Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 185 – 200
© TÜBİTAK
doi:10.3906/mat-2003-101

A Gompertz distribution for time scales

Tom CUCHTA[1,*], **Robert Jon NIICHEL**[1], **Sabrina STREIPERT**[2]
[1]Department of Computer Science and Mathematics, Fairmont State University, Fairmont, WV, USA
[2]Department of Mathematics and Statistics, McMaster University, Hamilton, ON, Canada

Received: 23.03.2020 • **Accepted/Published Online:** 12.11.2020 • **Final Version:** 21.01.2021

Abstract: We investigate a family of probability distributions, with three parameters associated with the dynamic Gompertz function. We prove its existence for various parameter sets and discuss the existence of its time scale moments. Afterwards, we investigate the special case of discrete time scales, where it is shown that the discrete Gompertz distribution is a q-geometric distribution of the second kind. Further, we find their q-binomial moments, we bound their expected value, and we show how a classical Gompertz distribution is obtained from them.

Key words: Time scales calculus, dynamic equations, difference equations, Gompertz, discrete q-distributions

1. Introduction

The Gompertz distribution is a family of two-parameter continuous distributions, typically with support on $(-\infty, \infty)$ or on $[0, \infty)$ [15]. There is significant interest in generalizations of the Gompertz distribution. In demography, recognizing the Gompertz distribution "as a member of families of models opens new perspectives" [18] in the field of mortality analysis. A five-parameter "beta generalized Gompertz distribution" was used to model lifetime of some devices [2]. A four-parameter "Gompertz Frêchet" distribution was used to model a data set of hauling times [17], and numerous three-parameter generalized Gompertz distributions have been defined in [1] [10], and [12].

On the other hand, the theory of discrete q-distributions has developed independently from the classical Gompertz distribution, culminating in the recent monograph [7]. The q-geometric distribution of the second kind, defined for parameters $0 < \theta < 1$ and $0 < q < 1$ with support $\{0, 1, 2, \ldots\}$, has a probability mass function

$$P(X = n) = \frac{\theta q^n}{1 - E_q\left(-\frac{\theta}{1-q}\right)} \prod_{j=1}^{n} \left(1 - \theta q^{j-1}\right), \tag{1.1}$$

where E_q denotes the so-called q-exponential function. Intuitively, the associated random variable counts the number of successes until the first failure when the probability of a success varies geometrically with the number of successes that have already been encountered. Compared to other discrete q-distributions, (1.1) has seen few applications in the literature, but relating it to the broadly applicable Gompertz distribution could invigorate new avenues of research in this area.

*Correspondence: tcuchta@fairmontstate.edu

2010 *AMS Mathematics Subject Classification:* 34N05, 39A12, 26D15

The synthesis of these two apparently disparate distributions is obtained in the theory of time scales calculus. The Gompertz dynamic equation is [8]

$$y^{\Delta} = (\ominus r)(t) y L_y, \quad y(t_0) = y_0, \tag{1.2}$$

whose solution was shown to be both qualitatively and quantitatively similar to the continuous Gompertz model. The Gompertz differential equation is a special case of (1.2) whose solution is normalized to obtain the Gompertz distribution with support on $(-\infty, \infty)$ or $[0, \infty)$. On the discrete time scale $\mathbb{T} = h\mathbb{N}_0 = \{h, 2h, 3h, \ldots\}$, $h > 0$, the unique solution of (1.2) normalizes to an instance of (1.1), unifying the Gompertz distribution with the discrete q-geometric distribution of the second kind.

2. Preliminaries

Discrete q-distributions are discrete distributions with the property that they contain a parameter $q \in (0,1)$ such that taking $q \to 1^-$ yields a traditional probability distribution. The q-numbers $[x]_q$ are defined by $[x]_q = \dfrac{1-q^x}{1-q}$. If $x = k$ is a nonnegative integer, then the q-factorial is defined by $[k]_q! = [k]_q[k-1]_q \ldots [2]_q[1]_q$, and we define the kth order q-factorial by the formula

$$[x]_{k,q} = [x]_q[x-1]_q \ldots [x-k+1]_q.$$

The q-binomial coefficients are $\begin{bmatrix} x \\ k \end{bmatrix}_q = \dfrac{[x]_q!}{[k]_q![x-k]_q!} = \dfrac{[x]_{k,q}}{[k]_q!}$, and for $x \geq m$, they obey the formula [13, (6.1)]

$$\begin{bmatrix} x \\ m \end{bmatrix}_q = \begin{bmatrix} x \\ x-m \end{bmatrix}_q. \tag{2.1}$$

The q-binomial moments of a discrete q random variable X are defined by [7, (1.55)] as

$$\mathbb{E}\left(\begin{bmatrix} X \\ m \end{bmatrix}_q \right) = \sum_{x=m}^{\infty} \begin{bmatrix} x \\ m \end{bmatrix}_q f(x), \tag{2.2}$$

where f is its probability mass function. For $0 < q < 1$, the q-exponential function E_q is defined [13, (9.10)] [7, (1.23)] by

$$E_q(t) = \prod_{j=1}^{\infty} \left(1 + t(1-q)q^{j-1}\right). \tag{2.3}$$

See [5] for the basic definitions of time scales calculus, which we now summarize. A time scale $\mathbb{T}$ is a nonempty closed (under the usual topology) subset of $\mathbb{R}$. We use the notation $\mathbb{T}^{\kappa}$ to mean $\mathbb{T} \backslash \sup \mathbb{T}$ when $\mathbb{T}$ has a left-scattered maximum; otherwise $\mathbb{T}^{\kappa} = \mathbb{T}$. The forward jump operator $\sigma \colon \mathbb{T} \to \mathbb{T}$ is $\sigma(t) = \inf\{s \in \mathbb{T} \colon s > t\}$, and the graininess operator $\mu \colon \mathbb{T} \to [0, \infty)$ is $\mu(t) = \sigma(t) - t$. Similarly a backwards jump operator $\rho \colon \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} \colon s < t\}$; if $\rho(t) < t$, then we say that t is left-scattered. If $f \colon \mathbb{T} \to \mathbb{R}$, then the Δ-derivative of f at $t \in \mathbb{T}^{\kappa}$ is defined to be the number $f^{\Delta}(t)$ with the property that for any $\epsilon > 0$, there is a neighborhood U of t such that for all $s \in U$,

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)\,(\sigma(t) - s) \right| \leq \epsilon |\sigma(t) - s|.$$

The Δ-integral is defined so that the fundamental theorem of calculus holds, i.e. for $t, s \in \mathbb{T}$,

$$\int_s^t f^{\Delta}(\tau)\Delta\tau = f(t) - f(s).$$

The reader should see [11] where a measure theoretic approach to the Δ-integral is defined via the so-called μ_Δ measure, but its construction is not emphasized or needed here.

The Taylor monomials h_n on a time scale are defined recursively by

$$\begin{cases} h_0(t,s) = 1, \\ h_{n+1}(t,s) = \displaystyle\int_s^t h_n(\tau,s)\Delta\tau, \end{cases}$$

and in particular, independent of the time scale, $h_1(t,s) = t - s$. If f is Δ-differentiable, then we define [8, Definition 7] the time scales logarithm L_f for $t \in \mathbb{T}, a \in \mathbb{R}$ by

$$L_f(t, t_0; a) = a + \int_{t_0}^t \frac{f^{\Delta}(\tau)}{f(\tau)}\Delta\tau.$$

A function $f\colon \mathbb{T} \to \mathbb{C}$ is called right-dense continuous ("rd-continuous") if it is continuous at right-dense points and its left-sided limits exist at left-dense points. The forward jump σ, and hence μ, is rd-continuous. By [5, Theorem 1.60 (ii), Theorem 1.65, and Remark 1.66], an rd-continuous function with compact domain is bounded, but may not necessarily achieve extreme values. An rd-continuous function is called regressive if for all $t \in \mathbb{T}$, $1 + \mu(t)f(t) \neq 0$. We write $f \in \mathcal{R}$ to mean that $f\colon \mathbb{T} \to \mathbb{C}$ is both rd-continuous and regressive. Moreover, we say $f \in \mathcal{R}^+$ if $f \in \mathcal{R}$ and for all $t \in \mathbb{T}$, $1 + \mu(t)f(t) > 0$. If f is regressive, then the circle minus function $\ominus f\colon \mathbb{T} \to \mathbb{C}$ is defined by $(\ominus f)(t) = \dfrac{-f(t)}{1 + \mu(t)f(t)}$. For $h \geq 0$, the cylinder transformation ξ_h is defined by

$$\xi_h(z) = \begin{cases} \dfrac{1}{h}\operatorname{Log}(1 + zh), & h > 0 \\ z, & h = 0. \end{cases}$$

If $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$, then the time scales exponential function $e_p(\cdot, t_0)$ is defined by the expression $e_p(t, t_0) = \exp\left(\displaystyle\int_{t_0}^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)$, and it is the unique solution of the initial value problem

$$y^{\Delta} = py, \quad y(t_0) = 1. \tag{2.4}$$

It is well-known that for $p \in \mathcal{R}$,

$$e_{\ominus p}(t,s) = \frac{1}{e_p(t,s)}, \tag{2.5}$$

$$e_p(t,s) = \frac{1}{e_p(s,t)},$$

and

$$e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s). \tag{2.6}$$

The dynamic exponential also obeys the semigroup property

$$e_p(t,s) = e_p(t,r)e_p(r,s). \tag{2.7}$$

If $\mathbb{T}$ is a time scale with $t_0 \in \mathbb{T}$ obeying the property that $\mu(t) > 0$ for all $t \in \mathbb{T}$ with $t > t_0$ and $t \neq \sup \mathbb{T}$, then

$$e_p(t,t_0) = \prod_{\tau \in [t_0,t) \cap \mathbb{T}} 1 + \mu(\tau)p(\tau).$$

If $r > 0$, then

$$e_{\ominus r}(t,t_0) = \prod_{\tau \in [t_0,t) \cap \mathbb{T}} \frac{1}{1+\mu(\tau)r}. \tag{2.8}$$

The following theorem shows sufficient requirements for the dynamic exponential function to be always positive.

Theorem 2.1 *If $p \in \mathcal{R}^+$, then for all $t \in \mathbb{T}$, $e_p(t,t_0) > 0$.*

If $p\colon \mathbb{T} \to \mathbb{R}$ is rd-continuous and nonnegative [3, Remark 2], then for $t \geq s$,

$$1 + \int_s^t p(\tau)\Delta\tau \leq e_p(t,s) \leq \exp\left(\int_s^t p(\tau)\Delta\tau\right). \tag{2.9}$$

Similary, it was shown in [4, Lemma 3.2] that if $p \in \mathcal{R}^+$, then the second inequality in (2.9) holds.

Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = \infty$. For a fixed $s \in \mathbb{T}$, an rd-continuous function $p\colon [s,\infty) \cap \mathbb{T} \to \mathbb{C}$ is said to be of exponential order $\alpha \in \mathbb{R}$ provided that the function $t \mapsto \alpha$ is in $\mathcal{R}^+$ for all $t \in [s,\infty) \cap \mathbb{T}$ and there exists $K > 0$ so that for all $t \in [s,\infty) \cap \mathbb{T}$, $|p(t)| \leq K e_\alpha(t,s)$. The time scales Laplace transform of a function $p\colon \mathbb{T} \to \mathbb{R}$, centered at $s \in \mathbb{T}$, is defined by

$$\mathscr{L}_{\mathbb{T}}\{p\}(z;s) = \int_s^\infty p(\tau)e_{\ominus z}(\sigma(\tau),s)\Delta\tau.$$

It is known that if p is of exponential order α, then $\mathscr{L}_{\mathbb{T}}\{p\}(z;s)$ exists for z in a certain open subset of $\mathbb{C}$ [4, Theorem 5.1]. It was shown in [4, Example 4.3] that the time scale Taylor monomials $h_n(\cdot,s)$ are of exponential order ϵ for each $\epsilon > 0$. In particular, [5, Theorem 3.90] shows that

$$\mathscr{L}_{\mathbb{T}}\{h_n(\cdot,s)\}(z;s) = \frac{1}{z^{n+1}}. \tag{2.10}$$

Now we review some information about random variables on time scales. Let X be a random variable with Δ-density function $f\colon \mathbb{T} \to [0,\infty)$. In [16, Remark 2], the nth time scale moments of X, centered at $s \in \mathbb{T}$, are defined by the formula

$$\mathbb{E}\left(n! h_n(X,s)\right) = n! \int_{\inf \mathbb{T}}^{\sup \mathbb{T}} h_n(t,s)f(t)\Delta t. \tag{2.11}$$

Equation (2.11) reduces to classical moments over the time scale $\mathbb{T} = \mathbb{R}$ because on that time scale, $h_n(t,s) = \dfrac{(t-s)^n}{n!}$. On the other hand, on the time scale $\mathbb{T} = \mathbb{N}_0$, (2.11) yields the so-called factorial moments [9, (5.2.1)] because in this case, $h_n(t,s) = \dbinom{t-s}{n}$.

3. Gompertz distribution on time scales

We now prove the existence of a Gompertz-type distribution on time scales. This distribution is closely related to the work in [8] where the Gompertz dynamic equation (1.2) was first introduced. Equation (1.2) has the unique solution

$$y(t) = y_0 e_p(t,t_0), \quad p(t) = a(\ominus r)(t) e_{\ominus r}(t,t_0) = \frac{-ar}{1+\mu(t)r} e_{\ominus r}(t,t_0). \tag{3.1}$$

Throughout the remainder of this article, we understand the function p as defined in (3.1) and assume that $p \in \mathcal{R}$. The following bounds on the limit were found when $r > 0$ and $a < 0$:

$$1 + |a| \leq \lim_{t\to\infty} e_p(t,t_0) \leq \exp(|a|). \tag{3.2}$$

Theorem 3.1 *If $\mathbb{T}$ is a bounded time scale, then for any $a, r \in \mathbb{R}$, $e_p^{\Delta}(\cdot,t_0)$ is Δ-integrable over $\mathbb{T}$.*

Proof By (2.4), $e_p^{\Delta}(t,t_0) = p(t)e_p(t,t_0)$ is rd-continuous since $e_p(\cdot,t_0)$ is Δ-differentiable (hence is continuous). By hypothesis, $\mathbb{T}$ is compact, so $e_p(\cdot,t_0)$ is bounded and the quantity $M := \sup_{t\in\mathbb{T}} e_p^{\Delta}(t,t_0)$ exists and is finite. Therefore

$$\int_{\mathbb{T}} e_p^{\Delta}(t,t_0)\Delta t \leq M(\sup(\mathbb{T}) - \inf(\mathbb{T})) < \infty,$$

completing the proof. $\square$

Theorem 3.1 guarantees the unsurprising result that the Gompertz distribution exists on any compact time scale, provided that $e_p^{\Delta}(\cdot,t_0)$ does not change sign. It is more interesting to consider unbounded time scales.

Theorem 3.2 *Let $\mathbb{T}$ be an arbitrary unbounded time scale. If $a < 0$ and $r > 0$, then $e_p^{\Delta}(t,t_0)$ is positive and*

$$0 < \int_{\inf\mathbb{T}}^{\sup\mathbb{T}} e_p^{\Delta}(t,t_0)\Delta t < \infty.$$

Proof Suppose $\sup\mathbb{T} = \infty$ and $\inf\mathbb{T} = -\infty$. Since $r > 0$, we know $1 + \mu(t)r > 0$ and hence the function $t \mapsto r$ is in $\mathcal{R}^+$. We calculate

$$1 + \mu(t)(\ominus r)(t) = 1 - \frac{\mu(t)r}{1+\mu(t)r} = \frac{1}{1+\mu(t)r} > 0,$$

and so $\ominus r \in \mathcal{R}^+$, allowing us to conclude by Theorem 2.1 that $e_{\ominus r}(t,t_0) > 0$ for all $t \in \mathbb{T}$. Since $r > 0$ and $a < 0$, we see that $a(\ominus r)(t) > 0$, $p = a(\ominus r)(t)e_{\ominus r}(t,t_0) > 0$, and $1 + \mu(t)p(t) > 0$, so we conclude by Theorem 2.1 that $e_p(t,t_0) > 0$ for all $t \in \mathbb{T}$. Therefore $e_p^{\Delta}(t,t_0) = p(t)e_p(t,t_0) > 0$, and hence $e_p(\cdot,t_0)$ is an increasing function. Therefore $L := \lim_{t\to-\infty} e_p(t,t_0)$ exists and is ≥ 0. We know from (3.2) that $U := \lim_{t\to\infty} e_p(t,t_0) \leq \exp(|a|)$. We conclude

$$0 < \int_{-\infty}^{\infty} e_p^{\Delta}(t,t_0)\Delta t = U - L < \infty.$$

If one of $\inf \mathbb{T}$ or $\sup \mathbb{T}$ is finite, a similar argument holds, completing the proof. □

We define the following condition on parameters for the Gompertz distribution that we will repeatedly use:

$$r > 0 \quad \text{and} \quad \forall t \in \mathbb{T},\ 0 < a < \frac{1}{|(\ominus r)(t)|\mu(t)e_{\ominus r}(t,t_0)}. \tag{3.3}$$

The following lemma is proven by routine algebraic manipulation.

Lemma 3.3 *If (3.3) holds, then $p \in \mathcal{R}^+$.*

Theorem 3.4 *Let $\mathbb{T}$ be an unbounded time scale such that $\inf \mathbb{T} = t_0 > -\infty$. If (3.3) holds, then the function $y(t) = e_p^{\Delta}(t,t_0)$ is negative and integrable on $\mathbb{T}$.*

Proof By Lemma 3.3, $p \in \mathcal{R}^+$ and so $e_p(t,t_0) > 0$ for all $t \in \mathbb{T}$. On the other hand,

$$e_p^{\Delta}(t,t_0) = a(\ominus r)(t)e_{\ominus r}(t,t_0) < 0,$$

and so we see that $e_p(\cdot,t_0)$ is a decreasing function. Therefore, $L := \lim\limits_{t\to\infty} e_p(t,t_0)$ exists and $L \geq 0$. Thus,

$$\int_{t_0}^{\infty} e_p^{\Delta}(t,t_0)\Delta t \quad = L - e_p(t_0,t_0) = L - 1 < 0,$$

completing the proof. □

We define the symbol $\mathcal{N}$ to denote the value

$$\mathcal{N} := \int_{\inf \mathbb{T}}^{\sup \mathbb{T}} e_p^{\Delta}(t,t_0)\Delta t,$$

provided it exists. Theorem 3.1, Theorem 3.2, and Theorem 3.4 show various conditions on r and a under which $\mathcal{N}$ is finite. In these situations, we define a time scales probability density function $g\colon \mathbb{T} \to [0,\infty)$ using p in (3.1) and three parameters a, r, and t_0 by

$$g(t) = \frac{1}{\mathcal{N}} e_p^{\Delta}(t,t_0) = \frac{1}{\mathcal{N}} p(t)e_p(t,t_0). \tag{3.4}$$

A random variable G shall be called a time scale Gompertz random variable on $\mathbb{T}$ if for every μ_{Δ}-measurable set $S \subseteq \mathbb{T}$,

$$P(G \in S) = \int_S g(\tau)\Delta\tau.$$

Theorem 3.5 *If $\mathbb{T}$ is an arbitrary time scale with $t_0 = \inf \mathbb{T} > -\infty$, $\sup \mathbb{T} = \infty$, $r > 0$, and $a < 0$, then the time scale moments of G, centered at s, exist. Moreover, for some constants m_s and M_s,*

$$\frac{n!m_s(s-t_0)}{\mathcal{N}} + \frac{n!|a|e_{\ominus r}(s,t_0)}{\mathcal{N}r^n} \leq \mathbb{E}(h_n(G,s)) \leq \frac{n!M_s(s-t_0)}{\mathcal{N}} + \frac{n!|a|\exp\left(|a|e_{\ominus r}(s,t_0)\right)}{\mathcal{N}r^n}.$$

Proof By (2.11), we must bound the expression

$$\mathbb{E}\left(h_n(G,s)\right) \quad = \frac{n!}{\mathcal{N}} \int_{t_0}^{\infty} h_n(t,s)p(t)e_p(t,t_0)\Delta t. \tag{3.5}$$

Rewrite this integral as

$$\frac{n!}{\mathcal{N}} \int_{t_0}^{s} h_n(t,s)p(t)e_p(t,t_0)\Delta t + \frac{n!}{\mathcal{N}} \int_{s}^{\infty} h_n(t,s)p(t)e_p(t,t_0)\Delta t. \tag{3.6}$$

Since the first integral is over a compact set, the integrand is bounded below and above by some constants m_s and M_s, respectively. Therefore the first integral in (3.6) obeys

$$\frac{n! m_s(s-t_0)}{\mathcal{N}} \leq \frac{n!}{\mathcal{N}} \int_{t_0}^{s} h_n(t,s)p(t)e_p(t,t_0)\Delta t \leq \frac{n! M_s(s-t_0)}{\mathcal{N}}. \tag{3.7}$$

Now we turn our attention to the second integral in (3.6). Using the semigroup property (2.7), we rewrite the integrand as

$$h_n(t,s)p(t)e_p(t,t_0) = a(\ominus r)(t)h_n(t,s)e_{\ominus r}(t,s)e_{\ominus r}(s,t_0)e_p(t,s)e_p(s,t_0),$$

and so the integral becomes

$$\frac{n!|a|re_p(s,t_0)e_{\ominus r}(s,t_0)}{\mathcal{N}} \int_s^{\infty} h_n(t,s)e_p(t,s)\frac{e_{\ominus r}(t,s)}{1+\mu(t)r}\Delta t, \tag{3.8}$$

which we now bound. Since $a<0$ and $r>0$, p is nonnegative and clearly rd-continuous, by (2.9),

$$\begin{aligned} e_p(t,s) \leq \exp\left(\int_s^t p(\tau)\Delta\tau\right) \quad &= \exp\left(ae_{\ominus r}(s,t_0)\int_s^t e_{\ominus r}^{\Delta}(\tau,s)\Delta\tau\right) \\ &= \exp\big(ae_{\ominus r}(s,t_0)(e_{\ominus r}(t,s)-1)\big) \\ &= \exp\big(|a|e_{\ominus r}(s,t_0)(1-e_{\ominus r}(t,s))\big). \end{aligned}$$

Using (2.5) and (2.6), we see

$$\frac{e_{\ominus r}(t,s)}{1+\mu(t)r} = \frac{1}{(1+\mu(t)r)e_r(t,s)} = \frac{1}{e_r(\sigma(t),s)} = e_{\ominus r}(\sigma(t),s).$$

Finally, since $|a|e_{\ominus r}(t,t_0) > 0$, it follows that $\exp(-|a|e_{\ominus r}(t,t_0)) < 1$. Using the above, (3.8), and (2.10), compute

$$\begin{aligned} &\int_s^{\infty} h_n(t,s)e_p(t,s)\frac{e_{\ominus r}(t,s)}{1+\mu(t)r}\Delta t \\ &\quad \leq \exp\left(|a|e_{\ominus r}(s,t_0)\right)\int_s^{\infty} h_n(t,s)e_{\ominus r}(\sigma(t),s)\exp\left(-|a|e_{\ominus r}(t,t_0)\right)\Delta t \\ &\quad \leq \exp\left(|a|e_{\ominus r}(s,t_0)\right)\int_s^{\infty} h_n(t,s)e_{\ominus r}(\sigma(t),s)\Delta t \\ &\quad = \exp\left(|a|e_{\ominus r}(s,t_0)\right)\mathscr{L}\{h_n(\cdot,s)\}(r;s) = \frac{\exp(|a|e_{\ominus r}(s,t_0))}{r^{n+1}}. \end{aligned} \tag{3.9}$$

Combining (3.6), (3.7), and (3.9) establishes

$$\mathbb{E}\left(h_n(G,s)\right) \leq \frac{n!M_s(s-t_0)}{\mathcal{N}} + \frac{n!|a|\exp\left(|a|e_{\ominus r}(s,t_0)\right)}{\mathcal{N}r^n},$$

completing the proof for the upper bound. For the lower bound, realize that (3.6) still applies, so it remains to bound the second integral there. By (2.9),

$$e_p(t,t_0) \geq 1 + \int_{t_0}^{t} p(s)\Delta s = 1 + a\Big(e_{\ominus r}(t,t_0) - 1\Big),$$

so we see

$$\begin{aligned}\int_s^\infty h_n(t,s)p(t)e_p(t,t_0)\Delta t &\geq \int_s^\infty h_n(t,s)p(t)\left(1 + a\Big(e_{\ominus r}(t,t_0) - 1\Big)\right)\Delta t \\ &= \int_s^\infty h_n(t,s)p(t)\Big((1-a) - |a|e_{\ominus r}(t,t_0)\Big)\Delta t.\end{aligned} \tag{3.10}$$

Now compute

$$\begin{aligned}(1-a)\int_s^\infty h_n(t,s)p(t)\Delta t &= |a|r(1-a)\int_s^\infty h_n(t,s)e_{\ominus r}(\sigma(t),t_0)\Delta t \\ &= |a|r(1-a)e_{\ominus r}(s,t_0)\mathscr{L}\{h_n(\cdot,s)\}(r;s) \\ &= \frac{|a|(1-a)e_{\ominus r}(s,t_0)}{r^n}.\end{aligned} \tag{3.11}$$

Since $t \mapsto (\ominus r)(t)$ is a function in $\mathcal{R}^+$, we use (2.9) to bound

$$e_{\ominus r}(t,t_0) \leq \exp\left(-r\int_{t_0}^{t} \frac{1}{1+\mu(\tau)r}\Delta\tau\right) \leq 1,$$

and hence $-e_{\ominus r}(t,t_0) \geq -1$. Therefore we compute

$$\begin{aligned}-|a|\int_s^\infty h_n(t,s)p(t)e_{\ominus r}(t,t_0)\Delta t &= -a^2 r\int_s^\infty \frac{h_n(t,s)}{1+\mu(t)r}e_{\ominus r}^2(t,t_0)\Delta t \\ &\geq -a^2 r\int_s^\infty \frac{h_n(t,s)}{1+\mu(t)r}e_{\ominus r}(t,t_0)\Delta t \\ &= -a^2 r e_{\ominus r}(s,t_0)\mathscr{L}\{h_n(\cdot,s)\}(r;s) \\ &= -\frac{a^2 e_{\ominus r}(s,t_0)}{r^n}.\end{aligned} \tag{3.12}$$

Recalling that $-|a| = a$ and combining (3.11) and (3.12) allows us to further bound (3.10) by

$$\int_s^\infty h_n(t,s)p(t)e_p(t,t_0)\Delta t \geq \frac{|a|e_{\ominus r}(s,t_0)}{r^n}.$$

Combining this in (3.6) with (3.7) yields

$$\mathbb{E}(h_n(G,s)) \geq \frac{n!m_s(s-t_0)}{\mathcal{N}} + \frac{n!|a|e_{\ominus r}(s,t_0)}{\mathcal{N}r^n},$$

completing the proof. □

We get simpler bounds if the moments are taken to be centered at $s = t_0$ in Theorem 3.5.

Corollary 3.6 *If $\mathbb{T}$ is an arbitrary time scale with $t_0 = \inf \mathbb{T} > -\infty$, $\sup \mathbb{T} = \infty$, $r > 0$, and $a < 0$, then the time scale moments of G, centered at t_0, exist. Moreover,*

$$\frac{n!|a|}{\mathcal{N} r^n} \leq \mathbb{E}(h_n(G, t_0)) \leq \frac{n!|a| e^{|a|}}{\mathcal{N} r^n}.$$

Classical moments of random variables on time scales are obtained by replacing $h_n(t, t_0)$ in (3.5) with $(t-s)^n$. In this case, the expression $\mathscr{L}_{\mathbb{T}}\{t^n\}(r)$ is obtained, but it appears to be unknown whether the function $t \mapsto (t-s)^n$ is of exponential order α on all time scales for every n. Since $h_1(t, s) = t - s$ on all time scales, we do immediately obtain a bound on the expected value of G.

Corollary 3.7 *If $\mathbb{T}$ is an arbitrary time scale with $t_0 = \inf \mathbb{T} > -\infty$, $\sup \mathbb{T} = \infty$, $r > 0$, and $a < 0$, then the expected value of G exists. Moreover,*

$$\frac{|a|}{\mathcal{N} r} + t_0 \leq \mathbb{E}(G) \leq \frac{|a| e^{|a|}}{\mathcal{N} r} + t_0.$$

In the sequel, we investigate a similar inequality on discrete time scales. In the following theorem, we refine the inequality in Corollary 3.7 in the same class of time scales investigated in the next section.

Theorem 3.8 *If $h > 0$, $\mathbb{T} = h\mathbb{N}_0 = \{0, h, 2h, \ldots\}$, $a < 0$, and $r > 0$, then*

$$\frac{|a|}{\mathcal{N} h r} < \mathbb{E}(G).$$

If, further, $|a| \leq h + h^2 r$, then

$$\mathbb{E}(G) \leq \frac{h|a| \left(1 + hr + |a|r\right)\left(1 + hr\right)}{\mathcal{N}\left(h^2 r + h - |a|\right)^2 r}.$$

Proof Under the hypothesis, for any $j \in \mathbb{N}_0$,

$$1 - \frac{ar}{(1+hr)^{j+1}} = 1 + \frac{|a|r}{(1+hr)^{j+1}} > 1,$$

so since we know by Theorem 3.2 that $\mathcal{N} > 0$, for $t \in h\mathbb{N}_0$,

$$\frac{|a|r}{\mathcal{N}(1+hr)^{j+1}} < \frac{|a|r}{\mathcal{N}(1+hr)^{j+1}} \prod_{k=0}^{j-1} \left(1 + \frac{|a|r}{(1+hr)^{k+1}}\right).$$

Thus,

$$\frac{|a|r}{\mathcal{N}} \sum_{j=0}^{\infty} \frac{hj}{(1+hr)^{j+1}} < \frac{|a|r}{\mathcal{N}} \sum_{j=0}^{\infty} \frac{hj}{(1+hr)^{j+1}} \prod_{k=0}^{j-1} \left(1 + \frac{|a|r}{(1+hr)^{k+1}}\right).$$

The right-hand side is the expected value $\mathbb{E}(G)$, so we obtain

$$\frac{h|a|r}{\mathcal{N}(1+hr)^2} \sum_{j=0}^{\infty} \frac{j}{(1+hr)^{j-1}} < \frac{1}{\mathcal{N}} \sum_{j=0}^{\infty} hj e_p^{\Delta}(j, 0) = \mathbb{E}(G). \tag{3.13}$$

It is well-known from elementary power series that when $|x| < 1$.

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}. \tag{3.14}$$

Taking $x = \frac{1}{1+hr}$ in (3.14) shows that (3.13) becomes

$$\mathbb{E}(G) > \frac{|a|rh}{\mathcal{N}(1+hr)^2} \frac{1}{\left(1 - \frac{1}{1+hr}\right)^2} = \frac{|a|}{\mathcal{N}hr},$$

which completes the first part of the proof. For the second part, the inequality $|a| \leq h + h^2 r$ can be algebraically rearranged to

$$\frac{1}{1+hr}\left(1 + \frac{|a|r}{1+hr}\right) < 1, \tag{3.15}$$

which implies

$$h + |a| < \frac{(1+hr)^2 - 1}{r} = 2h + h^2 r,$$

hence (3.15) is equivalent to the assumed condition. Since the function $f(j) = 1 + \frac{|a|r}{(1+hr)^{j+1}}$ is strictly decreasing, $\max_{j \geq 0} f(j) = f(0)$, so for $t \in h\mathbb{N}_0$,

$$\prod_{k=0}^{\frac{t}{h}-1}\left(1 + \frac{|a|r}{(1+hr)^{k+1}}\right) \leq \prod_{k=0}^{\frac{t}{h}-1}\left(1 + \frac{|a|r}{1+hr}\right) = \left(1 + \frac{|a|r}{1+hr}\right)^{\frac{t}{h}}.$$

Therefore, using (4.2),

$$P(G = t) \leq \frac{|a|r}{\mathcal{N}(1+hr)^{\frac{t}{h}+1}}\left(1 + \frac{|a|r}{1+hr}\right)^{\frac{t}{h}}. \tag{3.16}$$

Then

$$\begin{aligned}\sum_{k=0}^{\infty} k\frac{h|a|r}{\mathcal{N}(1+hr)^{k+1}}\left(1 + \frac{|a|r}{1+hr}\right)^k &= \frac{|a|r\left(1 + \frac{|a|r}{1+hr}\right)}{\mathcal{N}(1+hr)^2}\sum_{k=0}^{\infty} hk\left(\frac{1 + \frac{|a|r}{1+hr}}{1+hr}\right)^{k-1} \\ &= \frac{h|a|r\left(1 + \frac{|a|r}{1+hr}\right)}{\mathcal{N}(1+hr)^2\left(1 - \frac{1}{1+hr}\left(1 + \frac{|a|r}{1+hr}\right)\right)^2} = \frac{h|a|\left(1 + hr + |a|r\right)(1+hr)}{\mathcal{N}\left(h^2 r + h - |a|\right)^2 r}\end{aligned}$$

Combining this calculation with the inequality (3.16) completes the proof. □

4. Relation to the q-geometric distribution of the second kind

Throughout this section, we scale nonnegative integers by $h > 0$ to obtain the time scale $\mathbb{T} = h\mathbb{N}_0 = \{0, h, 2h, 3h, \ldots\}$ and, for convenience, we choose $t_0 = 0$. Here we have $\mu(t) \equiv h$ and $(\ominus r)(t) = \ominus r =$

$\dfrac{-r}{1+hr} \in \mathcal{R}$. Equation (2.8) becomes

$$e_{\ominus r}(t,0) = \prod_{j=0}^{\frac{t}{h}-1} \frac{1}{1+hr} = \left(\frac{1}{1+hr}\right)^{\frac{t}{h}},$$

where $e_{\ominus r}(0,0) = 1$. By (3.1),

$$p(t) = a(\ominus r) e_{\ominus r}(t,0) = \frac{-ar}{1+hr} e_{\ominus r}(t,0) = \frac{-ar}{(1+hr)^{\frac{t}{h}+1}}. \tag{4.1}$$

Consequently, for $t \in h\mathbb{N}_0$, (3.4) becomes

$$P(G=t) = \frac{-ar}{\mathcal{N}} \left(\frac{1}{1+hr}\right)^{\frac{t}{h}+1} \prod_{j=0}^{\frac{t}{h}-1} \left(1 - \frac{ar}{(1+hr)^{j+1}}\right). \tag{4.2}$$

By Theorem 3.4, if

$$r > 0 \text{ and } \forall t \in h\mathbb{N}_0, 0 < a < \frac{(1+hr)^{\frac{t}{h}+1}}{hr}, \tag{4.3}$$

then e_p^{Δ} is integrable with $\mathcal{N} = \displaystyle\int_0^\infty e_p^{\Delta}(\tau,0)\Delta\tau$. Since $r>0$ and $h>0$, the function $t \mapsto (1+hr)^{\frac{t}{h}+1}$ is a monotone increasing function, meaning that (4.3) reduces to

$$r > 0 \text{ and } 0 < a < 1 + \frac{1}{hr}. \tag{4.4}$$

We now find a closed form for the normalizing constant $\mathcal{N}$ by showing (4.2) is a q-geometric distribution of the second kind.

Theorem 4.1 *Let $\mathbb{T} = h\mathbb{N}_0$ and let G denote the associated time scales Gompertz distribution with probability mass function (4.2). If (4.4) holds, then G is a q-geometric distribution of the second kind with parameters $\theta = \dfrac{arh}{1+hr}$ and $q = \dfrac{1}{1+hr}$. Moreover, we obtain $\mathcal{N} = \dfrac{E_{\frac{1}{1+hr}}(-a) - 1}{h}$, where $E_{\frac{1}{1+hr}}$ is defined by (2.3).*

Proof First by (4.4), $\theta = \dfrac{arh}{1+hr} \in (0,1)$ and $q = \dfrac{1}{1+hr} \in (0,1)$. If X is a q-geometric distribution of the second kind with these parameters, then for $n \in \mathbb{N}_0$,

$$\begin{aligned} P(X=n) &= \frac{\left(\frac{arh}{1+hr}\right)\left(\frac{1}{1+hr}\right)^n}{1 - E_{\frac{1}{1+hr}}\left(-\frac{\frac{arh}{1+hr}}{1-\frac{1}{1+hr}}\right)} \prod_{j=1}^{n} \left(1 - \left(\frac{ar}{1+hr}\right)\left(\frac{1}{1+hr}\right)^{j-1}\right) \\ &= \frac{arh}{1 - E_{\frac{1}{1+hr}}(-a)} \left(\frac{1}{1+hr}\right)^{n+1} \prod_{j=0}^{n-1} \left(1 - \frac{ar}{(1+hr)^{j+1}}\right), \end{aligned}$$

which is identical to (4.2) for $\dfrac{t}{h} = n$ with

$$\mathcal{N} = \frac{E_{\frac{1}{1+hr}}(-a) - 1}{h},$$

completing the proof. $\square$

The following lemma does not appear to be stated in the literature.

Lemma 4.2 *If X has a q-geometric distribution of the second kind, then its q-binomial moments are given by*

$$\mathbb{E}\left(\begin{bmatrix} X \\ m \end{bmatrix}_q\right) = \frac{\theta^{-m} q^{-m^2} \prod_{j=1}^{m} (1-\theta q^{j-1})}{1 - E_q\left(-\frac{\theta}{1-q}\right)} \left(1 - E_q\left(-\frac{\theta q^m}{1-q}\right) \sum_{j=0}^{m} \frac{\theta^j q^{mj}}{(1-q)^j [j]_q!}\right).$$

Proof Let X have a q-geometric distribution of the second kind with parameters θ and q and probability mass function (1.1). The following identity is well-known [14, (3.9)]:

$$\sum_{r=0}^{\infty} \begin{bmatrix} n+r-1 \\ r \end{bmatrix}_q q^r \prod_{j=1}^{r} (1-\theta q^{j-1}) = \theta^{-n} \left(1 - E_q\left(\frac{-\theta}{1-q}\right) \sum_{j=0}^{n-1} \frac{\theta^j}{(1-q)^j [j]_q!}\right).$$

Choosing the values $x = n+r-1$ and $m = n-1$ implying $r = x-m$ and $n = m+1$ and using the relation (2.1), we compute

$$\sum_{x=m}^{\infty} \begin{bmatrix} x \\ m \end{bmatrix}_q q^{x-m} \prod_{j=1}^{x-m} (1-\theta q^{j-1}) = \theta^{-(m+1)} \left(1 - E_q\left(\frac{-\theta}{1-q}\right) \sum_{j=0}^{m} \frac{\theta^j}{(1-q)^j [j]_q!}\right).$$

Now using (2.2), we compute the mth q-binomial moment of X as

$$\begin{aligned}
\mathbb{E}\left[\begin{bmatrix} X \\ m \end{bmatrix}_q\right] &= \frac{\theta q^m \prod_{j=1}^{m} (1-\theta q^{j-1})}{1 - E_q\left(\frac{-\theta}{1-q}\right)} \sum_{x=m}^{\infty} \begin{bmatrix} x \\ m \end{bmatrix}_q q^{x-m} \prod_{j=1}^{x-m} (1-(\theta q^m) q^{j-1}) \\
&= \frac{\theta^{-m} q^{-m^2} \prod_{j=1}^{m} (1-\theta q^{j-1})}{1 - E_q\left(\frac{-\theta}{1-q}\right)} \left(1 - E_q\left(\frac{-\theta q^m}{1-q}\right) \sum_{j=0}^{m} \frac{\theta^j q^{mj}}{(1-q)^j [j]_q!}\right),
\end{aligned}$$

completing the proof. $\square$

The following theorem follows immediately from Theorem 4.1 and Lemma 4.2 when taking $\theta = \dfrac{arh}{1+hr}$ and $q = \dfrac{1}{1+hr}$.

Theorem 4.3 *If $\mathbb{T} = h\mathbb{N}_0$ and (4.4) holds, then the q-binomial moments of G are given by*

$$\mathbb{E}\left(\begin{bmatrix} G \\ m \end{bmatrix}_{\frac{1}{1+hr}}\right) = \frac{(1+hr)^{m+m^2} \prod_{j=1}^{m} \left(1 - \frac{arh}{(1+hr)^j}\right)}{(arh)^m \left(1 - E_{\frac{1}{1+hr}}(-a)\right)} \left(1 - E_{\frac{1}{1+hr}}\left(-\frac{a}{(1+hr)^m}\right) \sum_{j=0}^{m} \frac{a^j}{(1+hr)^{mj} [j]_{\frac{1}{1+hr}}!}\right).$$

Theorem 4.3 establishes the q-binomial moments of G under the condition (4.4) whose more general expression was used for Theorem 3.4. If instead, the condition from Theorem 3.2 is used, then G is not a q-geometric distribution of the second kind and so we do not know a formula for $\mathcal{N}$. However, Theorem 3.5 shows that bounds for the moments of G do exist.

If $f\colon h\mathbb{N}_0 \to \mathbb{R}$, then we will use the constant interpolation function $\overline{f}\colon \mathbb{R} \to \mathbb{C}$ given by

$$\overline{f}(t) = \begin{cases} 0, & t < 0, \\ f(hn), & t \in [hn, hn+h) \text{ for some } hn \in h\mathbb{N}_0. \end{cases}$$

It is known [6, Theorem 5.1] that

$$\int_{[a,b]\cap h\mathbb{N}_0} f(\tau)\Delta\tau = \int_{[a,b]} \overline{f}(\tau)\mathrm{d}\tau. \tag{4.5}$$

The following theorem establishes the convergence of the discrete distributions on $h\mathbb{N}_0$ to the classic Gompertz distribution with support on $[0,\infty)$ as $h \to 0^+$. We let $p(t;h\mathbb{N}_0)$ stand for (4.1), and we let $p(t;[0,\infty))$ stand for $p(t)$ in (3.1) on the time scale $\mathbb{T} = [0,\infty)$, i.e. $p(t;[0,\infty)) = -are^{-rt}$. Similarly we write $e_p(t,t_0;h\mathbb{N}_0)$ and $e_p(t,t_0;[0,\infty))$ to denote the time scales exponential functions over the time scales $h\mathbb{N}_0$ and $[0,\infty)$, respectively.

Theorem 4.4 *If $p(\cdot,h\mathbb{N}_0) \in \mathcal{R}^+$, (4.4) holds, and $t \in [0,\infty)$, then*

$$\lim_{h\to 0^+} \overline{e_p}(t,0;h\mathbb{N}_0) = e_p(t,0;[0,\infty)).$$

Proof From (4.4), we conclude that for all $\tau \in \mathbb{N}_0$,

$$0 > \frac{-arh}{(1+hr)^\tau} > \frac{-arh}{1+hr} > -1.$$

For $x > -1$, an elementary estimate for the logarithm is $\mathrm{Log}(1+x) \le x$. Using that, we see

$$\begin{aligned} \left|\xi_h\left(\frac{-ar}{(1+hr)^{\tau+1}}\right)\right| &= \left|\frac{1}{h}\mathrm{Log}\left(1+\left(\frac{-arh}{(1+hr)^{\tau+1}}\right)\right)\right| \\ &\le \left|\frac{1}{h}\left(\frac{-arh}{(1+hr)^{\tau+1}}\right)\right| \\ &= \frac{ar}{(1+hr)^{\tau+1}} \\ &< ar. \end{aligned}$$

So for $n = 1,2,3,\ldots$,

$$\left|\xi_{\frac{1}{n}}\left(\frac{-ar}{(1+\frac{1}{n}r)^{m+1}}\right)\right| < ar \quad \text{and} \quad \int_0^t ar\Delta t < art < \infty.$$

Therefore, using the Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_{n\to\infty}\int_0^t \xi_{\frac{1}{n}}\left(\frac{-ar}{\left(1+\frac{r}{n}\right)^{\tau+1}}\right)\Delta\tau &= \int_0^t \lim_{n\to\infty} \xi_{\frac{1}{n}}\left(\frac{-ar}{\left(1+\frac{r}{n}\right)^{\tau+1}}\Delta\tau\right) \\ &= \int_0^t \lim_{h\to 0^+} \xi_h\left(\frac{-ar}{(1+hr)^{\tau+1}}\Delta\tau\right). \end{aligned} \tag{4.6}$$

It remains to show that the limit inside the right-hand side equals $p(\tau;[0,\infty))$. Using L'Hôpital's rule and the chain rule, we get

$$\lim_{h\to 0^+}\xi_h(p(\tau;h\mathbb{N}_0)) = \lim_{h\to 0^+}\frac{1}{h}\mathrm{Log}(1+hp\,(\tau;h\mathbb{N}_0)) = \lim_{h\to 0^+}\frac{p(\tau;h\mathbb{N}_0)+h\frac{\mathrm{d}}{\mathrm{d}h}p(\tau;h\mathbb{N}_0)}{1+hp(\tau;h\mathbb{N}_0)}. \tag{4.7}$$

To obtain the limit, we further note that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}h}p(\tau;h\mathbb{N}_0) &= -ar\frac{\mathrm{d}}{\mathrm{d}h}\left(\frac{1}{1+hr}\right)^{\frac{\tau}{h}+1}\\
&= -ar\left(\frac{1}{1+hr}\right)^{\frac{\tau}{h}+1}\frac{\mathrm{d}}{\mathrm{d}h}\left[\left(\frac{\tau}{h}+1\right)\log\left(\frac{1}{1+hr}\right)\right]\\
&= p(\tau;h\mathbb{N}_0)\left[\left(-\frac{\tau}{h^2}\right)\log\left(\frac{1}{1+hr}\right)+\left(\frac{\tau}{h}+1\right)(1+hr)\left(\frac{-r}{(1+hr)^2}\right)\right].\\
&= p(\tau;h\mathbb{N}_0)\left[\frac{\tau(1+hr)\log(1+hr)-rh\tau-rh^2}{h^2+h^3r}\right] = p\,(\tau,h\mathbb{N}_0)\,G(h).
\end{aligned}$$

Equation (4.7) is now of the form

$$\lim_{h\to 0^+}\frac{p(\tau;h\mathbb{N}_0)+hp(\tau;h\mathbb{N}_0)G(h)}{1+hp(\tau;h\mathbb{N}_0)}.$$

Again using L'Hôpital's rule twice, we compute

$$\begin{aligned}
\lim_{h\to 0^+}G(h) = \lim_{h\to 0^+}\frac{\tau(1+hr)\log(1+hr)-rh\tau-rh^2}{h^2+h^3r} &= \lim_{h\to 0^+}\frac{r\tau\log(1+hr)-2rh}{2h+3h^2r}\\
&= \lim_{h\to 0^+}\frac{\frac{r^2\tau}{1+hr}-2r}{2+6hr} = \frac{1}{2}r(r\tau-2).
\end{aligned}$$

It is easy to see that

$$\lim_{h\to 0^+}p(\tau;h\mathbb{N}_0) = -are^{-rt},$$

hence

$$\lim_{h\to 0^+}h\frac{\mathrm{d}}{\mathrm{d}h}p(\tau;h\mathbb{N}_0) = 0.$$

We now conclude from (4.7) that

$$\lim_{h\to 0^+}\xi_h(p(\tau;h\mathbb{N}_0)) = p(\tau;[0,\infty)). \tag{4.8}$$

From (4.5), (4.6), (4.8), and $t\in[0,\infty)$, we see

$$\begin{aligned}
\lim_{h\to 0^+}\overline{e_p}(t,0;h\mathbb{N}_0) &= \exp\left(\int_{[0,t]}\lim_{h\to 0^+}\overline{\xi}_h(p(\tau;h\mathbb{N}_0))\mathrm{d}\tau\right)\\
&= \exp\left(\int_{[0,t]}p(\tau;[0,\infty))\mathrm{d}\tau\right) = \exp\left(\int_0^t -are^{-r\tau}\mathrm{d}\tau\right)\\
&= \exp\left(a\left(e^{-rt}-1\right)\right) = e_p(t,0;[0,\infty)),
\end{aligned}$$

as was to be shown. $\square$

5. Conclusion

We have shown that there is a class of probability distributions on time scales with varying parameter requirements that deserve the title “Gompertz distribution on time scales”. We have estimated classical moments, q-binomial moments, and time scales moments for these distributions. In Theorem 4.4, we showed that certain q-geometric distributions of the second kind converge to Gompertz distributions with support $[0, \infty)$ as the stepsize of the time scale approaches zero. Much work can still be done in this area: showing that $t \mapsto t^n$ is of exponential order α would establish bounds on the classical moments of G similar to Theorem 3.5, exploring the connections between the Gompertz distribution with support on $\mathbb{R}$ with the Gompertz distribution on $\mathbb{T} = h\mathbb{Z}$, and completely characterizing the parameter sets that yield a Gompertz distribution, to name a few.

6. Acknowledgments

The authors would like to thank Charalambos A. Charalambides for guidance in proving Lemma 4.2 and the Fairmont State Foundation for funding that contributed to the completion of this article.

References

[1] Ahuja JC, Nash SW. The generalized Gompertz-Verhulst family of distributions. Sankhyā: The Indian Journal of Statistics, Series A 1967; 29: 141-156.

[2] Benkhelifa L. The beta generalized Gompertz distribution. Applied Mathematical Modelling 2017; 52: 341-357.

[3] Bohner M. Some oscillation criteria for first order delay dynamic equations. Far East Journal of Applied Mathematics 2005; 18 (3): 289-304.

[4] Bohner M, Guseinov GS, Karpuz B. Properties of the Laplace transform on time scales with arbitrary graininess. Integral Transforms and Special Functions 2011; 22 (11): 785-800.

[5] Bohner M, Peterson A. Dynamic Equations on Time Scales: An Introduction with Applications. Boston, MA, USA: Birkhäuser, 2001.

[6] Cabada A, Vivero DR. Expression of the Lebesgue Δ-integral on time scales as a usual Lebesgue integral; application to the calculus of Δ-antiderivatives. Mathematical and Computer Modelling 2006; 43 (1-2): 194-207.

[7] Charalambides CA. Discrete q-distributions. Hoboken, NJ, USA: John Wiley & Sons, 2016.

[8] Cuchta T, Streipert S. Dynamic Gompertz model. Applied Mathematics and Information Science 2020; 14 (1): 1-9.

[9] Daley DJ, Vere-Jones D. An Introduction to the Theory of Point Processes: Volume I: Elementary Theory and Methods. 2nd ed. New York, NY, USA: Springer, 2003.

[10] El-Gohary A, Alshamrani A, Al-Otaibi AN. The generalized Gompertz distribution. Applied Mathematical Modelling 2013; 37 (1-2): 13-24.

[11] Guseinov GS. Integration on time scales. Journal of Mathematical Analysis and Applications 2003; 285 (1): 107-127.

[12] Jafari AA, Tahmasebi S. Gompertz-power series distributions. Communications in Statistics - Theory and Methods 2016; 45 (13): 3761-3781.

[13] Kac V, Cheung P. Quantum Calculus. New York, NY, USA: Springer, 2002.

[14] Kupershmidt BA. q-probability: I. basic discrete distributions. Journal of Nonlinear Mathematical Physics 2000; 7 (1): 73-93.

[15] Lenart A. The moments of the Gompertz distribution and maximum likelihood estimation of its parameters. Scandinavian Actuarial Journal 2014; 2014 (3): 255-277.

[16] Matthews T. Probability theory on time scales and applications to finance and inequalities. Doctoral Thesis, Missouri University of Science and Technology, Rolla, MO, USA, 2011.

[17] Oguntunde PE, Khaleel MA, Ahmed MT, Okagbue HI. The Gompertz Fréchet distribution: properties and applications. Cogent Mathematics & Statistics 2019; 6 (1): 1568662.

[18] Willekens F. Gompertz in context: the Gompertz and related distributions. In: Tabeau E, Van Den Berg Jeths A, Heathcote C (editors). Forecasting Mortality in Developed Countries Insights from a Statistical, Demographic and Epidemiological Perspective. Dordrecht, Netherlands: Kluwer Academic Publishers, 2002, pp. 105-126.