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Rotating periodic integrable solutions for second-order differential systems with nonresonance condition

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Abstract: In this paper, by using Parseval's formula and Schauder's fixed point theorem, we prove the existence and uniqueness of rotating periodic integrable solution of the second-order system $x'' + f(t,x) = 0$ with $x(t+T) = Qx(t)$ and $\int_{(k-1)T}^{kT} x(s)ds = 0$, $k \in Z^+$ for any orthogonal matrix Q when the nonlinearity f satisfies nonresonance condition.

Key words: Existence, uniqueness, rotating periodic integrable solution, Schauder's fixed point theorem

1. Introduction

This paper deals with the rotating periodic integrable problem of the following second-order differential equation:

$$x'' + f(t,x) = 0, \tag{1.1}$$

for all $t \in R, x \in R^n$, where $f(t,x) \in C^1([0,2\pi] \times R^n, R^n)$, and $f(t+T,x) = Qf(t,Q^{-1}x)$ with $T > 0$ and $Q \in O(n)$, where $O(n)$ denotes the group of orthogonal matrix. Our goal in this paper is to prove that the second-order differential equation (1.1) has a unique rotating periodic integrable solution, i.e. $x(t+T) = Qx(t)$ and $\int_{(k-1)T}^{kT} x(s)ds = 0$, $k \in Z^+$ for all $t \in R$. Under the adaptive conditions ($f(t+T,x) = Qf(t,Q^{-1}x)$), the rotating periodic integrable problem is equivalent to the problem (1.1) with rotating periodic integrable boundary value conditions (for short RPIBVP):

$$x(0) = Qx(T), \ \int_0^T x(s)ds = 0. \tag{1.2}$$

If the solutions of (1.1) only satisfies $x(t+T) = Qx(t)$, for all $t \in R$, generally, we call this kind of solutions the rotating periodic solutions of (1.1).

Periodic problems have always been the core issues in the theory of differential equations, since the concept of periodic solution was first proposed by Poincaré in the late 19th century. Recently, rotating periodic problem for second-order differential equations has arose more and more attention due to extensive attention in physics (for example, see [1]). In [7, 8], using the Maslov index theory, Hu et al. establish some important stability criteria for rotating periodic solutions of Hamiltonian systems. In 2016, using the coincidence degree

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theory Chang and Li [4] obtained the existence of rotating periodic solutions for the second-order dissipative differential systems. Later, Chang and Li [5] further had established that the second-order dynamical systems admits rotating periodic solutions under the Landesman–Lazer condition by exploiting the coincidence degree theory. Liu et al. [10] studied the asymptotically linear second-order Hamiltonian system, and by Morse theory and the technique of penalized functionals, the authors obtained the existence of rotating periodic solutions for system satisfying the resonance condition at infinity. Then, in [11, 12], using Morse theory and critical point theorems, they continued to study the existence of rotating periodic solutions for superlinear Hamiltonian systems and multiplicity of rotating periodic solutions for Hamiltonian systems with resonant conditions. In [13], they further studied the multiplicity of rotating periodic solutions for a second-order Hamiltonian systems with combined nonlinearities by using the Fountain theorem. Li et al. [14] investigated the rotating periodic problems of a class of second-order differential system, by applying on the homotopy continuation method, they proved the existence of this type solutions when the nonlinearity term satisfies the Hartman-type condition.

However, we study the existence of rotating periodic integrable solutions for problem (1.1), as far as we know, this kind of solution (rotating periodic integrable solution) is raised for the first time. When rotating periodic belongs to different ranges, different forms of solutions are obtained, such as periodic solutions if we take $Q = E_n$, where E_n stands for the identity matrix in R^n, antiperiodic solutions if we take $Q = -E_n$, subharmonic solutions if we take $Q^k = E_n$ for some $k \in Z^+$, and quasi-periodic solutions if we take $Q = diag(R(\theta_1), \cdots, R(\theta_k))$ for $n = 2k$ with $k \in Z^+$, or $Q = diag(R(\theta_1), \cdots, R(\theta_k), \pm 1)$ for $n = 2k+1$, where $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $\theta_i \in (0, 2\pi),\ i = 1, 2, \cdots, n$. Therefore, our results extend to those of [2, 3, 6] for the periodic integrable boundary value problems.

This paper is organized as follows. In Section 2, we present some lemmas, and prove the existence and uniqueness of rotating periodic integrable solution for linear differential equations. In Section 3, we get a prior estimate for nonlinear differential equations, then using Schauder's fixed point theorem we complete the proof of main results. Finally, an example is given in Section 4.

2. Linear equations

Consider the following rotation periodic integrable boundary value problem of second-order homogeneous differential equation

$$\begin{cases} x'' + A(t)x = 0, \\ x(0) = Qx(T), \int_0^T x(s)ds = 0, \end{cases} \tag{2.1}$$

where $A(t) = \mathrm{diag}(A_1(t), A_2(t), \cdots, A_n(t)) \in R^{n\times n}$ is a continuous for any $t \in [0, T]$, and satisfies

(H) there exist $N \in Z^+$ and $\varepsilon > 0$, such that

$$(\frac{2\pi}{T})^2(N^2 + \varepsilon)E \le A(t) \le (\frac{2\pi}{T})^2((N+1)^2 - \varepsilon)E,$$

for all $t \in [0, T]$, where E denotes the identity matrix.

We begin by introducing some notations and lemmas. A linear space K is defined as follows

$$K = \{l(t) \in L^2([0, T]; R^n), l'(t) \text{ is absolutely continuous on } [0, T]\},$$

with the norm

$$\|y\| = \max_{t\in[0,T]} |y(t)| + \max_{t\in[0,T]} |y'(t)|.$$

For any interval $[\alpha,\beta]\subset[0,T]$, a subspace $K_{\alpha,\beta}$ on K is given by

$$\begin{aligned} K_{\alpha,\beta} = \quad \{ \quad & l(t)\in L^2([0,T];R^n), l'(t) \text{ is absolutely continuous on } [\alpha,\beta] \text{ and} \\ & l(t)=0, \text{ for any } t\in[0,\alpha]\cup[\beta,T]\}. \end{aligned}$$

A bilinear function on $K_{\alpha,\beta}$ is given in the following

$$Z_{\alpha,\beta}(l(t),r(t)) = \int_0^T [\langle l'(t),r'(t)\rangle - \langle l(t),A(t)r(t)\rangle]dt,$$

where $l(t)$, $r(t)\in K_{\alpha,\beta}$ and $\langle\cdot,\cdot\rangle$ denotes the inner product in R^n. Let

$$X_{\alpha,\beta} := \{p(t)\in K_{\alpha,\beta} : p(t) = \sum_{m=N+1}^{\infty}(a_m\cos m\frac{2\pi}{T}t + b_m\sin m\frac{2\pi}{T}t)\},$$

$$Y_{\alpha,\beta} := \{q(t)\in K_{\alpha,\beta} : q(t) = c_0 + \sum_{k=1}^{N}(d_k\cos k\frac{2\pi}{T}t + e_k\sin k\frac{2\pi}{T}t)\}.$$

where $N\in Z^+$ is stated in assumption (H) and

$$a_m = (a_{m_1}, a_{m_2}, \cdots, a_{m_n})^\top,$$

$$b_m = (b_{m_1}, b_{m_2}, \cdots, b_{m_n})^\top,$$

$$c_0 = (c_{0_1}, c_{0_2}, \cdots, c_{0_n})^\top,$$

$$d_k = (d_{k_1}, d_{k_2}, \cdots, d_{k_n})^\top,$$

$$e_k = (e_{k_1}, e_{k_2}, \cdots, e_{k_n})^\top,$$

are constant vectors, elements of R^n. According to assumption (H), we derive that

$$\begin{cases} Z_{\alpha,\beta}(l(t),l(t)) \geq \int_0^T[|l'(t)|^2 - ((N+1)^2-\varepsilon)(\frac{2\pi}{T})^2|l(t)|^2]dt, \\ Z_{\alpha,\beta}(l(t),l(t)) \leq \int_0^T[|l'(t)|^2 - (N^2+\varepsilon)(\frac{2\pi}{T})^2|l(t)|^2]dt, \end{cases}$$

for any $l\in K_{\alpha,\beta}$ where N is given in assumption (H). For all $p\in X_{\alpha,\beta}, q\in Y_{\alpha,\beta}$, applying the Parseval's formula, we obtain that

$$\begin{cases} Z_{\alpha,\beta}(p,p) \geq \sum\limits_{m=N+1}^{\infty}\frac{2\pi^2}{T}[m^2-((N+1)^2-\varepsilon)](\sum\limits_{k=1}^{n}a_{m_k}^2 + \sum\limits_{i=1}^{n}b_{m_i}^2) \geq 0. \\ Z_{\alpha,\beta}(q,q) \leq \sum\limits_{k=1}^{N}\frac{2\pi^2}{T}[k^2-(N^2+\varepsilon)](\sum\limits_{j=1}^{n}d_{k_j}^2 + \sum\limits_{r=1}^{n}e_{k_r}^2) - \frac{4\pi^2}{T}(N^2+\varepsilon)\sum\limits_{h=1}^{n}c_{0_h}^2 \leq 0. \end{cases} \tag{2.2}$$

Therefore, $Z_{\alpha,\beta}$ is positive definite on $X_{\alpha,\beta}$, and negative on $Y_{\alpha,\beta}$. Now, we give some important lemmas.

Lemma 2.1 *(see [9], Lemma 1) Let X be a real vector space, and F is defined as a real symmetric bilinear operator on X. If $X = X_1 \oplus X_2$, where $\oplus$ means the direct sum. F is positive defined in X_1 and negative defined in X_2, then F is nondegenerate, i.e. if there are some $h \in X$, such that $F(g,h) = 0$, for all $g \in X$, then $h = 0$.*

Lemma 2.2 *Suppose that the assumption (H) holds, then the following rotating period boundary value problem (for short RPPVP)*

$$\begin{cases} x'' + A(t)x = 0, \\ x(0) = Qx(T), \end{cases} \tag{2.3}$$

has a unique trivial solution.

Proof It is easy to see that zero is a solution of RPPVB (2.3), let

$$r(t) = (r_1(t), r_2(t), \dots, r_n(t))^\top$$

be another solution of RPPVP (2.3), and

$$r_{\alpha,\beta}(t) := \begin{cases} r(t), & t \in [\alpha, \beta] \\ 0, & t \in [0, \alpha) \cup (\beta, 2\pi]. \end{cases}$$

For any $l(t) \in K_{\alpha,\beta}$, we find that

$$\int_\alpha^\beta \langle l(t), r''(t) + A(t)r(t)\rangle dt = 0,$$

which implies

$$-Z_{\alpha,\beta}(l(t), r_{\alpha,\beta}(t)) = -\int_0^T [\langle l'(t), r'_{\alpha,\beta}(t)\rangle - \langle l(t), A(t)r_{\alpha,\beta}(t)\rangle]dt = 0,$$

by using the integration by part. In light of (2.2), we obtain that $Z_{\alpha,\beta}$ is positive definite on $X_{\alpha,\beta}$ and negative definite on $Y_{\alpha,\beta}$. From Lemma 2.1, we get that $r_{\alpha,\beta} \equiv 0$, for any $t \in [\alpha, \beta]$, which means that $r(t) \equiv 0$ for any $t \in [\alpha, \beta]$. The proof is completed. □

Lemma 2.3 *If the assumption (H) holds, then RPIBVP (2.1) has a unique trivial solution.*

Proof From Lemma 2.2, we can see that RPIBVP (2.1) has at least one solution, e.g., $x_* = 0$, for reduction to absurdity, we suppose that there is nonzero solution $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^\top$. Then the following proof is decomposed into two cases.

Case 1 $x^*(0) = Qx^*(T) = 0$.

By Lemma 2.2 ($\alpha = 0$ and $\beta = T$), we observe that RPIBVP (2.1) has a unique trivial solution, which leads to a contradiction $x^* \neq 0$.

Case 2 $x^*(0) = Qx^*(T) = \eta$, where $\eta = (\eta_1, \eta_2, \ldots, \eta_n)^\top$.

Denote $S_i = \{t \in [0,T]; x_i^*(t) = 0\}$, and $a_i = \inf_{t\in S_i} t$ and $b_i = \sup_{t\in S_i} t$, for $i = 0, 1, \cdots, n$. For simplicity, considering its a component $x_i^* \not\equiv 0$. Here the rest part of proof is broken down into three sections:

- When $t \in [a_i, b_i] \subset [0,T]$, we claim $x_i^*(t) \equiv 0$. From the definition of S_i, we know that $x_i^*(a_i) = x_i^*(b_i) = 0$. The Lemma 2.2 ($\alpha = a_i, \beta = b_i$) yields that the following rotation periodic boundary value problem

$$\begin{cases} x_i'' + A_i(t)x_i = 0, \\ x_i(a_i) = Qx_i(b_i) = 0, \end{cases} \tag{2.4}$$

  has a unique solution. Hence $x_i^*(t) \equiv 0$, for $t \in [a_i, b_i]$.

- When $t \in [0, a_i]$, we claim $x_i^*(t) \equiv 0$. Due to $\int_0^T x_i^*(t)dt = 0$, then it follows from RPBVP (2.4) that $\int_0^{a_i} x_i^*(t)dt = -\int_{b_i}^T x_i^*(t)dt$, without loss of generality, assume $x_i(0) = \eta_i > 0$, which means that $x_i^*(t) > 0$ for any $t \in [0, a_i)$. For $t \in [0, a_i)$, consider the following boundary value problem

$$\begin{cases} x_i'' + A_i(t)x_i = 0, \quad t \in [0, a_i) \\ x_i(0) = \eta_i, x_i(a_i) = 0. \end{cases} \tag{2.5}$$

  From $x_i'' = -A_i(t)x_i$ ($A_i(t) > 0$), we have that $x_i''(t) < 0$, for $t \in [0, a_i)$. Since $x_i'(t)$ is continuous at a_i, so we infer from $x_{i+}'(a_i) = 0$ (right derivative) that $x_i'(a_i) = 0$, which gives $x_i'(t) > 0$ for any $t \in [0, a_i)$. Following $x_i(t) > 0$ for any $t \in [0, a_i)$, so x_i increases monotonously in $[0, a_i)$, which yields $x_i(a_i) > x_i(0) = \eta_i > 0$, with a contradict to $x_i(a_i) = 0$. If $\eta_i < 0$, the proof is similar to the aforementioned.

- When $t \in (b_i, T]$, the proof is similar to the case of $t \in [0, a_i)$, we have $x_i^*(t) \equiv 0$.

In conclusion, we have $x_i^* \equiv 0$ for any $t \in [0,T]$ and $i = 1, 2, \cdots, n$, i.e. RPIBVP (2.1) has a unique trivial solution. □

Lemma 2.4 *Assume that $P(t)$ is continuous in R^n, and $A(t)$ satisfies the assumption (H), then the following RPIBVP*

$$\begin{cases} x'' + A(t)x = P(t), \\ x(0) = Qx(T), \int_0^T x(s)ds = 0, \end{cases} \tag{2.6}$$

has a unique solution.

Proof Let $X_1(t), X_2(t), \ldots, X_{2n}(t)$ be $2n$ linear independent solutions of the equation as follows:

$$x'' + A(t)x = 0,$$

where $X_i(t) = (X_{i1}(t), X_{i2}(t), \ldots, X_{in}(t))^\top \in R^n$, for $i = 1, 2, \ldots, 2n$. Suppose that $Y(t) = C_1 \cdot X_1(t) + C_2 \cdot X_2(t) + \ldots + C_{2n} \cdot X_{2n}(t)$ is the general solutions of RPIBVP (2.1), where $C_1, C_2, \ldots, C_{2n}$ are constants in R. Thanks to the rotating periodic integrable boundary value condition of (2.1), we get

$$\begin{cases} (X_1(0) - QX_1(T), X_2(0) - QX_2(T), \ldots, X_{2n}(0) - QX_{2n}(T)) \cdot \mathbf{C} = 0, \\ (\int_0^T X_1(s)ds, \int_0^T X_2(s)ds, \ldots, \int_0^T X_{2n}(s)ds) \cdot \mathbf{C} = 0, \end{cases}$$

where $\mathbf{C} = (C_1, C_2, \ldots, C_{2n})^\top \in R^{2n}$. An application of the Lemma 2.3 leads to the RPIBVP (2.1) has a unique trivial solution, which implies the following determinant

$$\begin{vmatrix} X_1(0) - QX_1(T) & X_2(0) - QX_2(T) & \ldots & X_{2n}(0) - QX_{2n}(T) \\ \int_0^T X_1(s)ds & \int_0^T X_2(s)ds & \ldots & \int_0^T X_{2n}(s)ds \end{vmatrix} \neq 0 \tag{2.7}$$

Suppose that $X(t) = C_1X_1(t) + C_2X_2(t) + \ldots + C_{2n}X_{2n}(t) + X_0(t)$ is the general solutions of RPIBVP (2.6), where $X_0(t)$ is a special solution of RPIBVP (2.6). Invoking the rotating periodic integrable boundary value conditions, $X(t)$ satisfies the following second-order inhomogeneous linear equations

$$\left\{ \begin{array}{l} (X_1(0) - QX_1(T), X_2(0) - QX_2(T), \ldots, X_{2n}(0) - QX_{2n}(T)) \cdot \mathbf{C} \\ = QX_0(T) - X_0(0), \\ (\int_0^T X_1(s)ds, \int_0^T X_2(s)ds, \ldots, \int_0^T X_{2n}(s)ds) \cdot \mathbf{C} = -\int_0^T X_0(s)ds. \end{array} \right. \tag{2.8}$$

The determinant (2.7) implies that the constants $C_1, C_2, \ldots, C_{2n}$ in equations (2.8) are unique. Therefore the RPIBVP (2.1) has a unique solution, which complete the proof. □

3. Nonlinear equations

First we state our main result.

Theorem 3.1 *Assume that* $f(t,x) = (f_1(t,x_1), f_2(t,x_2), \cdots, f_n(t,x_n))^\top$, *and*
(A) *there exist* $N \in Z^+$ *and* $\varepsilon > 0$, *such that*

$$\left(\frac{2\pi}{T}\right)^2 (N^2 + \varepsilon) \leq \frac{\partial f_i(t,x_i)}{\partial x_i} \leq [(N+1)^2 - \varepsilon]\left(\frac{2\pi}{T}\right)^2,$$

for $i = 1, 2, \cdots, n$, *then the differential system (1.1) has a unique rotating periodic integrable solution.*

Generally speaking, the assumption (A) is called the nonresonance condition, where the set $\{N^2\}$ is called the set of resonance points. Under given the adaptive conditions, now we use the truncation technique to transform problem (1.1) into the following rotating periodic integral boundary value problem

$$\left\{ \begin{array}{l} x'' + B_h(x)x = f(t,0), \\ x(0) = Qx(T), \int_0^T x(s)ds = 0, \end{array} \right. \tag{3.1}$$

where $B_h(x) = \mathrm{diag}(h_1(t,x_1), h_2(t,x_2), \ldots, h_n(t,x_n))$, with

$$h_i(t,x_i) = \int_0^1 f_{x_i}(t, \theta x_i)d\theta.$$

A subspace of K is defined by

$$\begin{aligned} K^* = \{ \; & l(t) \in L^2([0,T]; R^n), l'(t) \text{ is absolutely continuous on } [0,T] \\ & l(0) = Ql(T) \text{ and } \int_0^T l(s)ds = 0\}. \end{aligned}$$

For any $y \in K^*$, consider an auxiliary RPIBVP

$$\begin{cases} x'' + B_h(y)x = f(t,0), \\ x(0) = Qx(T), \int_0^T x(s)ds = 0. \end{cases} \tag{3.2}$$

To complete the proof of Theorem 3.1, the following two lemmas are needed.

Lemma 3.2 *If f satisfies assumption (A), then for any $y \in K^*$, the solutions of RPIBVP (3.2) devoted as $x_y(t)$ satisfies $\|x_y\| \leq M$, where M is a positive constant.*

Proof In light of (A), we have

$$(\frac{2\pi}{T})^2(N^2+\varepsilon)E \leq B_h(y) \leq (\frac{2\pi}{T})^2((N+1)^2-\varepsilon)E. \tag{3.3}$$

Lemma 2.4 implies that RPIBVP (3.2) has only one solution $x_y(t)$ for each $y \in K^*$. For reduction to absurdity, it is assumed that there exist a sequence $\{y_m\}$, such that $\|x_{y_m}\| \to \infty$, as $m \to \infty$. Choose a subsequence of $\{B_h(y_m)\}_{m=1}^{\infty}$, without loss of generality, denoted by itself which is weakly convergent in $L^2([0,T], R^{n\times n})$. Let the limit be denoted by $B_{h_0}(t)$, it is easy to see $B_{h_0}(t) \in L^2([0,T], R^{n\times n})$. Since

$$\begin{aligned} \mathbb{S} := \quad \{ \quad & A(t) \in L^2([0,T], R^{n\times n}),\ A(t) \text{ is a diagonal matrix and} \\ & (\frac{2\pi}{T})^2(N^2+\varepsilon)E \leq A(t) \leq (\frac{2\pi}{T})^2((N+1)^2-\varepsilon)E\} \end{aligned}$$

is a bounded convex set in $L^2([0,T], R^{n\times n})$, the Mazur theorem gives $B_{h_0}(t) \in \mathbb{S}$. Hence we get

$$(\frac{2\pi}{T})^2(N^2+\varepsilon)E \leq B_{h_0}(t) \leq (\frac{2\pi}{T})^2((N+1)^2-\varepsilon)E.$$

Let $x_m := \frac{x_{y_m}}{\|x_{y_m}\|}$, then $\|x_m\| = 1$. On the basis of Arzela–Ascoli theorem, passing to the subsequence $x_{m_i} \to x_0$ and $x'_{m_i} \to z(t)$ in $C([0,T], R^n)$. Obviously, x_0 satisfies the boundary of $x_0(0) = Qx_0(T)$ and $\int_0^T x_0(s)ds = 0$. Then

$$x_{m_i}(t) = x_{m_i}(0) + \int_0^t x'_{m_i}(s)ds \to x_0(t) = x_0(0) + \int_0^t z(s)ds, \tag{3.4}$$

implies $z(t) = x'_0(t)$ for any $t \in [0,T]$ and $\|x_0\| = 1$. From RPIBVP (3.2), it follows that

$$\begin{cases} x''_{m_i}(t) + B_h(y_m)x_{m_i} = \frac{-f(t,0)}{\|x_{y_{m_i}}\|}, \\ x_{m_i}(0) = Qx_{m_i}(T), \int_0^T x_{m_i}(s)ds = 0. \end{cases} \tag{3.5}$$

As $m \to \infty$, RPIBVP (3.5) turns into the following RPIBVP,

$$\begin{cases} x''_0 + B_{h_0}(t)x_0 = 0, \\ x_0(0) = Qx_0(T), \int_0^T x_0(s)ds = 0. \end{cases} \tag{3.6}$$

By taking into account Lemma 2.3, RPIBVP (3.6) has a unique trivial solution, which leads to a contradiction with $\|x_0\| = 1$, which completes the proof. □

Set $B_M := \{x \in K^*, \|x\| \leq M\}$. An operator $\Omega : K^* \to K^*$ is defined by $\Omega(y) = x_y(t)$. Due to Lemma 3.2, we have $\Omega : B_M \to B_M$.

Lemma 3.3 *The operator Ω is completely continuous on K^*.*

Proof By Lemma 3.2, it suffices to show that the operator Ω is continuous in K^*. Let $\{y_m\}_{m\geq 1}\subset K^*$ such that $y_m\to y_0\in K^*$ as $m\to\infty$, and $l_m:=x_{y_m}-x_{y_0}$. From (3.2), one has that

$$\begin{cases} l_m''+B_h(y_m)l_m=(B_h(y_0)-B_h(y_m))x_{y_0}, \\ l_m(0)=Ql_m(T), \int_0^T l(s)ds=0, \end{cases} \tag{3.7}$$

where

$$x_{y_0}=(x_{y_{0_1}},x_{y_{0_2}},\dots,x_{y_{0_n}})^\top,$$

$$B_h(y_0)=\mathrm{diag}(h(t,y_{0_1}),h(t,y_{0_2}),\dots,h(t,y_{0_n})),$$

$$B_h(y_m)=\mathrm{diag}(h(t,y_{m_1}),h(t,y_{m_2}),\dots,h(t,y_{m_n})).$$

Now we will show that $l_m\to 0$ in $C([0,T],R^n)$. If not, there would be a constant $c>0$ such that $\lim\limits_{m\to\infty}\sup\|l_m\|\geq c$. Applying Lemma 3.2 and Arzela–Ascoli theorem, passing to a subsequence, we assume that $l_m\to l_0$ in $C([0,T],R^n)$. Evidently, we can easily see that $B_h(y_0)-B_h(y_m)\to 0$ as $m\to\infty$. Hence, the boundary value problem (3.7) becomes

$$\begin{cases} l_0''+B_h(y_0)l_0=0, \\ l_0(0)=Ql_0(T), \int_0^T l_0(s)ds=0, \end{cases} \tag{3.8}$$

where

$$(\frac{2\pi}{T})^2(N^2+\varepsilon)E\leq B_h(y_0)\leq(\frac{2\pi}{T})^2((N+1)^2-\varepsilon)E.$$

Therefore, following Lemma 2.3, we obtain that $l_0(t)\equiv 0$, which follows the conclusion. □

Now we present the proof of Theorem 3.1.

In view of Lemma 3.2, Lemma 3.3 and Schauder's fixed point theorem, we conclude that Ω has a fixed point on K^*, which is the solution of RPIBVP (1.1). The following is to prove uniqueness. Let $m(t)$ and $n(t)$ be any two different solutions of RPIBVP (1.1). Then let $x(t)=m(t)-n(t)$ be the solution of the equation $x''+B_{f_x}(t)x=0$ with the same boundary value condition of (2.3). In light of assumption (A), we deduce

$$(\frac{2\pi}{T})^2(N^2+\varepsilon)E\leq B_{f_x}(t)\leq(\frac{2\pi}{T})^2((N+1)^2-\varepsilon)E,$$

where $B_{f_x}(t)=$diag$(\theta_1(t),\theta_2(t),\dots,\theta_n(t))$ and $\theta_i(t)=\int_0^1 f_{x_i}(t,n_i(t)+sx_i(t))ds$. Hence by Lemma 2.4, we infer that $x(t)\equiv 0$. So the uniqueness is proved, which follows our desired result.

Remark 3.4 *If the assumption (A) is replaced with the following,*
(A_1) there exist $N\in Z^+$ and $\varepsilon>0$, such that

$$(\frac{2\pi}{T})^2(N^2+\varepsilon)\leq\frac{f_i(t,x_i)}{x_i}\leq((N+1)^2-\varepsilon)(\frac{2\pi}{T})^2,$$

for all $t\in[0,T]$, and $i=1,2,\dots,n$, then the differential system (1.1) has a unique rotating periodic integrable solution. The proof is similar to that of Theorem 1.

Remark 3.5 *Consider the following differential equation:*

$$(\lambda(t)x')' + f(t,x) = 0, \tag{3.9}$$

where $\lambda(t) = \mathrm{diag}(\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)) \in R^{n\times n}$, $\lambda_i(t) \in C(R, R^+)$, *satisfies* $\lambda(t+T) = Q^{-1}\lambda(t)Q$, *for all* $t \in R$, $f(t,x) = (f_1(t,x_1), f_2(t,x_2), \cdots, f_n(t,x_n))^\top$, *with* $f_i(t,x_i) \in C([0,2\pi]\times R, R)$, *for any* $i = 1,2,\cdots,n$, *and* $f(t+T,x) = Qf(t,Q^{-1}x)$ *with* $T>0$ *and* $Q \in O(n)$. *If the following assumption hold,*
(A_2) *there exist* $N \in Z^+$ *and* $\varepsilon > 0$, *such that*

$$(\frac{2\pi}{T})^2(N^2+\varepsilon) \le \frac{f_i(t,x_i)}{\lambda_i(t)} \le ((N+1)^2-\varepsilon)(\frac{2\pi}{T})^2,$$

then by Remark 3.4 the differential system (3.9) has a unique rotating periodic integrable solution.

4. Example

Consider (Q,T)-rotating periodic integrable problem of the following second-order nonlinear differential system

$$x'' + (N^2+\lambda-\frac{1}{1+|x|^2})x + \left(\sin t, \cos t, \sin\frac{2\pi t}{T}, \cos\frac{2\pi t}{T}\right)^\top = 0, \tag{4.1}$$

where $N \in Z^+$, $\lambda \in (N+1, (N+1)^2+2)$, and

$$Q = \begin{bmatrix} \cos(2\pi-T) & -\sin(2\pi-T) & 0 & 0 \\ \sin(2\pi-T) & \cos(2\pi-T) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The problem (4.1) is equivalent to the following RPBVP

$$\begin{cases} x'' + (N^2+\lambda-\frac{1}{1+|x|^2})x + \left(\sin t, \cos t, \sin\frac{2\pi t}{T}, \cos\frac{2\pi t}{T}\right)^\top = 0, \\ x(0) = Qx(T),\ \int_0^T x(s)ds = 0. \end{cases} \tag{4.2}$$

Let $f(t,x) = (N^2+\lambda-\frac{1}{1+|x|^2})x + (\sin t, \cos t, \sin\frac{2\pi t}{T}, \cos\frac{2\pi t}{T})^\top$. It is not difficult to check that the assumption (A) in Theorem 3.1 is satisfied. Next, we will show

$$Qf(t,Q^{-1}x) = f(t+T,x).$$

Note $Q \in O(n)$, then $Q^\top Q = E$. Thus, we have

$$
\begin{aligned}
&Qf(t, Q^{-1}x) \\
&= (N^2 + \lambda - \frac{1}{1 + |Q^{-1}x|^2})x + \begin{bmatrix} \cos(2\pi - T) & -\sin(2\pi - T) & 0 & 0 \\ \sin(2\pi - T) & \cos(2\pi - T) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sin t \\ \cos t \\ \sin \frac{2\pi t}{T} \\ \cos \frac{2\pi t}{T} \end{bmatrix} \\
&= (N^2 + \lambda - \frac{1}{1 + \langle Q^{-1}x, Q^{-1}x\rangle})x + \begin{bmatrix} \cos(2\pi - T)\sin t - \sin(2\pi - T)\cos t \\ \sin(2\pi - T)\sin t + \cos(2\pi - T)\cos t \\ \sin \frac{2\pi t}{T} \\ \cos \frac{2\pi t}{T} \end{bmatrix} \\
&= (N^2 + \lambda - \frac{1}{1 + x^\top Q Q^\top x})x + \begin{bmatrix} \cos(t + T - 2\pi) \\ \sin(t + T - 2\pi) \\ \sin \frac{2\pi t}{T} \\ \cos \frac{2\pi t}{T} \end{bmatrix} \\
&= (N^2 + \lambda - \frac{1}{1 + x^\top x})x + \begin{bmatrix} \cos(t + T) \\ \sin(t + T) \\ \sin \frac{2\pi}{T}(t + T) \\ \cos \frac{2\pi}{T}(t + T) \end{bmatrix} \\
&= (N^2 + \lambda - \frac{1}{1 + |x|^2})x + \begin{bmatrix} \cos(t + T) \\ \sin(t + T) \\ \sin \frac{2\pi}{T}(t + T) \\ \cos \frac{2\pi}{T}(t + T) \end{bmatrix} \\
&= f(t + T, x),
\end{aligned}
$$

so the adaptive condition of problem (4.1) clearly holds. Therefore, from Theorem 3.1, the system (4.1) has a unique rotating periodic integrable solution.

5. Conclusion

The present paper provides the existence and uniqueness of rotating periodic integrable solution based on Parseval's formula and Schauder's fixed point theorem for a second-order system under the nonresonance condition. However, the existence results for the second-order differential system with cross resonance condition is still an open problem which would be our next future work.

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