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Quasi-Cesàro matrix and associated sequence spaces

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Abstract: In the present study, we construct a new matrix which we call quasi-Cesàro matrix and is a generalization of the ordinary Cesàro matrix, and introduce BK-spaces C_k^q and C_∞^q as the domain of the quasi-Cesàro matrix C^q in the spaces ℓ_k and ℓ_∞, respectively. Furthermore, we exhibit some topological properties and inclusion relations related to these newly defined spaces. We determine the basis of the space C_k^q and obtain Köthe duals of the spaces C_k^q and C_∞^q. Based on the newly defined matrix, we present a factorization for the Hilbert matrix and generalize Hardy's inequality, as an application. Moreover we find the norm of this new matrix as an operator on several matrix domains.

Key words: Matrix operator, Hilbert matrix, Cesàro matrix, norm, sequence space

1. Introduction

Throughout this paper $1 \leq k < \infty$, unless mentioned otherwise, and s denote the set of all real-valued sequences. By sequence space, we mean any linear subspace of s. The Banach spaces ℓ_k and ℓ_∞ are the sets of all real sequences $x = (x_r)_{r=0}^\infty$ such that

$$\|x\|_{\ell_k} = \left(\sum_{v=0}^{\infty} |x_v|^k\right)^{1/k} < \infty \text{ and } \|x\|_{\ell_\infty} = \sup_{v\in\mathbb{N}} |x_v| < \infty, \tag{1.1}$$

respectively. A Banach space X is called BK space if it has continuous coordinates. The space ℓ_k is a BK space with respect to the ℓ_k-norm defined in (1.1). By ℓ_∞, c and c_0, we denote the spaces of all bounded, convergent and null sequences, respectively. Further bs and cs will denote the spaces of all bounded and convergent series. Throughout this paper, we use the notion $\mathbb{N}_0 = \{0, 1, 2, ...\}$.

Let X and Y be two sequence spaces and $\Phi = (\varphi_{rv})$ be an infinite matrix of real entries. We say that Φ defines a matrix mapping from X to Y if $\Phi x = \{(\Phi x)_v\} = \{\sum_{v=0}^\infty \varphi_{rv} x_v\} \in \mathsf{Y}$ for every sequence $x = (x_v) \in \mathsf{X}$. We call Φx as $\Phi-$transform of the sequence x. By (X, Y), we denote the family of all matrices that map from X to Y. Further the notion

$$\mathsf{X}_\Phi = \{x \in s : \Phi x \in \mathsf{X}\} \tag{1.2}$$

is called the domain of the matrix Φ in the space X. Moreover, X_Φ itself is a sequence space. Several authors have introduced new sequence spaces using the domain of some special triangular matrices. For relevant literature,

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one may see the papers [2, 3, 6, 7, 14, 20, 23, 24, 27] and textbooks [4, 18, 19]. For some recent publications, we refer [16, 17, 25, 26, 30–32].

If for every sequence $x \in \ell_k$, the inequality $\|\Phi x\|_{\ell_k} \leq K\|x\|_{\ell_k}$ is satisfied, then the operator Φ is called bounded in ℓ_k, where the constant K is independent of the choice of x. In this case the constant K is called an upper bound for the operator Φ and the smallest choice of K satisfying the relation $\|\Phi x\|_{\ell_k} \leq K\|x\|_{\ell_k}$ is called the norm of Φ. Throughout this paper, the notation $B(\mathsf{X} \to \mathsf{Y})$ shall represent the family of all bounded operators acting from the space X to the space Y. The upper bound and the norm of an operator plays important role in obtaining various inequalities in matrix domains. The celebrated Hardy's inequality and Hilbert's inequality

$$\sum_{r=0}^{\infty}\left(\sum_{v=0}^{r}\frac{|x_v|}{r+1}\right)^k \leq \left(\frac{k}{k-1}\right)^k\sum_{v=0}^{\infty}|x_v|^k \text{ and}$$

$$\sum_{r=0}^{\infty}\left(\sum_{v=0}^{\infty}\frac{|x_v|}{r+v+1}\right)^k \leq (\pi\csc(\pi/k))^k\sum_{v=0}^{\infty}|x_v|^k$$

are consequences of the boundedness of Cesàro and Hilbert operators, respectively.

Cesàro matrix. The infinite Cesàro operator $C=(c_{rv})$ is defined by

$$c_{rv} = \begin{cases} \frac{1}{r+1} & (0 \leq v \leq r), \\ 0 & \text{otherwise,} \end{cases}$$

for all $r, v \in \mathbb{N}_0$. Explicitly

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Further $\|C\|_{\ell_k} = k^*$, where $\frac{1}{k} + \frac{1}{k^*} = 1$.

Generalized Cesàro matrix. Let $Q \geq 1$ be a real number. Then, the generalized Cesàro matrix $C^Q = (c^Q_{rv})$ defined by

$$c^Q_{rv} = \begin{cases} \frac{1}{r+Q} & (0 \leq v \leq r), \\ 0 & \text{otherwise} \end{cases}$$

has the ℓ_k-norm $\|C^Q\|_{\ell_k} = k^*$ ([10, Lemma 2.3]). Note that $C^1 = C$, the well-known Cesàro matrix.

Quasi-Cesàro matrix. Let $(q_v)_{v=0}^{\infty}$ be a sequence with positive elements satisfying

$$\sup_v \frac{q_v}{v+1} = L < \infty \text{ and } \sup_v \frac{v+1}{q_v} = K < \infty, \tag{1.3}$$

where supremum runs over the set $\mathbb{N}_0$. Then the quasi-Cesàro operator $C^q = (c^q_{rv})$, generated by $(q_v)_{v=0}^{\infty}$, is defined by

$$c^q_{rv} = \begin{cases} \frac{1}{q_r} & (0 \leq v \leq r), \\ 0 & \text{otherwise} \end{cases}$$

or equivalently

$$C^q = \begin{pmatrix} 1/q_0 & 0 & 0 & \cdots \\ 1/q_1 & 1/q_1 & 0 & \cdots \\ 1/q_2 & 1/q_2 & 1/q_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Indeed quasi-Cesàro matrix C^q is invertible and its inverse $C^{-q} = (c_{rv}^{-q})$ is defined by

$$c_{rv}^{-q} = \begin{cases} q_v & (r = v), \\ -q_v & (r = v+1), \\ 0 & \text{otherwise}. \end{cases}$$

Define the diagonal matrix $D = (d_{rv})$ with entries $d_{vv} = \frac{q_v}{v+1}$ for all $v \in \mathbb{N}_0$. That is

$$D = \begin{pmatrix} q_0 & 0 & 0 & \cdots \\ 0 & \frac{q_1}{2} & 0 & \cdots \\ 0 & 0 & \frac{q_2}{3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.4}$$

Since D is diagonal, hence $\|D\|_{\ell_k} = \sup_v |d_{vv}| = \sup_v \frac{q_v}{v+1} = L$. It is easy to see that $C = DC^q$, where C is the Cesàro matrix. Hence we deduce that $\|C^q\|_{\ell_k} \geq \frac{1}{L}k^*$. Similarly, $C^q = D^{-1}C$ which results in $\|C^q\|_{\ell_k} \leq Kk^*$. Therefore, the quasi-Cesàro matrix is a bounded operator and

$$\frac{1}{L}k^* \leq \|C^q\|_{\ell_k} \leq Kk^*. \tag{1.5}$$

Further, we emphasize that in the special cases $q_v = v+1$ and $q_v = v+Q$, the quasi-Cesàro matrix C^q reduces to the well-known Cesàro and generalized Cesàro matrices C and C^Q, respectively.

We are ready to generalize Hardy's inequality.

Corollary 1.1 *Let $(q_v)_{v=0}^{\infty}$ satisfies the conditions (1.3). Then*

$$\sum_{r=0}^{\infty}\left(\sum_{v=0}^{r}\frac{|x_v|}{q_r}\right)^k \leq \left(\frac{Kk}{k-1}\right)^k \sum_{r=0}^{\infty}|x_r|^k.$$

In particular, for $q_v = v+1$, we get Hardy's inequality.

Proof Using relation (1.5), we obtain $\|C^q x\|_{\ell_k} \leq Kk^*\|x\|_{\ell_k}$ which is a generalization of Hardy's inequality. □

The Hilbert matrix $H = (h_{rv})$ is defined by

$$h_{rv} = \frac{1}{r+v+1} \quad (r, v \in \mathbb{N}_0).$$

It is known that H is a bounded operator on ℓ_k with $\|H\|_{\ell_k} = \pi\csc(\pi/k)$, ([13, Theorem 323]). As a consequence of Hardy's and Hilbert's inequalities, we have

$$\|Hx\|_{\ell_k} \leq \frac{\pi}{k^*}\csc(\pi/k)\|Cx\|_k, \tag{1.6}$$

which is Hardy's inequality versus Hilbert's.

2. Quasi-Cesàro sequence spaces C_k^q and C_∞^q

In this section, we introduce the sequence spaces C_k^q and C_∞^q, study their topological properties and some inclusion relations, and obtain the basis for the space C_k^q.

Before proceeding further, we define the sequence $y=(y_r)$ as the C^q-transform of the sequence $x=(x_r)$, that is

$$y_r=(C^q x)_r=\sum_{v=0}^{r}\frac{x_v}{q_r}, \tag{2.1}$$

for each $r\in\mathbb{N}_0$. Further on using (2.1), we define

$$x_v=q_v y_v-q_{v-1}y_{v-1} \tag{2.2}$$

for each $v\in\mathbb{N}_0$. Here, and in what follows, we use the conventions that any term with negative subscript like y_{-1}, q_{-1}, etc. shall be considered as naught.

Now we define the sequence spaces C_k^q and C_∞^q as follows:

$$C_k^q=\{x\in s: C^q x\in\ell_k\} \text{ and } C_\infty^q=\{x\in s: C^q x\in\ell_\infty\}.$$

One may observe that when $q_r=r+1$, the space C_k^q reduces to Cesàro sequence space X_k defined by Ng and Lee [20]. The above sequence spaces may also be defined in the notation of (1.2) by

$$C_k^q=(\ell_k)_{C^q} \quad\text{and}\quad C_\infty^q=(\ell_\infty)_{C^q}. \tag{2.3}$$

It is known that if X is BK space and Φ is a triangle, then X_Φ is also a BK space endowed with the norm $\|x\|_{\mathsf{X}_\Phi}=\|\Phi x\|_{\mathsf{X}}$. In the light of this and (2.3), we state that the sequence spaces C_k^q and C_∞^q are BK spaces under the norms defined by

$$\|x\|_{C_k^q}=\|C^q x\|_{\ell_k}=\left(\sum_{v=0}^{\infty}\left|\sum_{l=0}^{v}\frac{x_l}{q_v}\right|^k\right)^{1/k} \text{ and } \|x\|_{C_\infty^q}=\|C^q x\|_{\ell_\infty}=\sup_{v\in\mathbb{N}_0}\left|\sum_{l=0}^{v}\frac{x_l}{q_v}\right|,$$

respectively.

Theorem 2.1 *The spaces C_k^q and C_∞^q are linearly isomorphic to ℓ_k and ℓ_∞, respectively.*

Proof We define the mapping $\mathcal{T}: C_k^q\to\ell_k$ by $\mathcal{T}x=C^q x$ for all $x\in C_k^q$. It is easy to observe that $\mathcal{T}$ is linear and one-one. Let $y=(y_r)\in\ell_k$ and $x=(x_r)$ be defined as in (2.2). Then, for $1\le k<\infty$, we have

$$\begin{aligned}\|x\|_{C_k^q}&=\left(\sum_{r=0}^{\infty}\left|\sum_{v=0}^{r}\frac{x_v}{q_r}\right|^k\right)^{1/k}=\left(\sum_{r=0}^{\infty}\left|\sum_{v=0}^{r}\frac{1}{q_v}(q_v y_v-q_{v-1}y_{v-1})\right|^k\right)^{1/k}\\&=\left(\sum_{v=0}^{\infty}|y_v|^k\right)^{1/k}=\|y\|_{\ell_k}<\infty.\end{aligned}$$

Again for $k=\infty$, we have

$$\|x\|_{C_\infty^q}=\sup_{v\in\mathbb{N}_0}|(C^q x)_v|=\sup_{v\in\mathbb{N}_0}|y_v|=\|y\|_{\ell_\infty}<\infty.$$

Thus $x \in C^q_k$ for $1 \le k \le \infty$ and the mapping $\mathcal{T} : C^q_k \to \ell_k$ is onto and norm preserving. Hence, the spaces C^q_k and C^q_∞ are linearly isomorphic to ℓ_k and ℓ_∞, respectively. □

Theorem 2.2 *The space* C^q_k, $1 \le k \le \infty$, *is not a Hilbert space, except for the case* $k = 2$.

Proof We consider two sequences $x = (q_0, q_1 - q_0, -q_1, 0, \ldots)$ and $y = (q_0, -(q_0 + q_1), q_1, 0, \ldots)$. It is easy to check that $(C^q x)_r = (1, 1, 0, 0, \ldots)$ and $(C^q y)_r = (1, -1, 0, 0, \ldots)$. Then, we have

$$\|x + y\|^2_{C^q_k} + \|x - y\|^2_{C^q_k} = 8 \ne 2^{2+\frac{2}{k}} = 2\left(\|x\|^2_{C^q_k} + \|y\|^2_{C^q_k}\right).$$

Thus C^q_k norm violates the parallelogram law. Hence C^q_k is not a Hilbert space, except for the case $k = 2$. □

Theorem 2.3 *The inclusion* $C^q_k \subset C^q_\infty$ *strictly holds.*

Proof Since the inclusion $\ell_k \subset \ell_\infty$ holds, so the inclusion part is straightforward. To prove the strictness part, we consider the sequence $x = (q_0, -(q_1 + q_0), q_2 + q_1, -(q_3 + q_2), \ldots)$. Then one can easily verify that $C^q x = ((-1)^r) \in \ell_\infty \setminus \ell_k$. Eventually $x \in C^q_\infty \setminus C^q_k$. Therefore the inclusion $C^q_k \subset C^q_\infty$ strictly holds. □

Theorem 2.4 *Let* $1 \le k < t < \infty$. *Then the inclusion* $C^q_k \subset C^q_t$ *strictly holds.*

Proof It is known that the inclusion $\ell_k \subset \ell_t$ strictly holds for $1 \le k < t < \infty$. So the inclusion part is straight forward. To prove the strictness part, we choose $y \in \ell_t \setminus \ell_k$ and x as defined in (2.2), then $C^q x \in \ell_t \setminus \ell_k$. This implies $x \in C^q_t \setminus C^q_k$. Hence, the inclusion $C^q_k \subset C^q_t$ is strict. □

We recall that domain X_Φ of a triangle Φ has a basis if and only if X has a basis. This statement together with Theorem 2.1 gives us the following result:

Theorem 2.5 *Let* $\alpha_v = (C^q x)_v$ *for each* $v \in \mathbb{N}_0$. *We define the sequence* $b^{(r)}(q) = (b^{(r)}_v)(q)$ *of elements of the space* C^q_k *for every fixed* $r \in \mathbb{N}_0$ *by*

$$b^{(r)}_v(q) = \begin{cases} q_r & (v = r), \\ -q_r & (v = r + 1), \\ 0 & \textit{otherwise}. \end{cases}$$

Then the sequence $b^{(r)}(q)$ *forms a basis for the space* C^q_k *and every* $x \in C^q_k$ *can be uniquely expressed in the form* $x = \sum_{r=0}^{\infty} \alpha_r b^{(r)}(q)$ *for each* $k \in \mathbb{N}_0$.

3. Köthe duals

In this section we obtain Köthe duals (α-, β-, γ-duals) of the spaces C^q_k and C^q_∞. Since the proof for the cases $k = 1$ and $k = \infty$ is the same as that of $1 < k < \infty$, hence we provide the proof only for the later case. First we recall the definition of Köthe duals.

Definition 3.1 *The* $\alpha-$, $\beta-$ *and* $\gamma-$ *duals of subset* $\mathsf{X} \subset s$ *are defined by*

$$\mathsf{X}^{\alpha} = \{\varsigma = (\varsigma_r) \in s : \varsigma x = (\varsigma_r x_r) \in \ell_1 \text{ for all } x \in \mathsf{X}\},$$

$$\mathsf{X}^{\beta} = \{\varsigma = (\varsigma_r) \in s : \varsigma x = (\varsigma_r x_r) \in cs \text{ for all } x \in \mathsf{X}\},$$

$$\mathsf{X}^{\gamma} = \{\varsigma = (\varsigma_r) \in s : \varsigma x = (\varsigma_r x_r) \in bs \text{ for all } x \in \mathsf{X}\},$$

respectively.

Further $\mathcal{R}$ will denote the family of all finite subsets of $\mathbb{N}_0$. We state certain results due to Stielglitz and Tietz [29] that are necessary for our investigation.

Lemma 3.2 $\Phi = (\varphi_{rv}) \in (\ell_k, \ell_1)$ *if and only if*

$$\sup_{R \in \mathcal{R}} \sum_{v \in \mathbb{N}_0} \left| \sum_{r \in R} \varphi_{rv} \right|^{k^*} < \infty, \; 1 < k < \infty.$$

Lemma 3.3 $\Phi = (\varphi_{rv}) \in (\ell_k, c)$ *if and only if*

$$\lim_{r \to \infty} \varphi_{rv} \text{ exists for all } v \in \mathbb{N}_0, \tag{3.1}$$

$$\sup_{r \in \mathbb{N}_0} \sum_{v=0}^{\infty} |\varphi_{rv}|^{k^*} < \infty, \; 1 < k < \infty. \tag{3.2}$$

Lemma 3.4 $\Phi = (\varphi_{rv}) \in (\ell_k, \ell_\infty)$ *if and only if (3.2) holds.*

Theorem 3.5 *Let* $1 < k < \infty$ *and define the sets* $\mu^{[k^*]}(q)$ *by*

$$\mu^{[k^*]}(q) = \left\{ \varsigma = (\varsigma_r) \in s : \sum_{v=0}^{\infty} |q_v \varsigma_v|^{k^*} < \infty \right\},$$

$$\mu_\infty(q) = \left\{ \varsigma = (\varsigma_r) \in s : \sup_{v \in \mathbb{N}_0} |q_v \varsigma_v| < \infty \right\}.$$

Then $[C_1^q]^\alpha = \mu_\infty(q)$, $[C_k^q]^\alpha = \mu^{[k^*]}(q)$ *and* $[C_\infty^q]^\alpha = \mu^{[1]}(q)$.

Proof We give the proof for the space C_k^q, $1 < k < \infty$. Let $\varsigma = (\varsigma_r) \in s$ and $x = (x_r)$ be defined as in (2.2). Then, we have

$$\begin{aligned} \varsigma_v x_v &= \varsigma_v (q_v y_v - q_{v-1} y_{v-1}) \\ &= (\Lambda(q) y)_v \end{aligned} \tag{3.3}$$

for all $v \in \mathbb{N}_0$, where the matrix $\Lambda(q) = (\lambda_{rv}^q)$ is defined by

$$\lambda_{rv}^q = \begin{cases} q_v \varsigma_r & r = v, \\ -q_v \varsigma_r & r = v+1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Then we deduce from (3.3) that $\varsigma x \in \ell_1$ whenever $x \in C_k^q$ if and only if $\Lambda(q)y \in \ell_1$ whenever $y \in \ell_k$. This yields the fact that $\varsigma \in [C_k^q]^\alpha$ if and only if $\Lambda(q) \in (\ell_k, \ell_1)$. Thus by using Lemma 3.2, $\varsigma \in [C_k^q]^\alpha$ if and only if

$$\sup_{R\in\mathcal{R}} \sum_{v=0}^{\infty} \left| \sum_{r\in R} (-1)^{r-v} q_v \varsigma_r \right|^{k^*} < \infty. \tag{3.5}$$

Further, one may easily verify that the relation (3.5) is equivalent to the relation $\sum_{v=0}^{\infty} |q_v\varsigma_v|^{k^*} < \infty$. Thus we deduce that $[C_k^q]^\alpha = \mu^{[k^*]}(q)$. This completes the proof. □

Theorem 3.6 *Let* $1 < k < \infty$ *and define the sets* $\mu_{k^*}(q)$ *and* $\mu_0(q)$ *by*

$$\mu_{k^*}(q) = \left\{ \varsigma = (\varsigma_r) \in s : \sum_{v=0}^{\infty} |(\Delta\varsigma)_v q_v|^{k^*} < \infty \right\} \text{ and}$$
$$\mu_0(q) = \{\varsigma = (\varsigma_r) \in s : (\varsigma_v q_v) \in c_0\},$$

where $(\Delta\varsigma)_v = \varsigma_v - \varsigma_{v+1}$. *Then* $[C_1^q]^\beta = \mu_\infty(q)$, $[C_k^q]^\beta = \mu_\infty(q) \cap \mu_{k^*}(q)$ *and* $[C_\infty^q]^\beta = \mu_{k^*}(q) \cap \mu_0(q)$.

Proof Let $\varsigma = (\varsigma_v) \in s$ and $x = (x_v)$ be defined as in (2.2). Consider the following equality

$$\begin{aligned} \sum_{v=0}^{r} \varsigma_v x_v &= \sum_{v=0}^{r} \varsigma_v (q_v y_v - q_{v-1} y_{v-1}) \\ &= \sum_{v=0}^{r-1} (\varsigma_v - \varsigma_{v+1}) q_v y_v + \varsigma_r q_r y_r \qquad (3.6) \\ &= (\Omega(q)y)_v \qquad (3.7) \end{aligned}$$

for all $v \in \mathbb{N}_0$, where the matrix $\Omega(q) = (\omega_{rv}^q)$ is defined by

$$\omega_{rv}^q = \begin{cases} (\Delta\varsigma)_v q_v & (0 \le v < r), \\ \varsigma_v q_v & (v = r), \\ 0 & \text{otherwise}. \end{cases}$$

for all $r, v \in \mathbb{N}_0$. Since

$$\lim_{r\to\infty} \omega_{rv}^q = (\Delta\varsigma)_v q_v \tag{3.8}$$

for all $v \in \mathbb{N}_0$, $(\omega_{rv}^q)_{r=0}^{\infty} \in c$ for each $v \in \mathbb{N}_0$. Thus, we deduce from (3.7) that $\varsigma x \in cs$ whenever $x \in C_k^q$ if and only if $\Omega(q)y \in c$ whenever $y \in \ell_k$. This yields the fact that $\varsigma \in [C_k^q]^\beta$ if and only if $\Omega(q) \in (\ell_k, c)$. Thus by using Lemma 3.3, we get that $[C_k^q]^\beta = \mu_\infty(q) \cap \mu_{k^*}(q)$. This completes the proof. □

Theorem 3.7 *Let* $1 < k < \infty$. *Then,*
$[C_1^q]^\gamma = \mu_\infty(q)$, $[C_k^q]^\gamma = \mu_\infty(q) \cap \mu_{k^*}(q)$ *and* $[C_\infty^q]^\gamma = \mu_\infty(q) \cap \mu_0(q)$.

Proof The proof can be established by replacing Lemma 3.3 by Lemma 3.4 in the proof of the above theorem. Hence we omit details. □

4. Factorization of Hilbert matrix based on quasi-Cesàro matrix

Bennett [9] factorized the Hilbert matrix H in the form $H = BC$, where C is the Cesàro matrix and $B = (b_{rv})$ is defined by

$$b_{rv} = \frac{v+1}{(r+v+1)(r+v+2)} \quad (r, v = 0, 1, \ldots). \tag{4.1}$$

The matrix B is bounded on ℓ_k with $\|B\|_{\ell_k} = \frac{\pi}{k^*}\csc(\pi/k)$, ([9, Proposition 2]).

More recently, Roopaei [21] generalized Bennett's result and obtained another factorization of the Hilbert matrix which is summarized below:

Theorem 4.1 ([21], Theorem 2.2) *The Hilbert matrix H admits a factorization of the form $H = B^Q C^Q$, where $B^Q = (b^Q_{rv})$ has the entries*

$$b^Q_{rv} = \frac{v+Q}{(r+v+1)(r+v+2)} \quad (r, v \in \mathbb{N}_0). \tag{4.2}$$

and is bounded on ℓ_k with bounds

$$\frac{\pi}{k^*}\csc(\pi/k) \leq \|B^Q\|_{\ell_k} \leq \frac{\pi Q}{k^*}\csc(\pi/k).$$

In particular, when $Q = 1$, we arrive at Bennet's factorization [9, Proposition 2].

Let us define the matrix $B^q = (b^q_{rv})$ by

$$b^q_{rv} = \frac{q_v}{(r+v+1)(r+v+2)} \quad (r, v \in \mathbb{N}_0), \tag{4.3}$$

where $q_v \geq v+1$.

We observe that when $q_v = v+1$ and $q_v = v+Q$, B^q reduces to B and B^Q, where B and B^Q are defined by the relations (4.1) and (4.2), respectively.

It is obvious that $\|B^q\|_{\ell_k} \geq \|B\|_{\ell_k}$, but we will prove that the matrix B^q is a bounded operator on ℓ_k. We need the following theorem, also known as Schur's theorem, for obtaining our result.

Theorem 4.2 *[13, Theorem 275] Let $k > 1$ and $\Phi = (\varphi_{rv})$ be a matrix with $\varphi_{rv} \geq 0$ for all r, v. Suppose that S and T are two positive numbers satisfying*

$$\sum_{r=0}^{\infty} \varphi_{rv} \leq S \quad \textit{for all } k, \quad \sum_{v=0}^{\infty} \varphi_{rv} \leq T \quad \textit{for all } r,$$

(bounds for column and row sums respectively). Then

$$\|\Phi\|_{\ell_k} \leq T^{1/k^*} S^{1/k}.$$

Theorem 4.3 *The Hilbert operator admits a factorization of the form $H = B^q C^q$, where B^q is bounded on ℓ_k and*

$$\frac{\pi}{k^*}\csc(\pi/k) \leq \|B^q\|_{\ell_k} \leq \frac{L\pi}{k^*}\csc(\pi/k).$$

In particular,

(i) for $q_v = v+1$, the Hilbert operator has the Bennett's factorization $H = BC$.,

(ii) for $q_v = v+Q$, the Hilbert operator has the factorization $H = B^Q C^Q$, where C^Q is the generalized Cesàro matrix.

Proof Observe that

$$(B^q C^q)_{rv} = \sum_{l=v}^{\infty} \frac{q_l}{(r+l+1)(r+l+2)} \frac{1}{q_l} = \frac{1}{r+v+1} = h_{rv},$$

which proves the factorization $H = B^q C^q$. Let us recall the matrix D defined by (1.4) with $\|D\|_{\ell_k} = L$. It is noticed that $B^q = BD$, where the matrix B is defined in relation (4.1). Thus we have

$$\|B\|_{\ell_k} \leq \|B^q\|_{\ell_k} \leq \|B\|_{\ell_k} \|D\|_{\ell_k} = L\|B\|_{\ell_k}.$$

In particular, for $q_v = v+1$, $B^q = B$ and $C^q = C$. Consequently $H = BC$ and $\|B^q\|_{\ell_k} = \|B\|_{\ell_k} = \frac{\pi}{k^*}\csc(\pi/k)$. This completes the proof. □

As an immediate consequence of the above theorem, we generalize Hilbert's inequality versus Hardy's as follows:

Corollary 4.4 *Let $k > 1$ and C^q be quasi-Cesàro matrix. For every $x \in \ell_k$, we have*

$$\|Hx\|_{\ell_k} \leq \frac{L\pi}{k^*} \csc(\pi/k) \|C^q x\|_{\ell_k}.$$

In particular, for $q_v = v+1$, inequality (1.6) occurs.

Proof Since $H = B^q C^q$, we have

$$\|Hx\|_{\ell_k} = \|B^q C^q x\|_{\ell_k} \leq \frac{L\pi}{k^*} \csc(\pi/k) \|C^q x\|_{\ell_k}.$$

Now, for $q_v = v+1$, $B^q = B$ and $C^q = C$ which results in Hilbert's inequality versus Hardy's. □

5. Norm of quasi-Cesàro operator on matrix domains

In the current section, we evaluate the norm of transposed quasi-cesàro operator on the domain of difference matrix as well as computing the norm of Hilbert operator on quasi-Cesàro sequence space. The following lemma is essential for deducing our results.

Lemma 5.1 *[25, Lemma 3.1] Let the operator $\mathsf{U} \in B(\ell_k)$ and Λ_k and Ω_k be two matrix domains such that $\Lambda_k \simeq \ell_k$. Then*

(a) $\Phi \in B(\ell_k \to \Omega_k)$ if $\Omega\Phi \in B(\ell_k)$ and $\|\Phi\|_{(\ell_k \to \Omega_k)} = \|\Omega\Phi\|_{\ell_k}$.

(b) $\Phi \in B(\Lambda_k \to \ell_k)$ and $\|\Phi\|_{(\Lambda_k \to \ell_k)} = \|\mathsf{U}\|_{\ell_k}$.

In particular, when $\Lambda\Phi = \mathsf{U}\Lambda$, then $\Phi \in B(\Lambda_k)$ and $\|\Phi\|_{\Lambda_k} = \|\mathsf{U}\|_{\ell_k}$. Moreover, if Φ and Λ commute then $\|\Phi\|_{\Lambda_k} = \|\Phi\|_{\ell_k}$.

5.1. Norm of the transposed quasi-Cesàro operator on difference sequence spaces

In the rest of the paper, $\Delta^B = (\delta^B_{rv})$ and $\Delta^F = (\delta^F_{rv})$ represent the backward and forward difference matrices, respectively, defined by

$$\delta^B_{rv} = \begin{cases} 1 & (v = r), \\ -1 & (v = r-1), \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \delta^F_{rv} = \begin{cases} 1 & (v = r), \\ -1 & (v = r+1), \\ 0 & \text{otherwise} \end{cases}$$

and sequence spaces $\ell_k(\Delta^B)$ and $\ell_k(\Delta^F)$ obtained by the domain of the respective matrices in the space ℓ_k are defined by

$$\ell_k(\Delta^B) = \left\{ x = (x_r) \ : \ \sum_{r=1}^{\infty} |x_r - x_{r-1}|^k < \infty \right\},$$

and

$$\ell_k(\Delta^F) = \left\{ x = (x_r) \ : \ \sum_{r=1}^{\infty} |x_r - x_{r+1}|^k < \infty \right\},$$

respectively. The domains $c_0(\Delta^F)$, $c(\Delta^F)$ and $\ell_\infty(\Delta^F)$ of the forward difference matrix Δ^F in the spaces c_0, c and ℓ_∞ are introduced by Kızmaz [15]. Further, the domain bv_k of the backward difference matrix Δ^B in the space ℓ_k have recently been studied for $0 < k < 1$ by Altay and Başar [1], and for $1 \leq k \leq \infty$ by Başar and Altay [5].

Theorem 5.2 *Let $(q_v)_{v=0}^{\infty}$ satisfies the conditions given in (1.3). Then, the transposed quasi-Cesàro matrix $C^{q,t} \in B(\ell_k \to \ell_k(\Delta^F))$ and*

$$\|C^{q,t}\|_{(\ell_k \to \ell_k(\Delta^F))} = \sup_v \frac{1}{q_v}.$$

In particular, the Copson matrix $C^t \in B(\ell_k \to \ell_k(\Delta^F))$ and $\|C^t\|_{(\ell_k \to \ell_k(\Delta^F))} = 1$.

Proof The identity $\Delta^F C^{q,t} = D$ can be established easily, where $D = (d_{rv})$ is the diagonal matrix defined in (1.4) with $\|D\|_{\ell_k} = L$. Applying Lemma 5.1, we realise that

$$\|C^{q,t}\|_{(\ell_k \to \ell_k(\Delta^F))} = \|\Delta^F C^{q,t}\|_{\ell_k} = \|D\|_{\ell_k} = L.$$

□

Theorem 5.3 *The transposed quasi-Cesàro matrix $C^{q,t} \in B(\ell_k(\Delta^B) \to \ell_k(\Delta^F))$ and*

$$\|C^{q,t}\|_{(\ell_k(\Delta^B) \to \ell_k(\Delta^F))} \leq Kk^*.$$

In particular, the Copson matrix $C^t \in B(\ell_k(\Delta^B) \to \ell_k(\Delta^F))$ and

$$\|C^t\|_{(\ell_k(\Delta^B) \to \ell_k(\Delta^F))} = k^*.$$

Proof It is known from the proof of Theorem 5.2 that $\Delta^F C^{q,t} = D$, where D is a diagonal matrix as defined in relation (1.4). Since D is symmetric, therefore the identity $\Delta^F C^{q,t} = C^q \Delta^B$ also holds. Now employing Lemma 5.1, we arrive at the conclusion that

$$\|C^{q,t}\|_{(\ell_k(\Delta^B)\to\ell_k(\Delta^F))} = \|C^q\|_{\ell_k} \leq Kk^*.$$

In particular, taking $q_v = v+1$, $C^{q,t} = C^t$ is the Copson matrix. This completes the proof. □

5.2. Norm of Hilbert operator on quasi-Cesàro sequence space

Let l be a nonnegative integer, then the Hilbert matrix $H^l = (h^l_{rv})$ of order l is defined by

$$h^l_{rv} = \frac{1}{r+v+l+1} \quad (r, v \in \mathbb{N}_0).$$

Observe that when $l = 0$, the matrix H^l is the well-known Hilbert matrix H. Also

$$H^1 = \begin{pmatrix} 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ 1/4 & 1/5 & 1/6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \; H^2 = \begin{pmatrix} 1/3 & 1/4 & 1/5 & \cdots \\ 1/4 & 1/5 & 1/6 & \cdots \\ 1/5 & 1/6 & 1/7 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 5.4 *The Hilbert operator $H \in B(C^q_k \to \ell_k)$ and*

$$\|H\|_{(C^q_k\to\ell_k)} \leq \frac{L\pi}{k^*}\csc(\pi/k).$$

In particular, the Hilbert operator $H \in B(C_k \to \ell_k)$ and

$$\|H\|_{(C_k\to\ell_k)} = \frac{\pi}{k^*}\csc(\pi/k).$$

Proof It is known from Theorem 4.3 that the Hilbert matrix admits a factorization of the form $H = B^q C^q$, where B^q is a bounded operator on ℓ_k and

$$\frac{\pi}{k^*}\csc(\pi/k) \leq \|B^q\|_{\ell_k} \leq \frac{L\pi}{k^*}\csc(\pi/k).$$

We recall that C^q_k is isomorphic to ℓ_k. In the light of this, we have

$$\begin{aligned} \|H\|_{(C^q_k\to\ell_k)} &= \sup_{x\in C^q_k}\frac{\|Hx\|_{\ell_k}}{\|x\|_{C^q_k}} = \sup_{x\in C^q_k}\frac{\|B^qC^qx\|_{\ell_k}}{\|C^qx\|_{\ell_k}} = \sup_{y\in\ell_k}\frac{\|B^qy\|_{\ell_k}}{\|y\|_{\ell_k}} \\ &= \|B^q\|_{\ell_k} \leq \frac{L\pi}{k^*}\csc(\pi/k). \end{aligned}$$

In particular, taking $q_v = v+1$, $L = 1$, $C^q = C$ and $B^q = B$, where B is the factor in Bennett's factorization of the Hilbert operator. □

Theorem 5.5 *The Hilbert operator* $H \in B(C_k^q)$ *and*

$$\|H\|_{C_k^q} \leq KL\pi \csc(\pi/k).$$

In particular, Hilbert operator $H \in B(C_k)$ *and*

$$\|H\|_{C_k} = \pi \csc(\pi/k).$$

Proof Consider the identity $D^q = C^q B^q$, where B^q is as defined in the relation (4.3). Then

$$d_{rv}^q = \sum_{l=0}^{r} \frac{1}{q_r} \frac{q_v}{(l+v+1)(l+v+2)} = \left(\frac{q_v}{v+1}\right)\left(\frac{r+1}{q_r}\right)\frac{1}{r+v+2}.$$

Taking in account the condition (1.3) on the quasi-Cesàro matrix, we realise that $d_{rv}^q \leq KLh_{rv}^1$ which yields us

$$\|D^q\|_{\ell_k} \leq KL\|H^1\|_{\ell_k} = KL\pi \csc(\pi/k).$$

Since C_k^q is isomorphic to ℓ_k, we have by Lemma 5.1 that

$$\begin{aligned}\|H\|_{C_k^q} &= \sup_{x \in C_k^q} \frac{\|Hx\|_{C_k^q}}{\|x\|_{C_k^q}} = \sup_{x \in C_k^q} \frac{\|C^q H x\|_{\ell_k}}{\|C^q x\|_{\ell_k}} \\ &= \sup_{x \in C_k^q} \frac{\|D^q C^q x\|_{\ell_k}}{\|C^q x\|_{\ell_k}} = \sup_{y \in \ell_k} \frac{\|D^q y\|_{\ell_k}}{\|y\|_{\ell_k}} \\ &= \|D^q\|_{\ell_k} \leq KL\pi \csc(\pi/k).\end{aligned}$$

In particular, for $q_v = v+1$, $K = L = 1$, $C^q = C$ and $D^q = H^1$ which gives the desired result. □

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