}}\right)\sin u,\ \frac{1}{a}\mathrm{sd}\left(at,\frac{1}{\sqrt{2}}\right)\cos u\sin v,\ \frac{1}{a}\mathrm{sd}\left(at,\frac{1}{\sqrt{2}}\right)\cos u\cos v,\ \frac{1}{2}\int_0^t\mathrm{sd}^2\left(at,\frac{1}{\sqrt{2}}\right)dt\right) \tag{6.4}$$

for some positive real number a, where sd *is a Jacobi's elliptic function.*

Remark 6.5 *It was proved by the author in [34] that the hypersurface defined by (6.4) is one of the 3 types of $\delta(2)$ Chen ideal hypersurfaces in $\mathbb{E}^4$.*

7. Submanifolds in a Riemannian manifold of quasi-constant curvature

In this section we present results on δ-Casorati curvatures for submanifolds in spaces of quasi-constant curvature.

7.1. Riemannian manifolds of quasi-constant curvature

A Riemannian manifold (M^n,g) is said to be of quasi-constant curvature [47] if there exist a unit vector field ξ, called the generator, and two smooth functions κ,μ on M^n such that

$$R(X,Y)Z=\kappa\{g(Y,Z)X-g(X,Z)Y\}+\mu\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\xi+\mu\eta(Z)\{\eta(Y)X-\eta(X)Y\},$$

where η is the 1-form dual to ξ. We denote such a Riemannian manifold of quasi-constant curvature simply by $M^n_{\kappa,\mu}(\xi)$ as did in [66, p. 325].

For a Riemannian m-manifold $M^m_{\kappa,\mu}(\xi)$ of quasi-constant curvature, we have

$$R(X,Y)\xi=(\kappa+\mu)\{\eta(Y)X-\eta(X)Y\},\quad R(X,\xi)Z=(\kappa+\mu)\{\eta(Z)X-g(X,Z)\xi\},$$

while the Ricci curvature Ric satisfies

$$Ric(X,Y)=\{\kappa(n-1)+\mu\}g(X,Y)+\mu(n-2)\eta(X)\eta(Y).$$

Thus, $M^n_{\kappa,\mu}(\xi)$ is an η-Einstein manifold. In case κ,μ are constants, $M^n_{\kappa,\mu}(\xi)$ is a quasi-Einstein manifold (cf. [68]).

7.2. δ-Casorati curvatures in spaces of quasi-constant curvature

In [95], Pan et al. extended Theorem 5.3 to submanifolds in a Riemannian manifold of quasi-constant curvature as follows.

Theorem 7.1 *Let M^n be n-dimensional submanifold of a Riemannian manifold $M^m_{\kappa,\mu}(\xi)$ of quasi-constant curvature with generator ξ. Then for any real number $r \in (0, n(n-1))$, we have*

$$2\tau \leq \delta_{\mathcal{C}}(r; n-1) + n(n-1)\kappa + 2(n-1)\mu||\xi^T||^2; \tag{7.1}$$

and, for any real number $r > n(n-1)$, we have

$$2\tau \leq \hat{\delta}_{\mathcal{C}}(r; n-1) + n(n-1)\kappa + 2(n-1)\mu||\xi^T||^2, \tag{7.2}$$

where $||\xi^T||^2$ denotes the squared norm of the tangential component ξ^T of the generator ξ. The equality sign of (7.1) or (7.2) holds identically if and only if there exists an orthonormal frame $e_1, \ldots, e_n, \xi_{n+1}, \ldots, \xi_m$ such that with respect to this frame the shape operator satisfies (5.12).

8. δ-Casorati curvatures of Lagrangian submanifolds in complex space forms

An almost Hermitian manifold $(\widetilde{M}, \tilde{g}, J)$ is an almost complex manifold (M, J) endowed with a Riemannian metric $\tilde{g}$ which is compatible with the almost complex structure J. A submanifold M of $(\widetilde{M}, \tilde{g}, J)$ is called a complex submanifold (resp., totally real submanifold) if it satisfies $J(T_pM) \subseteq T_pM$ (resp., $J(T_pM) \subseteq T_p^{\perp}M$) for any $p \in M$. A totally real submanifold M in $\widetilde{M}$ is called Lagrangian if $\dim_{\mathbf{R}} M = \dim_{\mathbf{C}} \widetilde{M}$ (cf. [29]).

8.1. δ-Casorati curvatures for Lagrangian submanifolds

For Lagrangian submanifolds in complex space forms, Zhang et al. proved the following.

Theorem 8.1 *[119] Let M^n be a Lagrangian submanifold of a complex space form $\widetilde{M}^n(4c)$ with constant holomorphic sectional curvature $4c$ and of complex dimension n. We have*

$$\hat{\delta}_C(n-1) \geq \rho - c + \frac{2n(2n-3)}{(n-1)(2n+3)}H^2, \tag{8.1}$$

where ρ is the normalized scalar curvature and H^2 is the squared mean curvature of the submanifold. Moreover, the equality of (8.1) holds if and only if M^n is a Lagrangian totally geodesic submanifold.

This theorem implies the following.

Corollary 8.2 *[119] Let M^n be a Riemannian n-manifold. If there exists a point $p \in M^n$ such that*

$$\rho(p) > [\hat{\delta}_C(n-1)]_p,$$

then M^n does not admit any minimal Lagrangian isometric immersion into $\mathbb{C}^n$.

For Lagrangian submanifolds in complex space forms, we also have the following results obtained by Aquib et al.

Theorem 8.3 *[3] Let M^n be a Lagrangian submanifold of a complex space form $\widetilde{M}^n(4c)$. Then we have:*

(i) For a real number $r \in (0,(n-1))$, the generalized normalized δ-Casorati curvature $\delta_C(r;n-1)$ satisfies

$$\rho \leq \frac{\delta_C(r;n-1)}{n(n-1)} + c - \frac{2r^2}{(n-1)(n^2+n(r-1)+r)}H^2. \tag{8.2}$$

(ii) For a real number $r > n(n-1)$, the generalized normalized δ-Casorati curvature $\hat{\delta}_C(r;n-1)$ satisfies

$$\rho \leq \frac{\hat{\delta}_C(r;n-1)}{n(n-1)} + c - \frac{2n(n^2+n(r-1)-2r)}{(n-1)(n^2+n(r-1)+r)}H^2. \tag{8.3}$$

Moreover, the equality sign of (8.3) holds identically if and only if M^n is a Lagrangian totally geodesic submanifold.

Remark 8.4 *By taking $r = 2n(n-1)$ in (8.3) and using (6.2), we see that Theorem 8.3 implies Theorem 8.1.*

The following results are due to Vîlcu.

Theorem 8.5 *[117] Let M^n be a Lagrangian submanifold of a complex space form $\widetilde{M}^n(4c)$. Then we have*

$$\delta_C(n-1) \geq \rho - c + \frac{n}{n+3}H^2.$$

Corollary 8.6 *[117] If there exists a point p in a Riemannian n-manifold M^n such that $\rho(p) > [\delta_C(n-1)]_p$, then M^n does not admit any minimal Lagrangian isometric immersion into $\mathbb{C}^n$.*

8.2. H-umbilical submanifolds

The following notion of H-umbilical submanifolds was introduced by Chen in [25].

Definition 8.7 *A nontotally geodesic Lagrangian submanifold M^n of a Kaehler manifold $\widetilde{M}^n$ is called H-umbilical if its second fundamental form satisfies*

$$\begin{aligned} &h(e_i,e_i) = \mu J e_n,\ h(e_i,e_n) = \mu J e_i,\ i = 1,\dots,n-1,\\ &h(e_n,e_n) = \varphi J e_n,\ h(e_i,e_j) = 0,\ 1 \leq i \neq j \leq n-1, \end{aligned} \tag{8.4}$$

for some functions μ, φ with respect to an orthonormal frame $\{e_1,\dots,e_n\}$, where J is the complex structure of $\widetilde{M}^n$ If the ratio of $\varphi : \mu$ is a constant r, then the H-umbilical submanifold is said to be of ratio $r:1$.

Lagrangian H-umbilical submanifolds are the simplest Lagrangian submanifolds next to the totally geodesic ones. The next theorem was proved by Chen et al.

Theorem 8.8 *[45] For any real number r, there exist H-umbilical Lagrangian submanifolds of ratio $r:1$ in $CP^n(4)$ and in $CH^n(-4)$.*

8.3. Lagrangian Casorati ideal submanifolds

The following two results were proved by Aquib et al. [3].

Theorem 8.9 *Let M^n be a Lagrangian Casorati ideal submanifold for (8.2). Then it is either a totally geodesic Lagrangian submanifold or an H-umbilical Lagrangian submanifold satisfying (8.4) with $\varphi = (n^2 - n + 2t)\mu/t$.*

Corollary 8.10 *Let M^n be a Casorati ideal Lagrangian submanifold for (8.2) in a complex space form $\widetilde{M}^n(4c)$ without totally geodesic points. Then M^n is an H-umbilical Lagrangian submanifold of ratio $n^2 - n + 2t : t$.*

The following results are due to Vîlcu.

Theorem 8.11 *[117] If M^n is a Casorati $\delta_C(n-1)$-ideal Lagrangian submanifold of a complex space form $\widetilde{M}^n(4c)$, then it is either a totally geodesic Lagrangian submanifold or an H-umbilical Lagrangian submanifold satisfying (8.4) with $\varphi = 4\mu$.*

Corollary 8.12 *[117] Let M^n be a Casorati $\delta_C(n-1)$-ideal Lagrangian submanifold without totally geodesic points in a complex space form $\widetilde{M}^n(4c)$. Then M is an H-umbilical Lagrangian submanifold of ratio $4:1$.*

Corollary 8.13 *[117] Let M^n be a Casorati $\delta_C(n-1)$-ideal Lagrangian submanifold without totally geodesic points in the complex projective n-space $CP^n(4)$. Then M is congruent to an open portion of $\pi \circ \psi$, where $\pi : S^{2n+1}(1) \to CP^n(4)$ is the Hopf fibration, $\psi : M \to S^{2n+1}(1) \subset \mathbb{C}^{n+1}$ is given by*

$$\psi(t, y_1, \ldots, y_n) = (z_1(t), z_2(t)\boldsymbol{y}),\ \{\boldsymbol{y} \in \mathbb{R}^n : \langle \boldsymbol{y}, \boldsymbol{y} \rangle = 1\},$$

and $z : I \to S^3(1) \subset \mathbb{C}^2$ is a unit speed Legendre curve satisfying $z'' = 4\mathrm{i}\mu z' - z$, where μ is a nonzero solution of $2\mu\mu'' - \mu'^2 + 4\mu^2(3\mu^2 + 1) = 0$.

Corollary 8.14 *[117] Let M^n be a Casorati $\delta_C(n-1)$-ideal Lagrangian submanifold without totally geodesic points in the complex hyperbolic n-space $CH^n(-4)$. Then M is congruent to an open portion of $\pi \circ \psi$, where $\pi : H_1^{2n+1}(-1) \to CH^n(-4)$ is the Hopf fibration and $\psi : M \to H_1^{2n+1}(-1) \subset \mathbb{C}_1^{n+1}$ is either one of*

$$\psi(t, y_1, \ldots, y_n) = (z_1(t), z_2(t)\boldsymbol{y}),\ \{\boldsymbol{y} \in \mathbb{R}^n : \langle \boldsymbol{y}, \boldsymbol{y} \rangle = 1\},$$
$$\psi(t, y_1, \ldots, y_n) = (z_1(t)\boldsymbol{y}, z_2(t)),\ \{\boldsymbol{y} \in \mathbb{R}_1^n : \langle \boldsymbol{y}, \boldsymbol{y} \rangle = -1\},$$

where $z : I \to H_1^3(-1) \subset \mathbb{C}_1^2$ is a unit speed Legendre curve satisfying $z'' = 4\mathrm{i}\mu z' + z$, and μ is a nontrivial solution of $2\mu\mu'' - \mu'^2 + 4\mu^2(3\mu^2 - 1) = 0$; or ψ is

$$\psi(t, u_1, \ldots, u_{n-1}) = \frac{\sqrt{\mu(t)}}{\sqrt{\mu(0)}} e^{\mathrm{i}\int_0^t \mu(t)dt} \left(\frac{1}{2} + \frac{1}{2}\sum_{j=1}^{n-1} u_j^2 - \mathrm{i}t\mu(0) + \frac{\mu(0)}{2\mu(t)}, \right.$$
$$\left. \left(\mathrm{i}\mu(0) - \frac{\mu'(0)}{2\mu(0)}\right) \left(\frac{1}{2}\sum_{j=1}^{n-1} u_j^2 - it\mu(0) + \frac{\mu(0)}{2\mu(t)} - \frac{1}{2} \right), u_1, \ldots, u_{n-1} \right),$$

where $z : I \to H_1^3(-1) \subset \mathbb{C}_1^2$ is a unit speed Legendre curve and μ is a nontrivial solution of $\mu'^2 = 4\mu^2(1 - \mu^2)$.

9. δ-Casorati curvatures of slant submanifolds

Let M be a submanifold of an almost Hermitian manifold $(\widetilde{M}, \tilde{g}, J)$. For a nonzero vector $X \in T_pM$, we put

$$JX = PX + FX, \tag{9.1}$$

where PX and FX are the tangential and normal components of JX, respectively. The angle $\theta(X)$ between PX and T_pM is called the Wirtinger angle of X. The squared norm of the endomorphism P is defined by $||P||^2 = \sum_{i,j=1}^{n} \tilde{g}(Pe_i, e_j)^2$, where $\{e_1, \ldots, e_n\}$ is an orthonormal frame of the tangent bundle TM^n.

9.1. Slant submanifolds in almost Hermitian manifolds

In 1990, Chen [22] introduced the notion of slant submanifolds as follows:

Definition 9.1 *A submanifold M of an almost Hermitian manifold $(\widetilde{M}, \tilde{g}, J)$ is called* slant *if the Wirtinger angle $\theta(X)$ is independent of the choice of $X \in T_pM$ and of $p \in M$. The Wirtinger angle of a slant submanifold is called the* slant angle. *A slant submanifold with slant angle θ is simply called θ*-slant.

Obviously, complex and totally real submanifolds are exactly θ-slant submanifolds with $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. From J-action points of view, complex, totally real and slant submanifolds are the most natural submanifolds of almost Hermitian manifolds.

9.2. δ-Casorati curvatures of slant submanifolds in complex space forms

Ghişoiu [67] considered δ-Casorati curvatures of slant submanifolds in complex space forms and obtained the following.

Theorem 9.2 *Let M^n $(n \geq 3)$ be a θ-slant submanifold of a complex space form $\widetilde{M}^m(4c)$. We have*

$$\rho \leq \delta_C(n-1) + \left\{1 + \frac{3}{n-1}\cos^2\theta\right\} c \quad \left(\textit{resp., } \rho \leq \hat{\delta}_C(n-1) + \left\{1 + \frac{3}{n-1}\cos^2\theta\right\} c\right), \tag{9.2}$$

where ρ is the normalized scalar curvature.

In addition, if the equality of (9.2) holds identically, then there exists an orthonormal frame $e_1, \ldots, e_n$, $\xi_{n+1}, \ldots, \xi_{2m}$ such that with respect to this frame the shape operator satisfies

$$A_{n+1} = \begin{pmatrix} \lambda I_{n-1} & 0 \\ 0 & 2\lambda \end{pmatrix} \quad \left(\textit{resp., } A_{n+1} = \begin{pmatrix} \lambda I_{n-1} & 0 \\ 0 & \frac{1}{2}\lambda \end{pmatrix}\right) \ \textit{and}\ A_{n+2} = \ldots = A_{2m} = 0, \tag{9.3}$$

for some function λ.

9.3. δ-Casorati curvatures of bi-slant submanifolds in generalized complex space forms

Bi-slant immersions was introduced by Carriazo [17] as follows:

Definition 9.3 *A submanifold M of an almost Hermitian manifold $(\widetilde{M}, \tilde{g}, J)$ is called bi-slant if there is a pair of orthogonal distributions $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of M such that* (a) $TN = \mathfrak{D}_1 \oplus \mathfrak{D}_2$*;* (b) $J\mathfrak{D}_1 \perp \mathfrak{D}_2$ *and* $J\mathfrak{D}_2 \perp \mathfrak{D}_1$*; and* (c) *the distributions $\mathfrak{D}_1$, $\mathfrak{D}_2$ are slant with slant angle θ_1, θ_2, respectively.*

The pair $\{\theta_1, \theta_2\}$ of slant angles of a bi-slant submanifold is called bi-slant angles.

For a bi-slant submanifold M^n, we put $\operatorname{rank}\mathfrak{D}_1 = n_1$ and $\operatorname{rank}\mathfrak{D}_2 = n_2$. A bi-slant submanifold whose bi-slant angles satisfy $\theta_1 = \frac{\pi}{2}$ and $\theta_2 \in (0, \frac{\pi}{2})$ (resp., $\theta_1 = 0$ and $\theta_2 \in (0, \frac{\pi}{2})$) is called hemislant (resp., semislant). Further, a bi-slant submanifold is called proper if its bi-slant angles satisfies $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$.

Following Vanhecke [112], an almost Hermitian manifold $(\widetilde{M}, \tilde{g}, J)$ is called a RK-manifold if its curvature tensor $\tilde{R}$ satisfies $\tilde{R}(JX, JY, JZ, JW) = \tilde{R}(X, Y, Z, W)$ for any X, Y, Z, W tangent to $\widetilde{M}$. On a RK-manifold $\widetilde{M}$, we put

$$\lambda(X, Y) = \tilde{R}(X, Y, JX, JY) - \tilde{R}(X, Y, X, Y).$$

An almost Hermitian manifold $\widetilde{M}$ is said to be of pointwise constant type if at any point $p \in \widetilde{M}$ and any vector $X \in T_p\widetilde{M}$, we have $\lambda(X, Y) = \lambda(X, Z)$, where Y and Z are unit tangent vectors in $T_p\widetilde{M}$ which are orthogonal to X and JX. The $\widetilde{M}$ is said to be of constant type if for unit vectors fields X, Y with $g(X, Y) = g(JX, Y) = 0$, $\lambda(X, Y)$ is a constant function.

By definition, a generalized complex space form is a RK-manifold of constant holomorphic sectional curvature and it is of constant type (see [112]). Every complex space form is a generalized complex space form, but the converse is not true. The simplest example is the nearly Kaehler S^6 which is a generalized complex space form, but not a complex space form.

Let $\tilde{M}^m(c, \alpha)$ denote a generalized complex space form of constant holomorphic sectional curvature c and of constant type α. Then its curvature tensor $\tilde{R}$ satisfies

$$\tilde{R}(X, Y)Z = \frac{c + 3\alpha}{4}\{g(Y, Z)X - g(X, Z)Y\} + \frac{c - \alpha}{4}\{g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ\}.$$

The next result on bi-slant submanifold was proved by Aquib et al. [7].

Theorem 9.4 *Let M^n be a proper bi-slant submanifold of a generalized complex space form $\tilde{M}^m(c, \alpha)$. Then*

(i) For any real number r with $0 < r < n(n-1)$, the generalized normalized δ-Casorati curvature $\delta_C(r; n-1)$ and the scalar curvature of M^n satisfy

$$\tau \leq \frac{\delta_C(r; n-1)}{n(n-1)} + \frac{c + 3\alpha}{4} + \frac{3(c - \alpha)}{4n(n-1)}\left(n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2\right). \tag{9.4}$$

(ii) For any real number $r > n(n-1)$, the generalized normalized δ-Casorati curvature $\hat{\delta}_C(r; n-1)$ satisfies

$$\tau \leq \frac{\hat{\delta}_C(r; n-1)}{n(n-1)} + \frac{c + 3\alpha}{4} + \frac{3(c - \alpha)}{4n(n-1)}\left(n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2\right). \tag{9.5}$$

Moreover, the equality holds in (9.4) and (9.5) if and only if there exists an orthonormal frame $e_1, \ldots, e_n$, $\xi_{n+1}, \ldots, \xi_{2m}$ such that with respect to this frame the shape operator satisfies

$$A_{n+1} = \begin{pmatrix} \lambda I_{n-1} & 0 \\ 0 & \frac{n(n-1)}{r}\lambda \end{pmatrix}, \quad A_{n+2} = \ldots = A_{2m} = 0. \tag{9.6}$$

Theorem 9.4 implies the following result of Lone [86].

Theorem 9.5 *Let M^n be a θ-slant submanifold of a generalized complex space form $\tilde{M}^m(c,\alpha)$. Then*

(i) For a real number $r \in (0, n(n-1))$, the generalized normalized δ-Casorati curvature $\delta_C(r;n-1)$ satisfies

$$\tau \leq \frac{\delta_C(r;n-1)}{n(n-1)} + \frac{c+3\alpha}{4} + \frac{3(c-\alpha)}{4(n-1)}.\cos^2\theta_1, \tag{9.7}$$

(ii) For a real number $r > n(n-1)$, the generalized normalized δ-Casorati curvature $\hat{\delta}_C(r;n-1)$ satisfies

$$\tau \leq \frac{\hat{\delta}_C(r;n-1)}{n(n-1)} + \frac{c+3\alpha}{4} + \frac{3(c-\alpha)}{4(n-1)}\cos^2\theta_1. \tag{9.8}$$

Moreover, the equality holds in (9.7) and (9.8) if and only if there exists an orthonormal frame $e_1,\ldots,e_n$, $\xi_{n+1},\ldots,\xi_{2m}$ such that with respect to this frame the shape operator satisfies (9.6) for some function λ.

10. δ-Casorati curvatures in golden Riemannian manifolds

A tensor field F of type $(1,1)$ on a Riemannian manifold $(\widetilde{M},\tilde{g})$ is called almost product if it satisfies $F^2 = I$. A Riemannian manifold $(\widetilde{M},\tilde{g})$ endowed with an almost product structure F is called an almost product Riemannian manifold if it satisfies $\tilde{g}(FX,Y) = \tilde{g}(X,FY)$.

10.1. Golden Riemannian manifolds

Let $(\widetilde{M},\tilde{g})$ be a Riemannian m-manifold and let φ be a $(1,1)$-tensor field on $\widetilde{M}$. If φ satisfies

$$\varphi^2 - \varphi - I = 0, \tag{10.1}$$

then φ is called a golden structure. If the metric $\tilde{g}$ and φ are compatible, that is, $\widetilde{g}(\varphi X,Y) = \tilde{g}(X,\varphi Y)$ for $X,Y \in T\widetilde{M}$, then $(\tilde{M},\tilde{g},\varphi)$ is called a golden Riemannian manifold [51]. The real positive root ψ of the equation $x^2 - x - 1 = 0$, that is $\psi = \frac{1+\sqrt{5}}{2}$, is called the golden proportion.

Let M be a submanifold of a golden Riemannian manifold $(\widetilde{M},\tilde{g},\varphi)$. For any $X \in TM$ we put

$$\varphi X = PX + QX, \tag{10.2}$$

where PX and QX are the tangent and normal components of φX. The submanifold M is called slant if, for each $0 \neq X \in T_pM$, the angle $\theta(X)$ between φX and T_pM is constant, that is, $\theta(X)$ is independent of the choice of $p \in M$ and $X \in T_pM$. If the slant angle θ of a slant submanifold M satisfies $\theta = 0$ (resp., $\theta = \frac{\pi}{2}$), then M is called φ-invariant (resp., φ-antiinvariant). A slant submanifold which is neither invariant nor antiinvariant is called proper slant.

It was proved by Crasmareanu and Hretcanu [51] that an almost product structure F on $\widetilde{M}$ induces a Golden structure φ given by $\varphi = \frac{1}{2}(I + \sqrt{5}F)$. Conversely, every golden structure φ on $\widetilde{M}$ induces an almost product structure $F = \frac{1}{\sqrt{5}}(2\varphi - I)$. Hence, a locally product $M^p(c_p) \times M^q(c_q)$ of two real space forms $M^p(c_p)$ and $M^q(c_q)$ of constant curvature c_p and c_q is a golden Riemannian manifold, called a locally product golden space form. The Riemannian curvature tensor $\tilde{R}$ of a locally golden product space form $M^p(c_p)\times M^q(c_q)$ is derived by Poyraz and Yaşar [98] as follows:

$$\begin{aligned}\tilde{R}(X,Y)Z &= \left(\frac{(\psi-1)c_p+\psi c_q}{2\sqrt{5}}\right)\{g(Y,Z)X-g(X,Z)Y+g(\varphi Y,Z)\varphi X-g(\varphi X,Z)\varphi Y\}\\ &-\left(\frac{(1-\psi)c_p+\psi c_q}{4}\right)\{g(\varphi Y,Z)X-g(\varphi X,Z)Y+g(Y,Z)\varphi X-g(X,Z)\varphi Y\},\quad \psi=\frac{1+\sqrt{5}}{2}.\end{aligned}$$

10.2. δ-Casorati curvatures of slant submanifolds in golden Riemannian manifolds

The following results on δ-Casorati curvatures of slant submanifolds of locally product golden space forms are obtained by Choudhary and Park.

Theorem 10.1 *[50] Let M^n be a θ-slant proper submanifold of a locally product golden space form $\bar{M}^m=(M^p(c_p)\times M^q(c_q),\tilde{g},\varphi)$. Then we have*

(i) For a real number $r\in(0,n(n-1))$, the generalized normalized δ-Casorati curvature $\delta_C(r;n-1)$ satisfies

$$\begin{aligned}\rho \le \frac{\delta_C(r;n-1)}{n(n-1)} &- \left[\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\right]\left[1+\frac{\mathrm{trace}^2\varphi}{n(n-1)}-\cos^2\theta\left[\frac{1}{n-1}+\frac{1}{n(n-1)}\mathrm{trace}\,P\right]\right]\\ &-\left[\frac{(1-\psi)c_p-\psi c_q}{4}\right]\frac{2}{n}\mathrm{trace}\,\psi,\end{aligned}\tag{10.3}$$

where $\psi=\frac{1+\sqrt{5}}{2}$ is the golden proportion.

(ii) For a real number $r>n(n-1)$, the generalized normalized δ-Casorati curvature $\hat{\delta}_C(r;n-1)$ satisfies

$$\begin{aligned}\rho \le \frac{\hat{\delta}_C(r;n-1)}{n(n-1)} &- \left[\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\right]\left[1+\frac{\mathrm{trace}^2\varphi}{n(n-1)}-\cos^2\theta\left[\frac{1}{n-1}+\frac{\mathrm{trace}\,P}{n(n-1)}\right]\right]\\ &-\left[\frac{(1-\psi)c_p-\psi c_q}{4}\right]\frac{2}{n}\mathrm{trace}\,\psi.\end{aligned}\tag{10.4}$$

Moreover, the equality holds in (10.3) and (10.4) if and only if there exists an orthonormal frame $e_1,\dots,e_n,\xi_{n+1},\dots,\xi_m$ such that with respect to this frame the shape operator satisfies (9.6).

Remark 10.2 *Choudhary and Park [50] derived similar results for invariant and antiinvariant submanifolds of locally product golden space forms.*

11. δ-Casorati curvatures in metallic Riemannian space forms

11.1. Metallic Riemannian space forms

A $(1,1)$-tensor field φ on a Riemannian manifold $(\widetilde{M},g)$ satisfying

$$\varphi^2=p\varphi+qI\tag{11.1}$$

with $p, q \in \mathbb{N}^*$ is called a metallic structure. A Riemannian manifold $(\widetilde{M}, g)$ endowed with a metallic structure φ is called a metallic Riemannian manifold if the Riemannian metric g is φ-compatible, that is,

$$g(\varphi X, Y) = g(X, \varphi Y). \tag{11.2}$$

Since φ is a self-adjoint, after interchanging X by φX, we obtain from (11.2) that

$$g(\varphi X, \varphi Y) = g(\varphi^2 X, Y) = pg(X, \varphi Y) + qg(X, Y).$$

Note that if $p = q = 1$ in (11.1), then a metallic structure becomes a golden structure.

It is known that each metallic structure φ gives rise two almost product structures (cf. [49])

$$F_1 = \frac{2}{2\sigma_{p,q} - p}\varphi - \frac{p}{2\sigma_{p,q} - p}I, \quad F_2 = \frac{-2}{2\sigma_{p,q} - p}\varphi + \frac{p}{2\sigma_{p,q} - p}I,$$

where $\sigma_{p,q} = \frac{1}{2}(p + \sqrt{p^2 + 4q})$ is called the metallic proportion. Conversely, each almost product structure F on $\widetilde{M}$ induces two metallic structures

$$\varphi_1 = \frac{p}{2}I + \frac{2\sigma_{p,q} - p}{2}F, \quad \varphi_2 = \frac{p}{2}I - \frac{2\sigma_{p,q} - p}{2}F.$$

A metallic Riemannian manifold $(\widetilde{M}, g, \varphi)$ is said to be a locally metallic Riemannian manifold if the Levi–Civita connection ∇ of g is a φ-connection, that is, $\nabla\varphi = 0$.

Note that slant submanifolds of a metallic Riemannian manifold can be defined exactly in the same way as slant submanifolds of a golden Riemannian manifold given in Section 10 (cf. [49]).

11.2. δ-Casorati curvatures of slant submanifolds in metallic Riemannian manifolds

The following results on δ-Casorati curvatures of slant submanifolds of locally product metallic Riemannian manifolds are proved by Choudhary and Blaga.

Theorem 11.1 *[49] Let M^n be an n-dimensional θ-slant proper submanifold of a metallic product space form $\widetilde{M}^m = (M_1(c_1) \times M_2(c_2), g, \varphi)$. Then we have*

(i) For a real number $r \in (0, n(n-1))$, the generalized normalized δ-Casorati curvature $\delta_C(r; n-1)$ satisfies

$$\begin{aligned}\rho \leq \frac{\delta_C(r; n-1)}{n(n-1)} &+ \frac{c_1 + c_2}{2(p^2 + 4q)}\left[p^2 + 2q + \frac{2(\operatorname{trace}^2\varphi - (p \operatorname{trace} P + nq)\cos^2\theta)}{n(n-1)} - \frac{2p}{n}\operatorname{trace}\varphi\right] \\ &+ \frac{c_1 - c_2}{2\sqrt{p^2 + 4q}}\left(\frac{2}{n}\operatorname{trace}\varphi - p\right).\end{aligned} \tag{11.3}$$

where P is defined by (10.2).

(ii) For a real number $r > n(n-1)$, the generalized normalized δ-Casorati curvature $\hat{\delta}_C(r; n-1)$ satisfies

$$\begin{aligned}\rho \leq \frac{\hat{\delta}_C(r; n-1)}{n(n-1)} &+ \frac{c_1 + c_2}{2(p^2 + 4q)}\left[p^2 + 2q + \frac{2(\operatorname{trace}^2\varphi - (p \operatorname{trace} P + nq)\cos^2\theta)}{n(n-1)} - \frac{2p}{n}\operatorname{trace}\varphi\right] \\ &+ \frac{c_1 - c_2}{2\sqrt{p^2 + 4q}}\left(\frac{2}{n}\operatorname{trace}\varphi - p\right).\end{aligned} \tag{11.4}$$

Moreover, the equality holds in (11.3) *and* (11.4) *if and only if there exists an orthonormal frame* $e_1, \ldots, e_n, \xi_{n+1}, \ldots, \xi_m$ *such that with respect to this frame the shape operator satisfies* (5.12).

Remark 11.2 *Choudhary and Blaga also derived in [49] the corresponding results for invariant and antiinvariant submanifolds of metallic product space forms and as well as for* $\delta_C(n-1)$ *and* $\hat{\delta}_C(n-1)$.

12. δ-Casorati curvatures in Bochner–Kaehler manifolds

Bochner introduced the Bochner curvature tensor on Kaehler manifolds in [13] as an analogue of the Weyl conformal curvature tensor in Riemannian geometry.

12.1. Bochner tensor and Bochner–Kaehler manifolds

Let $(\widetilde{M}^m, g, J)$ be a Kaehler manifold of real dimension m. As before, let R, Ric, S, and τ be the Riemann curvature tensor, Ricci tensor, Ricci operator, and scalar curvature of $\widetilde{M}^m$, respectively. Then the Bochner curvature tensor B is given by

$$\begin{aligned} B(X,Y)Z = R(X,Y)Z &- \frac{1}{m+4}[g(Y,Z)SX - g(SX,Z)Y + g(JY,Z)SJX - g(SJX,Z)JY \\ &+ g(SY,Z)X - g(X,Z)SY + g(SJY,Z)JX - g(JX,Z)SJY - 2g(JX,SY)JZ - 2g(JX,Y)SJZ] \\ &+ \frac{2\tau}{(m+2)(m+4)}[g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ]. \end{aligned}$$

A Kaehler manifold $\widetilde{M}$ is called Bochner–Kaehler if its Bochner tensor vanishes identically (cf. [21]). Bochner–Kaehler manifolds can be regarded as the Kaehlerian version of conformally flat spaces in Riemannian geometry. Some simple characterizations of Bochner–Kaehler manifolds can be found in [37, 48, 110].

12.2. Inequalities of δ-Casorati curvatures in Bochner–Kaehler manifolds

The following result was obtained by Liu et al.

Theorem 12.1 *[83] Let* M^n *be a submanifold of a Bochner–Kaehler manifold* $(\widetilde{M}^m, \tilde{g}, J)$. *Then*

(i) For any real number r *with* $0 < r < n(n-1)$, *the generalized normalized* δ*-Casorati curvature* $\delta_C(r; n-1)$ *satisfies*

$$\left(1 - \frac{3n^2+n-4-3||P||^2}{8(n+1)(n+2)}\right)\rho \leq \frac{\delta_C(r;n-1)}{n(n-1)} + \frac{3}{(n-1)n(n+2)} \sum_{1\leq i\neq j\leq n} \tilde{R}ic(e_i, Je_j)\tilde{g}(e_i, Je_j), \quad (12.1)$$

where ρ *is the normalized scalar curvature,* $\tilde{R}ic$ *is the Ricci tensor of* $\widetilde{M}^m$, *and* $||P||^2$ *is defined in §9.*

(ii) For any real number $r > n(n-1)$, *the generalized normalized* δ*-Casorati curvature* $\hat{\delta}_C(r; n-1)$ *satisfies*

$$\left(1 - \frac{3n^2+n-4-3||P||^2}{8(n+1)(n+2)}\right)\rho \leq \frac{\hat{\delta}_C(r;n-1)}{n(n-1)} + \frac{3}{(n-1)n(n+2)} \sum_{1\leq i\neq j\leq n} \tilde{R}ic(e_i, Je_j)\tilde{g}(e_i, Je_j). \quad (12.2)$$

Moreover, the equality holds in (12.1) *and* (12.2) *if and only if there exists an orthonormal frame* $e_1, \dots, e_n$, $\xi_{n+1}, \dots, \xi_{2m}$ *such that with respect to this frame the shape operator satisfies* (9.6).

Consequently, we have the following.

Corollary 12.2 *[83] Let* M^n *be a submanifold of a Bochner–Kaehler manifold* $(\widetilde{M}^m, \tilde{g}, J)$. *Then*

(i) The δ*-Casorati curvature* $\delta_C(n-1)$ *satisfies*

$$\left(1 - \frac{3n^2+n-4-3||P||^2}{8(n+1)(n+2)}\right)\rho \leq \delta_C(n-1) + \frac{3}{(n-1)n(n+2)} \sum_{1\leq i\neq j\leq n} \tilde{R}ic(e_i, Je_j)\tilde{g}(e_i, Je_j). \tag{12.3}$$

(ii) The δ*-Casorati curvature* $\hat{\delta}_C(n-1)$ *satisfies*

$$\left(1 - \frac{3n^2+n-4-3||P||^2}{8(n+1)(n+2)}\right)\rho \leq \hat{\delta}_C(n-1) + \frac{3}{(n-1)n(n+2)} \sum_{1\leq i\neq j\leq n} \tilde{R}ic(e_i, Je_j)\tilde{g}(e_i, Je_j). \tag{12.4}$$

Moreover, the equality holds in (12.3) *and* (12.4) *if and only if there exists an orthonormal frame* $e_1, \dots, e_n$, $\xi_{n+1}, \dots, \xi_m$ *such that with respect to this frame the shape operator satisfies* (9.3).

Remark 12.3 *For* δ*-Casorati curvatures of submanifolds in Bochner–Kaehler manifolds, see also [4, 74, 83].*

13. δ-Casorati curvatures in quaternionic space forms

A quaternionic Kaehler manifold is a Riemannian $4m$-manifold whose Riemannian holonomy group is a subgroup of $Sp(m)\cdot Sp(1)$.

13.1. Quaternionic Kaehler manifolds and quaternionic space forms

An almost quaternionic Hermitian manifold $(\overline{M}, \overline{g}, \Sigma)$ is a Riemannian manifold equipped with a rank 3-subbundle Σ of $End(T\overline{M})$ with local basis $\{J_1, J_2, J_3\}$ satisfying

$$\overline{g}(J_\alpha X, J_\alpha Y) = \overline{g}(X, Y), \quad J_\alpha^2 = -I, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1}J_\alpha = J_{\alpha+2}, \quad X, Y \in T\overline{M},$$

for all $\alpha \in \{1,2,3\}$, where I is the identity transformation on $T\overline{M}$ and the indices are taken from $\{1,2,3\}$ modulo 3. Such manifold is of dimension $4m$, $m \geq 1$. Moreover, if the bundle Σ is parallel with respect to the Levi-Civita connection of $\overline{g}$, then $(\overline{M}, \sigma, \overline{g})$ is said to be a quaternionic Kaehler manifold.

For a quaternionic Kaehler manifold $(\overline{M}, \sigma, \overline{g})$, let X be a nonzero vector in $T\overline{M}$. The 4-plane $\overline{Q}(X)$ spanned by $\{X, J_1X, J_2X, J_3X\}$, is called a quaternionic 4-plane. Any 2-plane in $\overline{Q}(X)$ is called a quaternionic plane. The sectional curvature of a quaternionic plane is called a quaternionic sectional curvature. A quaternionic Kaehler manifold is said to be a *quaternionic space form* if its quaternionic sectional curvatures are equal to a constant. A quaternionic space form of constant quaternionic sectional curvature c is denoted by $\overline{M}(c)$. The curvature tensor $\overline{R}$ of $\overline{M}(c)$ satisfies

$$\overline{R}(X,Y)Z = \frac{c}{4}\{\overline{g}(Z,Y)X - \overline{g}(X,Z)Y + \sum_{\alpha=1}^{3}[\overline{g}(Z, J_\alpha Y)J_\alpha X - \overline{g}(Z, J_\alpha X)J_\alpha Y + 2\overline{g}(X, J_\alpha Y)J_\alpha Z]\}.$$

A submanifold M of a quaternionic Kaehler manifold $\overline{M}$ is called slant [100], if for each nonzero vector $X \in T_pM$, the angle $\theta(X)$ between $J_\alpha(X)$ and T_pM, $\alpha \in \{1,2,3\}$, is a global constant, so that it is independent of the choice of $p \in M$ and $X \in T_pM$. A slant submanifold of a quaternionic Kaehler manifold is called proper (or proper θ-slant) if $\theta \neq 0, \frac{\pi}{2}$.

Let M be a submanifold of a quaternionic Kaehler manifold $\overline{M}$. For any $X \in T_pM$, put

$$J_\alpha X = P_\alpha X + F_\alpha X, \quad P_\alpha X \in T_pM, \quad F_\alpha X \in T_p^\perp M.$$

The squared norm of P_α is $||P_\alpha||^2 = \sum_{i,j=1}^n g(P_\alpha e_i, e_j)^2$, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM.

13.2. δ-Casorati curvatures for slant submanifolds in quaternionic space forms

For proper θ-slant submanifolds of a quaternionic space form $\overline{M}^{4m}(c)$, we have the following.

Theorem 13.1 *[75, 82] Let M^n be a proper θ-slant submanifold of a quaternionic space form $\overline{M}^{4m}(c)$. Then*

(i) For a real number $r \in (0, n(n-1))$, the generalized normalized δ-Casorati curvature $\delta_C(r; n-1)$ satisfies

$$\rho \leq \frac{\delta_C(r; n-1)}{n(n-1)} + \frac{c}{4}\left(1 + \frac{9}{n-1}\cos^2\theta\right). \tag{13.1}$$

(ii) For any real number $r > n(n-1)$, the generalized normalized δ-Casorati curvature $\hat{\delta}_C(r; n-1)$ satisfies

$$\rho \leq \frac{\hat{\delta}_C(r; n-1)}{n(n-1)} + \frac{c}{4}\left(1 + \frac{9}{n-1}\cos^2\theta\right). \tag{13.2}$$

The equality case of either inequality holds if and only if there exists an orthonormal frame $e_1, \ldots, e_n$, $\xi_{n+1}, \ldots, \xi_{4m}$ such that with respect to this frame the shape operator takes the form

$$A_{n+1} = \begin{pmatrix} \lambda I_{n-1} & 0 \\ 0 & \frac{n(n-1)}{r}\lambda \end{pmatrix}, \quad A_{n+2} = \ldots = A_{4m} = 0. \tag{13.3}$$

Theorem 13.1 implies the following result from [107, 120] .

Corollary 13.2 *Let M^n be a θ-slant proper submanifold of a quaternionic space form $\overline{M}^{4m}(c)$. Then the normalized δ-Casorati curvature $\delta_c(n-1)$ (resp., $\hat{\delta}_c(n-1)$) satisfies*

$$\rho \leq \delta_c(n-1) + \frac{c}{4}\left(1 + \frac{9}{n-1}\cos^2\theta\right) \quad \left(resp.,\ \rho \leq \hat{\delta}_c(n-1) + \frac{c}{4}\left(1 + \frac{9}{n-1}\cos^2\theta\right)\right). \tag{13.4}$$

The equality holds if and only if there exists an orthonormal frame $e_1, \ldots, e_n, \xi_{n+1}, \ldots, \xi_{4m}$ such that with respect to this frame the shape operator satisfies

$$A_{n+1} = \begin{pmatrix} \lambda I_{n-1} & 0 \\ 0 & 2\lambda \end{pmatrix} \ \left(resp.,\ A_{n+1} = \begin{pmatrix} \lambda I_{n-1} & 0 \\ 0 & \frac{1}{2}\lambda \end{pmatrix}\right) \ and\ A_{n+2} = \ldots = A_{4m} = 0. \tag{13.5}$$

Remark 13.3 *Corollary 13.2 improves Theorem 4.1 of [53].*

13.3. δ-Casorati curvatures for submanifolds in quaternionic space forms

For submanifolds in a quaternionic space form, Suh and Tripathi proved the following.

Theorem 13.4 *[109] Let M^n be a submanifold of a quaternionic space form $\overline{M}^{4m}(c)$. Then*

(i) For any real number $r \in (0, n(n-1))$, the generalized normalized δ-Casorati curvature $\delta_C(r; n-1)$ satisfies

$$\rho \leq \frac{\delta_C(r;n-1)}{n(n-1)} + \frac{c}{4}\left(1 + \frac{3}{n(n-1)}\sum_{\alpha=1}^{3}||P_\alpha||^2\right). \tag{13.6}$$

(ii) For any real number $r > n(n-1)$, the generalized normalized δ-Casorati curvature $\hat{\delta}_C(r; n-1)$ satisfies

$$\rho \leq \frac{\hat{\delta}_C(r;n-1)}{n(n-1)} + \frac{c}{4}\left(1 + \frac{3}{n(n-1)}\sum_{\alpha=1}^{3}||P_\alpha||^2\right). \tag{13.7}$$

The equality sign holds in the inequalities (13.6) and (13.7) if and only if there exists an orthonormal frame $e_1, \ldots, e_n, \xi_{n+1}, \ldots, \xi_{4m}$ such that with respect to this frame the shape operator takes the form (13.3).

Theorem 10.3 implies the following.

Corollary 13.5 *[109] Let M^n be a submanifold of a quaternionic space form $\overline{M}^{4m}(c)$. Then*

(i) The normalized δ-Casorati curvature $\delta_C(n-1)$ satisfies

$$\rho \leq \delta_C(n-1) + \frac{c}{4}\left(1 + \frac{3}{n(n-1)}\sum_{\alpha=1}^{3}||P_\alpha||^2\right). \tag{13.8}$$

(ii) The normalized δ-Casorati curvature $\hat{\delta}_C(n-1)$ satisfies

$$\rho \leq \hat{\delta}_C(n-1) + \frac{c}{4}\left(1 + \frac{3}{n(n-1)}\sum_{\alpha=1}^{3}||P_\alpha||^2\right). \tag{13.9}$$

The equality holds in (13.8) [resp., (13.9)] if and only if there exists an orthonormal frame $e_1, \ldots, e_n$, $\xi_{n+1}, \ldots, \xi_{4m}$ such that with respect to this frame the shape operator satisfies (13.5).

14. δ-Casorati curvatures in Sasakian space forms

14.1. Almost contact metric manifolds, Kenmotsu space forms and Sasakian space forms

A Riemannian $(2n+1)$-manifold $(\overline{M}^{2n+1}, g)$ is called an almost contact metric manifold [12] if there exist a $(1,1)$-tensor field φ, a vector field ξ (called the structure vector field), and a 1-form η on $\overline{M}^{2n+1}$ such that

$$\eta(\xi) = 1, \ \ \varphi^2(X) = -X + \eta(X)\xi, \ \ \varphi\xi = 0, \ \ \eta \circ \varphi = 0,$$
$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi)$$

for X, Y tangent to $\overline{M}^{2n+1}$. An almost contact metric manifold $(\overline{M}^{2n+1}, \varphi, \xi, \eta, g)$ is called a Kenmotsu manifold if it satisfies $(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X$, where ∇ is the Levi–Civita connection of $\bar{M}^{2n+1}$. An almost contact metric manifold $(\overline{M}^{2n+1}, \varphi, \xi, \eta, g)$ is called a Sasakian manifold if it satisfies

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X.$$

A Sasakian manifold is called Sasakian space form if it has constant φ-sectional curvature. The curvature tensor of a Sasakian space form $\overline{M}^{2n+1}(c)$ of constant φ-sectional curvature c is given by

$$\begin{aligned} R(X,Y)Z = {} & \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c-1}{4}\{g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\} \\ & + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned}$$

for vector fields X, Y, Z tangent to $\overline{M}^{2n+1}$. Sasakian space forms $\overline{M}^{2n+1}(c)$ can be modeled based on $c > -3$, $c = -3$ or $c < -3$. It is known that $\mathbb{R}^{2m+1}$ has constant φ-sectional curvature -3, while S^{2m+1} is of constant φ-sectional curvature 1. Denote the above Sasakian space forms by $\mathbb{R}^{2m+1}(-3)$ and $S^{2m+1}(1)$, respectively.

A Riemannian submanifold M^n of an almost contact metric manifold $(\overline{M}^{2m+1}, \varphi, \xi, \eta, g)$ is called C-totally real if the structure vector field ξ is normal to M. It follows that $\varphi(T_pM^n) \subset T_p^{\perp}M^n$ for C-totally real submanifolds.

A C-totally real submanifold is called Legendrian if $n = m$. Hence, a Legendrian submanifold is a C-totally real submanifold with the smallest possible codimension.

Remark 14.1 *[65, Proposition 3.2] Any Kenmotsu manifold can be obtained locally as follows. Let (M_0, g_0, J) be an almost Hermitian manifold. Put $\overline{M} = M_0 \times \mathbb{R}$, $\bar{g} = e^{2t}g_0 + dt^2$, $\bar{\xi} = \frac{\partial}{\partial t}$ and define φ by $\varphi X = JX$ for X tangent to $\overline{M}$ and $\varphi\bar{\xi} = 0$. Then we have*

(1) The triple $(\bar{g}, \varphi, \bar{\xi})$ is an almost contact metric structure on $\overline{M}$.

(2) (M_0, g_0, J) is a Kaehler manifold if and only if $(\bar{g}, \varphi, \bar{\xi})$ is a Kenmotsu structure on $\overline{M}$.

14.2. δ-Casorati curvatures for Legendrian submanifolds in Sasakian space forms

For Legendrian submanifolds, Lee et al. proved the following.

Theorem 14.2 *[80] Let M^n be a Legendrian submanifold of a Sasakian space form $\bar{M}^{2n+1}(c)$. Then:*

(i) The δ-Casorati curvature $\delta_C(n-1)$ satisfies

$$\rho \le \delta_C(n-1) + \frac{c+3}{4} - \frac{n}{n+3}H^2. \tag{14.1}$$

Moreover, if the equality sign of (14.1) holds identically, then M^n is either a totally geodesic Legendrian submanifold or an H-umbilical Legendrian submanifold satisfying (8.4) with $\mu = 4\lambda$.

(ii) The δ-Casorati curvature $\hat{\delta}_C(n-1)$ satisfies

$$\rho \le \hat{\delta}_C(n-1) + \frac{c+3}{4} - \frac{2n(2n-3)}{(n-1)(2n+3)}H^2. \tag{14.2}$$

Further, the equality sign of (14.2) holds identically, then M^n is a totally geodesic Legendrian submanifold.

Note that the unit n-sphere $S^n(1)$ is a Legendrian totally geodesic submanifold of $S^{2n+1}(1)$. It is easy to verify that this submanifold attains equality in the inequalities (13.1) and (13.2) identically. For the equality case of (13.1) Lee et al. also provided the following example.

Example 14.3 *[80] Let I be an open interval in $\mathbb{R}$. Consider the isometric immersion $\psi : M^{12} \to S^{25}$ of the Riemannian 12-manifold $M^{12} = I \times S^{11}$ into S^{25}, equipped with standard Sasakian structure, given by*

$$\psi(x, y_1, \ldots, y_{12}) = \frac{1}{5}\left(4e^{-\frac{i\sqrt{3}}{2}x}, 3e^{\frac{2i\sqrt{3}}{3}x}y_1, \ldots, 3e^{\frac{2i\sqrt{3}}{3}x}y_{12}\right),$$

where $y_1^2 + \ldots + y_{12}^2 = 1$. Then $(M^{12}, \psi^* g_0)$ is an H-Legendrian submanifold of S^{25} satisfying (8.4) with $\lambda = \frac{1}{2\sqrt{3}}$ and $\mu = \frac{2}{\sqrt{3}}$. The M^{12} satisfies the equality case of inequality (13.1) identically.

14.3. δ-Casorati curvatures for slant submanifolds in Sasakian space forms

A submanifold M^n of an almost contact metric manifold $\left(\overline{M}^{2m+1}, \varphi, \xi, \eta, g\right)$ is a *slant submanifold* if the angle $\theta(X)$ between φX and T_pM^n is constant for all $0 \neq X \in T_pM^n \setminus \xi_p$ and all $p \in M^n$.

For slant submanifolds of a Sasakian space form, Lone [85] obtained the following.

Theorem 14.4 *Let M^n be a θ-slant submanifold of a Sasakian space form $\overline{M}^{2m+1}(c)$. Then*

(i) For a real number $r \in (0, n(n-1))$, the generalized normalized δ-Casorati curvature $\delta_C(r; n-1)$ satisfies

$$\rho \leq \frac{\delta_C(r; n-1)}{n(n-1)} + \frac{c+3}{4} + \frac{3(c-1)}{4n}\cos^2\theta + \frac{c-1}{n}||\xi^T||^2, \tag{14.3}$$

where $||\xi^T||^2$ is the squared norm of the tangent component ξ^T of the structure vector field ξ.

(ii) For a real number $r > n(n-1)$, the generalized normalized δ-Casorati curvature $\hat{\delta}_C(r; n-1)$ satisfies

$$\rho \leq \frac{\hat{\delta}_C(r; n-1)}{n(n-1)} + \frac{c+3}{4} + \frac{3(c-1)}{4n}\cos^2\theta + \frac{c-1}{n}||\xi^T||^2. \tag{14.4}$$

The equality case of either inequality holds if and only if there exists an orthonormal frame $e_1, \ldots, e_n$, $\xi_{n+1}, \ldots, \xi_{2m+1}$ such that with respect to this frame the shape operator takes the form:

$$A_{n+1} = \begin{pmatrix} \lambda I_{n-1} & 0 \\ 0 & \frac{n(n-1)}{r}\lambda \end{pmatrix}, \quad A_{n+2} = \ldots = A_{2m+1} = 0. \tag{14.5}$$

15. Slant submanifolds in generalized Sasakian space forms

15.1. Generalized Sasakian space forms

The notion of a generalized Sasakian space form was introduced by Alegre et al. in [8]. An odd-dimensional manifold $\overline{M}^{2n+1}$ equipped with an almost contact metric structure (φ, ξ, η, g) is called generalized Sasakian space form if there exist three functions f_1, f_2, f_3 on $\overline{M}^{2n+1}$ such that

$$\overline{R}(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\}$$
$$+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}.$$

We denote such a manifold by $\overline{M}^{2n+1}(f_1,f_2,f_3)$. A generalized Sasakian space form $\overline{M}^{2n+1}(f_1,f_2,f_3)$ is a Sasakian space form if $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$, where c is a constant. The generalized Sasakian space forms also generalize the concept of Kenmotsu space forms and cosymplectic space forms as follows:

(i) A Kenmotsu space form is a generalized Sasakian space form with $f_1 = \frac{c-3}{4}$ and $f_2 = f_3 = \frac{c+1}{4}$.

(ii) A cosymplectic space form is a generalized Sasakian space form with $f_1 = f_2 = f_3 = \frac{c}{4}$.

15.2. Slant submanifolds in generalized Sasakian space forms

For θ-slant submanifolds of a generalized Sasakian space form $\overline{M}^{2m+1}(f_1,f_2,f_3)$, we have the following result of Lone obtained in [87].

Theorem 15.1 *Let M^n be a θ-slant submanifold of a generalized Sasakian space form $\overline{M}^{2m+1}(f_1,f_2,f_3)$. Then*

(i) For any real number $r \in (0, n(n-1))$, the generalized normalized δ-Casorati curvature $\delta_C(r;n-1)$ satisfies

$$\rho \leq \frac{\delta_C(r;n-1)}{n(n-1)} + f_1 + \frac{3f_2}{n}\cos^2\theta + \frac{2f_3}{n}||\xi^T||^2, \tag{15.1}$$

(ii) For any real number $r > n(n-1)$, the generalized normalized δ-Casorati curvature $\hat{\delta}_C(r;n-1)$ satisfies

$$\rho \leq \frac{\hat{\delta}_C(r;n-1)}{n(n-1)} + f_1 + \frac{3f_2}{n}\cos^2\theta + \frac{2f_3}{n}||\xi^T||^2. \tag{15.2}$$

The equality case of either inequality holds if and only if there exists an orthonormal frame $e_1,\ldots,e_n$, $\xi_{n+1},\ldots,\xi_{2m+1}$ such that with respect to this frame the shape operator takes the form (13.5).

Remark 15.2 *Theorem 15.1 implies Theorem 4.3 of [105].*

15.3. δ-Casorati curvature of bi-slant submanifolds in generalized Sasakian space forms

A submanifold M of an almost contact metric manifold $\overline{M}^{2m+1}$ is called bi-slant [17] if there exists a pair of orthogonal distributions $\mathcal{D}_{\theta_1}$ and $\mathcal{D}_{\theta_2}$ of M such that (i) $TM^n = \mathcal{D}_{\theta_1} \oplus \mathcal{D}_{\theta_2} \oplus \text{Span}\{\xi\}$, (ii) $\varphi\mathcal{D}_{\theta_1} \perp \mathcal{D}_{\theta_2}$ and $\varphi\mathcal{D}_{\theta_2} \perp \mathcal{D}_{\theta_1}$, and (iii) each $\mathcal{D}_{\theta_i}$ is slant distribution with the slant angle θ_i for $i=1,2$.

A bi-slant submanifold with bi-slant angles θ_1,θ_2 is called a semislant (resp., hemislant) if $\theta_1 = 0$ and $\theta_2 \neq 0, \frac{\pi}{2}$ (resp., $\theta_1 = \frac{\pi}{2}$ and $\theta_2 \neq 0, \frac{\pi}{2}$). For a bi-slant submanifold, we put $n_i = \frac{1}{2}\text{rank}\, D_i$ for $i = 1,2$.

15.4. Bi-slant submanifolds in generalized Sasakian space forms

For bi-slant submanifolds of a generalized Sasakian space form, Siddiqui and Shahid [104, 105] proved the following.

Theorem 15.3 *Let M^n be a bi-slant submanifold of a generalized Sasakian space form $\overline{M}^{2m+1}(f_1, f_2, f_3)$. Then we have*

(i) The normalized δ-Casorati curvature $\delta_C(n-1)$ satisfies

$$\rho \leq \delta_C(n-1) + f_1 + \frac{6f_2}{n(n-1)}\left(n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2\right) - \frac{2f_3}{n}||\xi^T||^2. \tag{15.3}$$

(ii) The normalized δ-Casorati curvature $\hat{\delta}_C(n)$ satisfies

$$\rho \leq \hat{\delta}_C(n-1) + f_1 + \frac{6f_2}{n(n-1)}\left(n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2\right) - \frac{2f_3}{n}||\xi^T||^2. \tag{15.4}$$

The equality case of either inequality holds if and only if there exists an orthonormal frame $e_1, \ldots, e_n$, $\xi_{n+1}, \ldots, \xi_{2m+1}$ such that with respect to this frame the shape operator takes the form

$$A_{n+1} = \begin{pmatrix} \lambda I_{n-1} & 0 \\ 0 & 2\lambda \end{pmatrix} \ \left(\textit{resp.},\ A_{n+1} = \begin{pmatrix} \lambda I_{n-1} & 0 \\ 0 & \frac{1}{2}\lambda \end{pmatrix}\right) \ \textit{and}\ A_{n+2} = \ldots = A_{2m+1} = 0. \tag{15.5}$$

It was shown in [105] that if M^n is a hemislant (resp., semislant) submanifold in a generalized Sasakian space form $\overline{M}^{2m+1}(f_1, f_2, f_3)$, then Theorem 15.1 holds after replacing "$\frac{6f_2}{n(n-1)}(n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2)$" in (15.3) and (15.4) by "$\frac{6f_2}{n(n-1)} n_1 \cos^2\theta_1$" (resp., by "$\frac{6f_2}{n(n-1)}(n_1 + n_2 \cos^2\theta_2)$").

15.5. Slant submanifolds in cosymplectic space forms

In particular, Theorem 15.1 implies the next result for slant submanifolds in cosymplectic space forms.

Theorem 15.4 *Let M^n be a θ-slant submanifold of a cosymplectic space form $\overline{M}^{2m+1}$. Then*

(i) For a real number $r \in (0, n(n-1))$, the generalized normalized δ-Casorati curvature $\delta_C(r; n-1)$ satisfies

$$\rho \leq \frac{\delta_C(r; n-1)}{n(n-1)} + \frac{c}{4} + \frac{3c}{4n}\cos^2\theta + \frac{c}{2n}||\xi^T||^2, \tag{15.6}$$

(ii) For any real number $r > n(n-1)$, the generalized normalized δ-Casorati curvature $\hat{\delta}_C(r; n-1)$ satisfies

$$\rho \leq \frac{\hat{\delta}_C(r; n-1)}{n(n-1)} + \frac{c}{4} + \frac{3c}{4n}\cos^2\theta + \frac{c}{2n}||\xi^T||^2. \tag{15.7}$$

The equality case of either inequality holds if and only if there exists an orthonormal frame $e_1, \ldots, e_n$, $\xi_{n+1}, \ldots, \xi_{2m+1}$ such that with respect to this frame the shape operator takes the form (13.5).

16. Slant submanifolds in pointwise Kenmotsu space forms

Let $(\overline{M}^{2m+1}, \varphi, \xi, \eta, \overline{g})$ be a Kenmotsu manifold of dimension ≥ 5. Then $\overline{M}^{2m+1}$ is said to be a pointwise Kenmotsu space form, denoted by $\overline{M}^{2m+1}(c)$, if its φ-sectional curvature function $c(X)$ of φ-holomorphic plane Span$\{X, \varphi X\} \subset T_p\overline{M}$ depends only on the point $p \in \overline{M}$, but not on the φ-holomorphic plane at p. In particular, if c is global constant, then $\overline{M}^{2m+1}(c)$ is called a Kenmotsu space form.

It is known that a Kenmotsu manifold $\overline{M}^{2m+1}$ is a pointwise Kenmotsu space form if and only if there exists a function c such that the Riemann curvature tensor $\overline{R}$ of $\overline{M}^{2m+1}$ satisfies (cf. [97])

$$\begin{aligned}\overline{R}(X,Y)Z = \frac{c-3}{4}\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\} + \frac{c+1}{4}\{&\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi \\ &- \eta(X)g(Y,Z)\xi - g(\varphi X, Z)\varphi Y + g(\varphi Y, Z)\varphi X + 2g(X,\varphi Y)\varphi Z\}.\end{aligned}$$

16.1. δ-Casorati curvature of submanifolds in pointwise Kenmotsu space forms

For submanifolds in a pointwise Kenmotsu space form, we have the following result of Lone et al.

Theorem 16.1 *[89] Let M^n be a submanifold of a $(2m+1)$-dimensional (pointwise) Kenmotsu space form $\overline{M}^{2m+1}(c)$. If the structure vector field ξ is tangent to M^n, then*

(i) For a real number $r \in (0, n(n-1))$, the generalized normalized δ-Casorati curvature $\delta_C(r; n-1)$ satisfies

$$\delta_C(r;n-1) \geq n(n-1)\left(\rho - \frac{c-3}{4}\right) + \frac{(n-1)(c+1)}{2} - \frac{3(c+1)}{4}||P||^2. \tag{16.1}$$

(ii) For any real number $r > n(n-1)$, the generalized normalized δ-Casorati curvature $\hat{\delta}_C(r;m)$ satisfies

$$\hat{\delta}_C(r;n-1) \geq n(n-1)\left(\rho - \frac{c-3}{4}\right) + \frac{(n-1)(c+1)}{2} - \frac{3(c+1)}{4}||P||^2. \tag{16.2}$$

In addition, the equality cases of (16.1) and (16.2) hold identically at a point $p \in M^n$ if and only if p is a totally geodesic point.

Theorem 16.1 implies the following corollary.

Corollary 16.2 *[76, 89] Let M^n be a submanifold of a pointwise Kenmotsu space form $\overline{M}^{2m+1}(c)$. If the structure vector field ξ is tangent to M, then*

(i) The normalized δ-Casorati curvature $\delta_C(n-1)$ satisfies

$$\delta_C(n-1) \geq \rho + \frac{c+1}{2n} - \frac{c-3}{4} - \frac{3(c+1)}{4n(n-1)}||P||^2. \tag{16.3}$$

(ii) The normalized δ-Casorati curvature $\hat{\delta}_C(m)$ satisfies

$$\hat{\delta}_C(n-1) \geq \rho + \frac{c+1}{2n} - \frac{c-3}{4} - \frac{3(c+1)}{4n(n-1)}||P||^2. \tag{16.4}$$

Moreover, the equality cases of (16.3) and (16.4) hold identically at a point $p \in M^n$ if and only if p is a totally geodesic point.

Remark 16.3 *It is important to notice that although both the generalized normalized δ-Casorati curvatures $\delta_C(r;n-1)$ and $\hat{\delta}_C(r;n-1)$ in Theorem 16.1 and Corollary 16.2 satisfy the same inequality, they are in general different, except the case when the submanifold is totally geodesic (see Remark 2 of [76]).*

For a θ-slant submanifold M^n of a pointwise Kenmotsu space form $\overline{M}^{2m+1}(c)$ such that the structure vector field ξ is tangent to M^n, we have

$$||P||^2 = n\cos^2\theta. \tag{16.5}$$

It follows from Theorem 16.1 and (16.5) that the following result holds.

Corollary 16.4 *[76, 89] Let M^n be a θ-slant submanifold of a pointwise Kenmotsu space form $\overline{M}^{2m+1}(c)$. If the structure vector field ξ is tangent to M, then we have*

(i) The normalized δ-Casorati curvature $\delta_C(n-1)$ satisfies

$$\rho \leq \delta_C(n-1) + \frac{c-3}{4} + \frac{3(c+1)}{4n}\cos^2\theta - \frac{c+1}{2n}. \tag{16.6}$$

(ii) The normalized δ-Casorati curvature $\hat{\delta}_C(n-1)$ satisfies

$$\rho \leq \hat{\delta}_C(n-1) + \frac{c-3}{4} + \frac{3(c+1)}{4n}\cos^2\theta - \frac{c+1}{2n}. \tag{16.7}$$

Moreover, the equality cases of (16.6) and (16.7) hold identically at a point $p \in M^n$ if and only if p is a totally geodesic point.

16.2. Slant submanifolds in Kenmotsu space forms

For slant submanifolds in Kenmotsu space forms, we have the following result of Lone.

Theorem 16.5 *[85] Let M^n be a θ-slant submanifold of a Kenmotsu space form $\overline{M}^{2m+1}(c)$. Then*

(i) For a real number $r \in (0, n(n-1))$, the generalized normalized δ-Casorati curvature $\delta_C(r;n-1)$ satisfies

$$\rho \leq \frac{\delta_C(r;n-1)}{n(n-1)} + \frac{c-3}{4} + \frac{3(c+1)}{4n}\cos^2\theta + \frac{c+1}{2n}||\xi^T||^2. \tag{16.8}$$

(ii) For any real number $r > n(n-1)$, the generalized normalized δ-Casorati curvature $\hat{\delta}_C(r;n-1)$ satisfies

$$\rho \leq \frac{\hat{\delta}_C(r;n-1)}{n(n-1)} + \frac{c-3}{4} + \frac{3(c+1)}{4n}\cos^2\theta + \frac{c+1}{2n}||\xi^T||^2. \tag{16.9}$$

The equality cases of (16.8) and (16.9) hold if and only if there exists an orthonormal frame $e_1,\ldots,e_n$, $\xi_{n+1},\ldots,\xi_{2m+1}$ such that with respect to this frame the shape operator takes the form (13.5).

Remark 16.6 *Theorem 16.5 was extended to bi-slant submanifolds by Lone in [88].*

17. Submanifolds in generalized (κ,μ)–space forms

17.1. Generalized (κ,μ)-space forms

A contact metric manifold $\left(\overline{M}^{2m+1},\varphi,\xi,\eta,g\right)$ is called *a generalized* (κ,μ)*-space* if its curvature tensor satisfies

$$\overline{R}\left(X,Y\right)\xi=\kappa\left\{\eta\left(Y\right)X-\eta\left(X\right)Y\right\}+\mu\left\{\eta\left(Y\right)hX-\eta\left(X\right)hY\right\}$$

for some functions κ,μ on $\overline{M}$, where $h=\frac{1}{2}\mathcal{L}_{\xi}\varphi$ and $\mathcal{L}$ denotes the Lie derivative. If κ,μ are constant, $\overline{M}^{2m+1}$ is called a (κ,μ)*-space*. In particular, if a (κ,μ)-space has constant φ-sectional curvature, then it is said to be a (κ,μ)*–space form*.

In [18], Carriazo et al. defined a generalized (κ,μ)–space form as an almost contact metric manifold $(\overline{M}^{2m+1},\varphi,\xi,\eta,g)$ whose curvature tensor $\overline{R}$ satisfies

$$\overline{R}(X,Y)Z=f_1R_1(X,Y)Z+f_2R_2(X,Y)Z+f_3R_3(X,Y)Z+f_4R_4(X,Y)Z+f_5R_5(X,Y)Z+f_6R_6(X,Y)Z,$$

where $f_1,\dots,f_6$ are smooth functions and $R_1,\dots,R_6$ are tensor fields defined by

$$\begin{aligned}
R_1(X,Y)Z&=g(Y,Z)X-g(X,Z)Y,\\
R_2(X,Y)Z&=g(X,\varphi Z)\varphi Y-g(Y,\varphi Z)\varphi X+2g(X,\varphi Y)\varphi Z,\\
R_3(X,Y)Z&=\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi,\\
R_4(X,Y)Z&=g(Y,Z)hX-g(X,Z)hY+g(hY,Z)X-g(hX,Z)Y,\\
R_5(X,Y)Z&=g(hY,Z)hX-g(hX,Z)hY+g(\varphi hX,Z)\varphi hY-g(\varphi hY,Z)\varphi hX,\\
R_6(X,Y)Z&=\eta(X)\eta(Z)hY-\eta(Y)\eta(Z)hX+g(hX,Z)\eta(Y)\xi-g(hY,Z)\eta(X)\xi.
\end{aligned}$$

17.2. δ-Casorati curvatures in generalized (κ,μ)–space forms

For submanifolds of generalized (κ,μ)–space forms, Aquib and Shahid proved the following.

Theorem 17.1 *[6] Let M^n be a submanifold of a generalized (κ,μ)–space form $\overline{M}^{2m+1}$. Then*

(i) For a real number $r\in(0,n(n-1))$, the generalized normalized δ-Casorati curvature $\delta_C(r;n-1)$ satisfies

$$\begin{aligned}\rho\le{}&\frac{\delta_C(r;n-1)}{n(n-1)}+f_1+\frac{3}{n(n-1)}f_2||P||^2-\frac{2}{n}f_3+\frac{2}{n}f_4\mathrm{tr}\,(h^T)\\&+\frac{1}{n(n-1)}f_5\big\{(\mathrm{tr}\,(h^T))^2-||h^T||^2-||(\varphi h)^T||^2-(\mathrm{tr}\,(\varphi h)^T))^2\big\}-\frac{2}{n(n-1)}f_6\mathrm{tr}\,(h^T).\end{aligned}\tag{17.1}$$

(ii) For any real number $r>n(n-1)$, the generalized normalized δ-Casorati curvature $\hat{\delta}_C(r;n-1)$ satisfies

$$\begin{aligned}\rho\le{}&\frac{\hat{\delta}_C(r;n-1)}{n(n-1)}+f_1+\frac{3}{n(n-1)}f_2||P||^2-\frac{2}{n}f_3+\frac{2}{n}f_4\mathrm{tr}\,(h^T)\\&+\frac{1}{n(n-1)}f_5\big\{(\mathrm{tr}\,(h^T))^2-||h^T||^2-||(\varphi h)^T||^2-(\mathrm{tr}\,(\varphi h)^T))^2\big\}-\frac{2}{n(n-1)}f_6\mathrm{tr}\,(h^T).\end{aligned}\tag{17.2}$$

The equality cases of (17.1) *and* (17.2) *hold if and only if there exists an orthonormal frame* $e_1, \ldots, e_n$, $\xi_{n+1}, \ldots, \xi_{2m+1}$ *such that with respect to this frame the shape operator takes the form* (13.5).

As special cases of Theorem 17.1, Aquib and Shahid derived the corresponding results in [6] for (i) submanifolds of (κ, μ)–space forms, (ii) bi-slant submanifolds of generalized (κ, μ)–space forms, (iii) CR-submanifolds of generalized (κ, μ)–space forms, and (iv) slant submanifolds of generalized (κ, μ)–space forms.

Remark 17.2 *Hui et al. proved independently in [72] the special case of Theorem 17.1 for* $\delta_C(n-1)$ *and* $\hat{\delta}_C(n-1)$. *They also derived the corresponding result for slant submanifolds.*

18. δ-Casorati curvatures in statistical space forms

The notion of statistical manifolds was introduced by Amari [1] in 1985, which provided a setting for the field of information geometry and it also associates a dual connection (known as conjugate connection). The nice applications of statistical manifolds in applied science and engineering have attracted the attention of many geometers. The theory of statistical model as statistical manifold is a fast growing research subject in differential geometry. Many articles have been published in the setting of statistical manifold in recent years.

18.1. Statistical manifolds and statistical space forms

Let $(\widetilde{M}, \tilde{g})$ be a Riemannian manifold with Levi–Civita connection $\widetilde{\nabla}^0$. For a torsion free affine connection $\widetilde{\nabla}$ on $(\widetilde{M}, \tilde{g})$, let $\widetilde{\nabla}^*$ be the torsion free connection defined by

$$Z\tilde{g}\,(X,Y) = \tilde{g}\big(\widetilde{\nabla}_Z X, Y\big) + \tilde{g}\big(X, \widetilde{\nabla}^*_Z Y\big), \tag{18.1}$$

which is called the dual connection of $\widetilde{\nabla}$ with respect to $\tilde{g}$. It is easily shown that $(\widetilde{\nabla}^*)^* = \widetilde{\nabla}$. The Riemannian manifold $(\widetilde{M}, \tilde{g})$ equipped with a such pair of torsion free affine connections $\widetilde{\nabla}$, $\widetilde{\nabla}^*$ is called a statistical manifold. And the pair $(\widetilde{\nabla}, \tilde{g})$ is called a statistical structure on $\tilde{M}$. If $(\widetilde{\nabla}, \tilde{g})$ is a statistical structure on $\widetilde{M}$, then $(\widetilde{\nabla}^*, \tilde{g})$ is also a statistical structure. For the statistical manifold we have

$$\widetilde{\nabla} + \widetilde{\nabla}^* = 2\widetilde{\nabla}^0, \tag{18.2}$$

A statistical structure $(\widetilde{\nabla}, \tilde{g})$ is said to be of constant curvature c if

$$\widetilde{R}^{\widetilde{\nabla}}(X,Y)Z = c\{\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y\} \tag{18.3}$$

holds, where $\widetilde{R}^{\widetilde{\nabla}}$ denotes the curvature tensor associated with $\widetilde{\nabla}$. A statistical structure $(\widetilde{\nabla}, \tilde{g})$ of constant curvature 0 is called a Hessian structure (cf. e.g., [44, 63]).

Since the curvature tensor $\widetilde{R}^{\widetilde{\nabla}}$ and $\widetilde{R}^{\widetilde{\nabla}^*}$ of the dual connections $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ on $\widetilde{M}$ satisfy

$$g(\widetilde{R}^{\widetilde{\nabla}^*}(X,Y)Z, W) = -\tilde{g}(Z, \widetilde{R}^{\widetilde{\nabla}}(X,Y)W).$$

it follows that if $(\widetilde{\nabla}, \tilde{g})$ is a statistical structure of constant curvature c, then $(\widetilde{\nabla}^*, \tilde{g})$ is also a statistical structure of constant curvature c. In particular, if $(\widetilde{\nabla}, \tilde{g})$ is Hessian, then $(\widetilde{\nabla}^*, \tilde{g})$ is also Hessian (cf. [102]).

18.2. Basics on statistical submanifolds in statistical manifolds

Let M^n be an n-dimensional submanifold of a statistical m-manifold $(\widetilde{M}^m, \tilde{g})$, then (M^n, g) is also a statistical manifold with the induced connection by ∇ and the induced metric g. The fundamental equations for statistical submanifolds have been derived by Vos [118] in 1989 as follows.

For tangent vector fields X, Y of M^n the corresponding Gauss formulas are

$$\widetilde{\nabla}_X Y = \nabla_X Y + \zeta(X,Y), \quad \widetilde{\nabla}^*_X Y = \nabla^*_X Y + \zeta^*(X,Y), \tag{18.4}$$

where ζ, ζ^* are symmetric and bilinear, called the second fundamental forms (cf. [11]) or imbedding curvature tensors (cf. [57]). Since ζ and ζ^* are bilinear, there exist linear transformations Λ_ξ and Λ^*_ξ on TM^n, known as the shaper operators, defined by

$$g(\Lambda_\xi X, Y) = \tilde{g}(\zeta(X,Y), \xi), \quad g(A^*_\xi X, Y) = \tilde{g}(\zeta^*(X,Y), \xi), \tag{18.5}$$

for any normal vector field ξ. Further, the corresponding Weingarten formulas are given by

$$\widetilde{\nabla}_X \xi = -\Lambda_\xi X + \nabla^\perp_X \xi, \quad \widetilde{\nabla}^*_X \xi = -\Lambda^*_\xi X + \nabla^{*\perp}_X \xi. \tag{18.6}$$

Let R and $\widetilde{R}$ denote the curvature tensor fields of ∇ and $\widetilde{\nabla}$, respectively. Then the *Gauss equation* for $\widetilde{\nabla}$ is given by (cf. [118])

$$\tilde{g}(\widetilde{R}(X,Y)Z,W) = g(R(X,Y)Z,W) + \tilde{g}(h(X,Z), h^*(Y,W)) - \tilde{g}(h^*(X,W), h(Y,Z)). \tag{18.7}$$

Similarly, the Gauss equation for $\tilde{\nabla}^*$ is given by

$$\tilde{g}(\widetilde{R}^*(X,Y)Z,W) = g(R^*(X,Y)Z,W) + \tilde{g}(h^*(X,Z), h(Y,W)) - \tilde{g}(h(X,W), h^*(Y,Z)), \tag{18.8}$$

where R^* and $\widetilde{R}^*$ denote the curvature tensor fields of ∇^* and $\tilde{\nabla}^*$, respectively.

Let S denote the *statistical curvature tensor field of a statistical manifold* (M, g, ∇), where S is defined by [64]

$$S(X,Y)Z = \frac{1}{2}\{R(X,Y)Z + R^*(X,Y)Z\}. \tag{18.9}$$

If $\pi = \mathrm{Span}_{\mathbb{R}}\{u_1, u_2\}$ is a plane section of T_pM, then the *sectional curvature* of M is defined by [64]:

$$K_S(\pi) = \frac{g(S(u_1,u_2)u_2, u_1)}{g(u_1,u_1)g(u_2,u_2) - g^2(u_1,u_2)}. \tag{18.10}$$

Let $\{e_1, ..., e_n\}$ and $\{\xi_{n+1}, ..., \xi_m\}$ be orthonormal bases of the tangent space T_pM^n and of the normal bundle $T^\perp_p M^n$, respectively. Then scalar curvature τ and normalized scalar curvature ρ of M^n are given by

$$\tau(p) = \sum_{1\le i<j\le n} K_S(e_i \wedge e_j), \quad \rho = \frac{2\tau}{n(n-1)}. \tag{18.11}$$

And the mean curvature vectors H and H^* of M^n with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ are given respectively by

$$H = \frac{1}{n}\sum_{i=1}^n h(e_i,e_i) = \frac{1}{n}\sum_{\alpha=n+1}^m \left(\sum_{i=1}^n \zeta^\alpha_{ii}\right)\xi_\alpha, \quad H^* = \frac{1}{n}\sum_{i=1}^n \zeta^*(e_i,e_i) = \frac{1}{n}\sum_{\alpha=n+1}^m \left(\sum_{i=1}^n \zeta^{*\alpha}_{ii}\right)\xi_\alpha, \tag{18.12}$$

where $1 \le i,j \le n$ and $n+1 \le \alpha \le m$, $\zeta_{ij}^{\alpha} = \tilde{g}(\zeta(e_i,e_j),\xi_\alpha)$ and $\zeta_{ij}^{*\alpha} = \tilde{g}(\zeta^*(e_i,e_j),\xi_\alpha)$.

The Casorati curvatures $\mathcal{C}$ and $\mathcal{C}^*$ of the submanifold M^n are defined as

$$\mathcal{C} = \frac{1}{n}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n}(\zeta_{ij}^{\alpha})^2, \quad \mathcal{C}^* = \frac{1}{n}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n}(\zeta_{ij}^{*\alpha})^2.$$

The Casorati curvatures of an r-plane field L, spanned by $\{e_{q+1},\dots,e_{q+r}\}$, $q < n-\ell$, $r \ge 2$, are defined by

$$\mathcal{C}(L) = \frac{1}{\ell}\sum_{\alpha=n+1}^{m}\left(\sum_{i,j=k+1}^{k+\ell}(\zeta_{ij}^{\alpha})^2\right), \quad \mathcal{C}^*(L) = \frac{1}{\ell}\sum_{\alpha=n+1}^{m}\left(\sum_{i,j=k+1}^{k+\ell}(\zeta_{ij}^{*\alpha})^2\right). \tag{18.13}$$

18.3. δ-Casorati curvatures for submanifolds in statistical space forms

The modified normalized δ-Casorati curvature $\tilde{\delta}_C(n-1)$ and the normalized δ-Casorati curvature $\hat{\delta}_C(n-1)$ of M^n are given as follows (see Section 5):

$$\left[\tilde{\delta}_C(n-1)\right]_p = \frac{1}{2}\mathcal{C}_p + \frac{(n+1)}{2n(n-1)}\inf\{\mathcal{C}(L) \mid L \text{ a hyperplane of } T_pM^n\}, \tag{18.14}$$

$$\left[\hat{\delta}_{\mathcal{C}}(n-1)\right]_p = 2\mathcal{C}_p - \frac{(2n-1)}{2n}\sup\{\mathcal{C}(L) \mid L \text{ a hyperplane of } T_pM^n\}. \tag{18.15}$$

The same definitions for $\left[\tilde{\delta}^*_{\mathcal{C}}(n-1)\right]_p$ and $\left[\hat{\delta}^*_{\mathcal{C}}(n-1)\right]_p$. Similarly, the same definitions of $\delta_C(r;n-1), \hat{\delta}_C(r;n-1)$ and $\delta^*_C(r;n-1), \hat{\delta}^*_C(r;n-1)$ given in Subsection 5.2 applied to statistical submanifolds as well. For simplicity, put

$$\delta^o_C(r;n-1) = \frac{1}{2}\{\delta_C(r;n-1) + \delta^*_C(r;n-1)\}, \ \ H^o = \frac{1}{2}(H+H^*), \ \ \mathcal{C}^o = \frac{1}{2}(\mathcal{C}+\mathcal{C}^*). \tag{18.16}$$

For statistical submanifolds of statistical space forms, Bansal et al. obtained the following two results.

Theorem 18.1 *[11] Let M^n be a statistical submanifold of a statistical space form $\widetilde{M}^m(c)$ of constant curvature c. Then the generalized normalized δ-Casorati curvature $\delta^o_C(r;n-1)$ satisfies*

$$\rho \le \frac{2\delta^o_C(r;n-1)}{n(n-1)} + \frac{\mathcal{C}^o}{n-1} - \frac{2n}{n-1}||H^o||^2 + \frac{n}{n-1}\tilde{g}(H,H^*) + c. \tag{18.17}$$

Theorem 18.2 *[11] Let M^n be a statistical submanifold of a statistical space form $\widetilde{M}^m(c)$ of constant curvature c. Then the generalized normalized δ-Casorati curvature $\delta^o_C(r;n-1)$ satisfies*

$$\rho \le -\frac{\delta^o_C(r;n-1)}{n(n-1)} + \frac{2n}{n-1}||H^o||^2 - \frac{2\mathcal{C}^o}{n-1} + c. \tag{18.18}$$

Remark 18.3 *Theorem 18.1 and Theorem 18.2 imply that the normalized scalar curvature ρ of M^n was bounded above and below by* (18.14) *and* (18.15)*, respectively.*

In particular, for the normalized δ-Casorati curvature $\delta^o_C(n-1)$, the following two results are obtained by Lee et al. in [79], and Cai et al. in [16].

Theorem 18.4 *[16, 79] Let M^n be a statistical submanifold of a statistical space form $\widetilde{M}^m(c)$ of constant curvature c. Then the normalized δ-Casorati curvature $\delta^o_C(r; n-1)$ satisfies*

$$\rho \leq 2\delta^o_C(n-1) + \frac{\mathcal{C}^o}{n-1} - \frac{2n}{n-1}||H^o||^2 + \frac{n}{n-1}\tilde{g}(H, H^*) + c. \tag{18.19}$$

Theorem 18.5 *[16, 79] Let M^n be a statistical submanifold of a statistical space form $\widetilde{M}^m(c)$ of constant curvature c. Then the normalized δ-Casorati curvature $\delta^o_C(r; n-1)$ satisfies*

$$\rho \leq -\frac{1}{2}\delta^o_C(n-1) + \frac{2n}{n-1}||H^o||^2 - \frac{2\mathcal{C}^o}{n-1} + c. \tag{18.20}$$

18.4. δ-Casorati curvatures in statistical complex space forms

Let $(\widetilde{M}, J)$ be an almost complex manifold. A quadruplet $(\widetilde{M}, \widetilde{\nabla}, \tilde{g}, J)$ is called a holomorphic statistical manifold if

(a) $(\widetilde{\nabla}, \tilde{g})$ is a statistical structure on $\tilde{M}$, and

(b) ω is a $\widetilde{\nabla}$-parallel 2-form on $\tilde{M}$,

where ω is defined by $\omega(X, Y) = \tilde{g}(X, JY)$ for X, Y tangent to $T\widetilde{M}$.

For a holomorphic statistical manifold, we have

$$\tilde{g}(\tilde{S}(Z, W)JY, JX) = \tilde{g}(\tilde{S}(JZ, JW)Y, X) = \tilde{g}(\tilde{S}(Z, W)Y, X). \tag{18.21}$$

A holomorphic statistical manifold $(\tilde{M}, \tilde{\nabla}, \tilde{g}, J)$ is said to be of constant holomorphic sectional curvature c if the following formula holds (cf. [65]):

$$\tilde{S}(X, Y)Z = \frac{c}{4}\{\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y + \tilde{g}(JY, Z)JX - \tilde{g}(JX, Z)JY + 2\tilde{g}(X, JY)JZ\}, \tag{18.22}$$

where $\tilde{S}$ is the statistical curvature tensor field of $\tilde{M}$.

For statistical submanifolds of holomorphic statistical manifolds of constant holomorphic sectional curvature, Decu et al. proved the following.

Theorem 18.6 *[56]. Let M be an n-dimensional statistical submanifold of a $2m$-dimensional holomorphic statistical manifold $(\tilde{M}, \tilde{\nabla}, \tilde{g}, J)$ of constant holomorphic sectional curvature c. Then we have*

(i) For any real number r such that $0 < r < n(n-1)$,

$$2\tau \leq \delta^0_{\mathcal{C}}(r; n-1) + n\mathcal{C}^0 - 2n^2\|H^0\|^2 + n^2\tilde{g}(H, H^*) + \frac{3c}{4}\|P\|^2 + \frac{c}{4}n(n-1). \tag{18.23}$$

(ii) For any real number $r > n(n-1)$,

$$2\tau \leq \hat{\delta}^0_{\mathcal{C}}(r; n-1) + nC^0 - 2n^2\|H^0\|^2 + n^2\tilde{g}(H, H^*) + \frac{3c}{4}\|P\|^2 + \frac{c}{4}n(n-1). \tag{18.24}$$

Moreover, the equality cases of (18.23) and (18.24) hold identically if and only if the second fundamental forms ζ and ζ^* with respect to the dual connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ satisfy $\zeta + \zeta^* = 0$.

Remark 18.7 *Decu et al. also provided in [56] an example which satisfies the equality cases of* (18.23) *and* (18.24) *identically.*

Theorem 18.6 implies the following.

Corollary 18.8 *Let M be an n-dimensional statistical submanifold of a $2m$-dimensional holomorphic statistical manifold $(\tilde{M}, \tilde{\nabla}, \tilde{g}, J)$ of constant holomorphic sectional curvature c. Then, we have*

$$\rho \leq \delta^0_{\mathcal{C}}(m-1) + \frac{1}{m-1}\mathcal{C}^0 - \frac{2m}{m-1}\|H^0\|^2 + \frac{m}{m-1}\tilde{g}(H, H^*) + \frac{3c}{4m(m-1)}\|P\|^2 + \frac{c}{4}, \tag{18.25}$$

$$\rho \leq \hat{\delta}^0_{\mathcal{C}}(m-1) + \frac{1}{m-1}\mathcal{C}^0 - \frac{2m}{m-1}\|H^0\|^2 + \frac{m}{m-1}\tilde{g}(H, H^*) + \frac{3c}{4m(m-1)}\|P\|^2 + \frac{c}{4}. \tag{18.26}$$

Moreover, the equality cases of (18.25) *and* (18.26) *hold identically if and only if ζ and ζ^* satisfy $\zeta + \zeta^* = 0$, which implies that M is a totally geodesic submanifold with respect to the Levi-Civita connection.*

18.5. δ-Casorati curvatures in Kenmotsu statistical space forms

For a statistical structure $(\tilde{\nabla}, \tilde{g})$ on $\widetilde{M}$, put $\tilde{K} = \tilde{\nabla} - \nabla^0$. Then $\tilde{K}$ satisfies (cf. [57])

$$\tilde{K}_X Y = \tilde{K}_Y X, \quad \tilde{g}(K_X Y, Z) = \tilde{g}(Y, K_X Z).$$

Let $(\widetilde{M}, \tilde{g}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$ be a Kenmotsu manifold and $(\tilde{\nabla}, \tilde{g})$ a statistical structure on $\widetilde{M}$. Then the quadruple $(\tilde{\nabla}, \tilde{g}, \tilde{\varphi}, \tilde{\xi})$ is called a *Kenmotsu statistical structure* on $\tilde{M}$ if

$$\tilde{K}(X, \varphi Y) + \varphi \tilde{K}(X, Y) = 0 \tag{18.27}$$

holds for vector fields X, Y tangent to $\tilde{M}$. A manifold equipped with a Kenmotsu statistical structure is called a Kenmotsu statistical manifold. A Kenmotsu statistical manifold $(\widetilde{M}, \tilde{g}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$ is called a Kenmotsu statistical space form if it has constant φ-sectional curvature.

Decu et al. [57] proved the following result.

Theorem 18.9 *Let M^n be a statistical submanifold of a Kenmotsu statistical-space form $(\widetilde{M}^{2m+1}(c), \tilde{g}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$ of constant φ-sectional curvature c. Then*

(i) For $0 < r < n(n-1)$, the generalized normalized δ-Casorati curvature $\delta^o_C(r; n-1)$ satisfies

$$2\tau \leq \delta^o_C(r; n-1) + n\mathcal{C}^o - 2n^2||H^o||^2 + n^2\tilde{g}(H, H^*) + \frac{3(c+1)}{4}||P||^2 + \frac{n-1}{4}[(n-2)c - 3(n-1) - 5]. \tag{18.28}$$

(ii) For $r > n(n-1)$, the generalized normalized δ-Casorati curvature $\hat{\delta}^o_C(r; n-1)$ satisfies

$$2\tau \leq \hat{\delta}^o_C(r; n-1) + n\mathcal{C}^o - 2n^2||H^o||^2 + n^2\tilde{g}(H, H^*) + \frac{3(c+1)}{4}||P||^2 + \frac{n-1}{4}[(n-2)c - 3(n-1) - 5]. \tag{18.29}$$

In addition, the equality cases of (18.28) *and* (18.29) *hold identically if and only if the second fundamental forms ζ and ζ^* associated with the dual connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ satisfy $\zeta = -\zeta^*$.*

19. Some links between Chen and Casorati ideal submanifolds and principal directions

The Casorati operator A^C for a submanifold M^n in a Riemannian m-manifold $\widetilde{M}^m$ is a extrinsic operator. The eigenvectors of A^C are (extrinsic) principal Casorati directions. On the other hand, the eigenvectors of Ricci operator on a Riemannian manifold define (intrinsic) Ricci principal directions.

A submanifold of a Riemannian manifold is called pseudo-umbilical if the shape operator $A_{\overrightarrow{H}}$ at mean curvature vector $\overrightarrow{H}$ is proportional to the identity transformation (cf. [36]).

The following link between Casorati and Ricci principal directions was discovered by Haesen et al. in [70].

Theorem 19.1 *For submanifolds M^n in real space forms $\widetilde{M}^m(c)$, under each of the following three conditions the Casorati principal directions and Ricci principal directions do coincide: (1) M is minimal in $\widetilde{M}^m(c)$, (2) M is pseudo-umbilical in $\widetilde{M}^m(c)$, and (3) M has flat normal connection in $\widetilde{M}^m(c)$.*

For Chen ideal submanifolds, Decu et al. proved the following link between Casorati and Ricci principal directions.

Theorem 19.2 *[58] Let n and k be natural numbers satisfying $2k \le n$ and $n \ge 3$. For any $\delta(\overbrace{2,\dots,2}^{k})$-Chen ideal submanifold M^n in a Euclidean m-space $\mathbb{E}^m$, the principal Casorati directions and the principal Ricci directions coincide. In particular, if $n \ge 3$, then for any $\delta(2)$-Chen ideal submanifold M^n in $\mathbb{E}^m$, the principal Casorati directions and the principal Ricci directions coincide.*

Theorem 19.3 *[52] For Casorati ideal submanifold M^n with $n > 3$ in a real space form $R^m(c)$, the principal Casorati directions and the principal Ricci directions coincide.*

For a submanifold M^n in a Riemannian m-manifold $\widetilde{M}^m$, there is an operator $\mathfrak{a} : T^\perp M^n \to T^\perp M^n$ defined by $\mathfrak{a}(\xi) = \frac{1}{n}||\xi|| \sum_{\alpha=n+1}^{m} \operatorname{trace}(A_\xi A_\alpha)\xi_\alpha$ for $\xi \in T^\perp M^n$, where $\xi_{n+1},\dots,\xi_m$ is an orthonormal frame of the normal bundle. The vector $\mathfrak{a}(\xi)$ is called the allied normal vector field of ξ (see [36, p. 122]). An eigenvector of $\mathfrak{a}$ is called a normal principal Casorati vector in [91]. The following two results were proved by Moruz and Verstraelen in [91].

Theorem 19.4 *Let M^n be a Lagrangian submanifold of a Kaehler manifold $\widetilde{M}^m$. Then a vector $v \in TM^n$ is a principal Casorati vector with Casorati principal curvature $c^T > 0$ if and only if Jv is a normal principal Casorati vector with corresponding normal Casorati principal curvature $c^\perp = c^T > 0$.*

Theorem 19.5 *Let M^n be a Lagrangian submanifold of a Kaehler manifold $\widetilde{M}^m$ with first normal space of maximal dimension. Then M^n admits an adapted orthonormal frame field $\{F_1,\dots,F_n,\xi_{n+1} = JF_1,\dots,\xi_{2n} = JF_n\}$ in $\widetilde{M}^m$ of which the n normal vector fields are the principal Casorati normal vector fields of M^n in $\widetilde{M}^n$, and the corresponding tangential and normal principal curvatures are equal, that is, $c_i^T = c_i^\perp, i = 1,\dots,n$.*

In [59], Decu et al. proved the following.

Theorem 19.6 *On every Wintgen ideal submanifold in a real space form the Casorati and the Ricci principal directions do coincide.*

20. Further results on δ-Casorati curvatures and an open problem

There are further development in the last decade on δ-Casorati curvatures for submanifolds in various ambient manifolds.

(1) The notion quaternionic Kaehler-like statistical manifolds was introduced by Vîlcu and Vîlcu [116]. Aquib and Shahid [5] obtained the lower bounds for generalized normalized δ-Casorati curvatures $\delta^o_C(r;n-1)$ and $\hat{\delta}^o_C(r;n-1)$ of statistical submanifolds in quaternion Kaehler-like statistical space forms.

(2) Inequalities for generalized normalized δ-Casorati curvatures of statistical submanifolds in cosymplectic statistical space forms were obtained by Malek and Akbari [90]. Also, Aquib [2] proved inequalities for generalized normalized δ-Casorati curvatures for bi-slant submanifolds in T-space forms.

(3) Shahid and Siddiqui [99] obtained inequalities involving generalized normalized δ-Casorati curvatures on totally real submanifolds in LCS-manifolds. In [85], Lone obtained inequalities for δ-Casorati curvature of submanifolds in locally conformal Kaehler manifolds.

(4) Bansal and Shahid [10] obtained lower bounds of generalized normalized δ-Casorati curvatures for real hypersurfaces in the complex quadric. They also obtained similar results for real hypersurfaces in complex quadric endowed with semi-symmetric metric connection in [9]. Also, Park [96] proved two inequalities for the δ-Casorati curvatures of real hypersurfaces in some Grassmannians.

(5) Inequalities for generalized δ-Casorati curvatures of submanifolds in real space forms were extended by Lee et al. [78, 81] to submanifolds in real space form endowed with a semisymmetric metric connection.

(6) Inequalities for δ-Casorati curvatures of submanifolds in a Riemannian manifold of quasi-constant curvature equipped with a semisymmetric metric connection were obtained in [122] by Zhang and Zhang and also in [84] by Liu et al.

(7) Optimal inequalities for δ-Casorati curvatures of submanifolds in generalized complex space forms and also in generalized Sasakian space forms endowed with semisymmetric metric connections were given by Lee et al. [77] and Siddiqui [103]. Further, inequalities for δ-Casorati curvatures of submanifolds in (κ,μ)-space-forms endowed with semisymmetric metric connections were obtained by Hui et al. in [72].

(8) Optimal inequalities for δ-Casorati curvatures of submanifolds in generalized space forms endowed with semisymmetric nonmetric connections were obtain by He et al. [71].

(19) Siddiqui and Shahid [106] obtained optimal inequalities for generalized normalized δ-Casorati curvatures of statistical hypersurfaces in statistical complex space forms.

An open problem. *The δ-Casorati curvatures $\delta_C(n-1)$, $\hat{\delta}_C(n-1)$, $\delta_C(r;n-1)$ and $\hat{\delta}_C(r;n-1)$ introduced in [54, 55] by Decu et al. were in the spirit of δ-invariants $\delta(n-1)$ and $\hat{\delta}(n-1)$ for Riemannian n-manifolds M^n. In author's opinion it is quite interesting to define and study other Casorati curvatures such as $\delta_C(k)$, $\hat{\delta}_C(k)$, $\delta_C(r;k)$ and $\hat{\delta}_C(r;k)$ with $2 \le k < n-1$ for submanifolds M^n isometrically immersed in various space forms.*

21. Historical remarks

Casorati's first article on his curvature was [20]. The term of "Casorati curvature" was given by Verstraelen in his article [115] for his study on visual sensation surfaces in human vision using the critical values of $\kappa_1^2 + \kappa_2^2$ to determine the contours of the images. According to Verstraelen, he learned Casorati's first article from Bernard Rouxel (in a personal communication between the author and L. Verstraelen). The notion of Casorati curvature was extended by Haesen et al. in [70] to the tangential Casorati curvatures; and by Verstraelen in [113] to the normal Casorati curvatures. Also, δ-Casorati curvatures was defined for the first time in [54, 55] by Decu et al.

22. Conclusion

Author's δ-invariants introduced in the early 1990s are intrinsic invariants defined on Riemannian manifolds of dimension ≥ 3 which are quite different in natural from "classical" Ricci and scalar curvatures. The main feature of author's δ-invariants is to remove certain portions of sectional curvatures from the scalar curvature. Via δ-invariants the author was able to introduce the notion of ideal submanifolds in the sense that they receive the least amount of tension from the surrounding space. During the last three decades, many interesting results and applications in δ-invariants and ideal submanifolds have been achieved by many mathematicians.

On the other hand, in the spirit of δ-invariants, Decu et al. initiated the study of "extrinsic" δ-Casorati curvatures in 2007. Since then the study of δ-Casorati curvatures and related Casorati ideal submanifolds has attracting more and more researchers and a lot of interesting results have been obtained. It is author's intention that this comprehensive survey on δ-Casorati curvatures will provide a useful reference for graduate students and beginning researchers who want to study this subject, as well as for researchers who have already working in the field.

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