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Polyhedral optimization of second-order discrete and differential inclusions with delay

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Abstract: The present paper studies the optimal control theory of second-order polyhedral delay discrete and delay differential inclusions with state constraints. We formulate the conditions of optimality for the problems with the second-order polyhedral delay discrete (PD_d) and the delay differential (PC_d) in terms of the Euler–Lagrange inclusions and the distinctive "transversality" conditions. Moreover, some linear control problem with second-order delay differential inclusions is given to illustrate the effectiveness and usefulness of the main theoretic results.

Key words: Polyhedral, Euler–Lagrange inclusion, second-order delay differential inclusions, transversality

1. Introduction

Optimal control problems with ordinary and partial differential inclusions, set valued mappings cover one of the intensive developmental areas of optimal processes in the mathematical theory (see [1, 2, 4, 6, 10, 13, 30, 36, 37, 39] and their references). This paper deals with optimal control problems described by polyhedral discrete and differential inclusions with delay constraints, and this class of problems is truly challenging and underinvestigated in control theory while being highly important for various applications; Moreau's sweeping process and Von Neumann–Gale dynamical systems, etc. Note that these problems also occur not only in mechanics, aerospace engineering, management science, and economy but also automatic control, auto vibration, and biophysics [19, 34, 35].

Neutral systems have some similarities with the so-called hybrid and differential-algebraic equations which are significant for applications in engineering control. In the paper [16] sufficient conditions for the controllability of neutral functional differential and integrodifferential inclusions with infinite delay are derived in a Banach space. The paper [33] discusses optimal control problems for dynamical systems controlled by constrained neutral type functional differential inclusions. These control systems include time delays not only in state variables but also in velocity variables, making them much more complex than delay differential inclusions.

The existence of state constraints on an optimal control trajectory is sufficiently demonstrated by producing discontinuities in the corresponding adjoint arc. The existence of mild solutions for a first-order impulsive

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neutral evolution differential inclusion with state-dependent delay is proved in the paper [15]. The study [11] is devoted to the problem of robust stability for a class of time-delay systems and it is presumed that the uncertainties in the systems are time-invariant and belong to a convex bounded polytopic domain. The concept of set invariance for time-delay systems is introduced in the paper [17] with special attention to the linear discrete-time case and the description of a delay invariant set concerning for to a bounded polyhedral subset of the state space is given.

We aim to establish well verifiable necessary and sufficient conditions of optimality for polyhedral delay discrete and delay differential inclusions with state constraints. These conditions are more precise than any previously published as they provide useful forms of adjoint inclusions of Euler–Lagrange-type and ”transversality” conditions. As we have already suggested, we expect all these developments to support the future development of optimal control theory with delay discrete and delay differential inclusions.

Due to higher-order derivatives, the problems associated with higher-order differential inclusions are more complicated. The practical approach for eliminating this complexity in the optimal control theory involving higher order derivatives is the systematic transformation of such problems into a set of first-order differential inclusions or equations. It happens that, in practice, it is very difficult to return to the original higher-order problem and express the optimized conditions usually obtained from the original problem data set. Although the construction of adjoint inclusions and transversality conditions is more complicated, in the paper [27], the conditions of optimality are formulated for the optimal control problem of higher-order differential inclusions with functional constraints.

For second-order discrete and differential inclusions, the existence of solutions and other qualitative properties has been intensively analyzed in the recent literature (see [3, 5, 7, 12, 14, 29] and their references). The paper [8] deals with the existence of viable solutions to the Cauchy problem $x'' \in F(x,x')$, $x(0)=x_0$, $x'(0)=y_0$, where F is a set valued map defined on a locally compact set $M \subset \mathbb{R}^{2n}$, contained in the Fr$\acute{e}$chet subdifferential of a φ-convex function of order two.

Moreover the paper [9] examines the existence and controllability of fractional order evolution class inclusions with varying time delays. Then by applying the Glicksberg–Ky Fan fixed point theorem, the controllability of this system with a nonlocal condition is established. For a nonlinear control problem with constant delay in phase coordinates and with general function, the existence theorems of the optimal element are proved in the work [38].

The key methods of our investigations, along with the generalized differential calculation, focus on the extremal theory and its modifications. Using the method of discrete approximations and generalized differentiation principles, we establish the conditions of optimality for problems in the Euler–Lagrange forms. The paper [32] is dedicated to studying a general class of optimal control problems defined by delay differential inclusions with endpoint constraints of equality and inequality and set valued initial conditions. The method of discrete approximations and advanced methods of variational analysis and generalized differentiation in infinite dimensions is used to derive necessary optimality conditions in the extended Euler–Lagrange form. The approach for the delay differential systems under consideration is fully implemented.

Therefore, we can use the results for discrete approximation problems to obtain sufficient conditions of optimality for second-order polyhedral delay differential inclusions. Besides, many of the necessary and sufficient conditions for optimization of differential inclusions inevitably require the creation of new types of equivalence in the Mahmudov survey papers [20]–[26]. The paper [28] develops the theory of duality for the Mayer problem given by second-order evolution differential inclusions with delay and state constraints. Although all the proofs in that work relating to dual problems are carried out in the case of delay, the results are also considered without delay. So we formulate the conditions of optimality in Corollary 5.2 for the problem (PC_d) with nondelayed second-order differential inclusions.

The present paper relates with delay to one of the complicated and important field – polyhedral optimization of the second-order discrete and differential inclusions. The problems presented and the corresponding conditions of optimality are new. The paper is organized into the order that follows.

For the convenience of the reader, the necessary definitions and additional results from Mahmudov's book [18] are summarized in Section 2. Specifically, set valued mapping F and local adjoint mapping (LAM) are given and polyhedral optimization problems are introduced for discrete and differential second-order inclusions.

In Section 3, the problem (PD_d) is reduced to a polyhedral minimization problem with geometric polyhedral constraints. Due to the use of convex and nonsmooth analysis structures, the necessary and sufficient conditions of optimality are derived for this problem in terms of the Euler–Lagrange polyhedral inclusions. Note that the reasons for adopting discrete modeling are as follows: First, statistics are collected at discrete times (day, week, month, or year). Thus, discrete-time models can be described in a more straightforward, more accurate and timely manner than continuous-time models. Secondly, the use of discrete-time models avoids some mathematical complexities such as the choice of the function space and the regularity of the solution. Third, the numerical simulation of continuous-time models is obtained through discretization.

In Section 4, the second-order discrete approximation problem is associated with the (PC_d) problem by using the first and the second-order difference operators. By reducing this problem to the form of problem (PD_d), we apply Theorem 3.2 to a discrete approximation problem and obtain the necessary and sufficient conditions of optimality for a discrete approximation problem. It is clear that this method, which is unquestionable of independent interest from a qualitative point of view, can also play an important role in numerical procedures.

In Section 5, the results obtained in Section 4 are used to provide sufficient conditions of optimality for the (PC_d) problem. The derivation of these conditions is implemented by passing to the limit, where the discrete steps tend to be zero. Thus, using the discretization method, we deduce the sufficient conditions of optimality for the (PC_d) problem in the Euler–Lagrange forms. Of course, using the suggested methods for ordinary differential inclusions of Mordukhovich [31], it can be shown that sufficient conditions are also necessary to achieve optimum results. However, proof of the necessary conditions is a separate subject of discussion and is omitted from this article.

2. Necessary concepts and preliminary problem statements

The necessary concepts can be found in [18]. Let $\mathbb{R}^n$ be a n-dimensional Euclidean space, $\langle x,v\rangle$ be an inner product of $x,v\in\mathbb{R}^n$, (x,v) be a pair of x,v elements. Suppose that $F:\mathbb{R}^{3n}\rightrightarrows\mathbb{R}^n$ is a set valued mapping from $\mathbb{R}^{3n}$ into the set of subsets of $\mathbb{R}^n$. Then $F:\mathbb{R}^{3n}\rightrightarrows\mathbb{R}^n$ is convex if its graph $gphF=\{(x,y,z,v):v\in F(x,y,z)\}$

is a convex subset of $\mathbb{R}^{4n}$. A set-valued mapping F is convex closed if its $gphF$ is a convex closed set in $\mathbb{R}^{4n}$. It is convex valued if $F(x,y,z)$ is a convex set for each $(x,y,z) \in domF = \{(x,y,z) : F(x,y,z) \neq \emptyset\}$.

A polyhedral convex set in $\mathbb{R}^n$ is a set that can be expressed as the intersection of some finite family of closed half-spaces, that is, as the set of solutions to some finite system of inequalities of the form $\langle x, x_k^* \rangle \leq \beta_k, \; k = 1, \ldots, l$. The definition of a polyhedral set makes it immediately clear why such sets play a leading role in linear programming. In particular, if the finite system of inequalities is homogeneous, the set of solutions to this finite system of inequalities is called the polyhedral cone. A bounded polyhedral convex set is a polytope (polyhedron) that is a convex hull of finitely many points. The polyhedral set is closed.

The convex cone $K_A(z_0)$, $z_0 = (x^0, v_1^0, v_2^0, v_3^0)$ is called the cone of tangent directions at the point $z_0 \in A$ if from $\overline{z} = (\overline{x}, \overline{v}_1, \overline{v}_2, \overline{v}_3) \in K_A(z_0)$ it follows that $\overline{z}$ is a tangent vector to the set $A \subset \mathbb{R}^{4n}$ at the point z_0. In other words, there exists such function $\mu : \mathbb{R} \to \mathbb{R}^{4n}$ such that $z_0 + \lambda\overline{z} + \mu(\lambda) \in A$ for sufficiently small $\lambda > 0$ and $\lambda^{-1}\mu(\lambda) \to 0$ as $\lambda \downarrow 0$.

It should be noted that the $K_A(z_0)$ cone is not uniquely defined. In any case, we can see that the wider the cone of tangent directions, we have a basic condition for a minimum. Clearly, for the convex set A at the point $(x^0, v_1^0, v_2^0, v_3^0) \in A$ setting $\mu(\lambda) \equiv 0$, we have for all $(x, v_1, v_2, v_3) \in A$,

$$K_A(z_0) = \Big\{(\overline{x}, \overline{v}_1, \overline{v}_2, \overline{v}_3) : \overline{x} = \lambda(x - x^0), \;\; \overline{v}_1 = \lambda(v_1 - v_1^0), \; \overline{v}_2 = \lambda(v_2 - v_2^0), \overline{v}_3 = \lambda(v_3 - v_3^0)\;, \lambda > 0\Big\}.$$

In general, convex functions cannot be differentiated. However, these functions have many useful differential properties and one of them is the universal existence of directional derivatives. Moreover, in the case of convex functions, the notion of subgradient can be defined and the set of subgradients provides the "subdifferential" concept.

The Mordukhovich's subdifferential consists of the main class of generalized differentials and plays a vital role in pure and applied analysis. Mordukhovich published two volumes of the book [31], which provide key issues for these subdifferentials from theoretical and applied points of view in modern variational analysis.

In the case of set valued mappings, a similar role is played by an important concept the "locally adjoint mappings". In general, for a set valued mapping $F : \mathbb{R}^{3n} \rightrightarrows \mathbb{R}^n$, a set-valued mapping $F^*(\cdot, x, u_1, u_2, v) : \mathbb{R}^n \rightrightarrows \mathbb{R}^{3n}$ defined by the following:

$$F^*\big(v^*; (x, u_1, u_2, v)\big) := \Big\{(x^*, u_1^*, u_2^*) : (x^*, u_1^*, u_2^*, -v^*) \in K^*_{gphF}(x, u_1, u_2, v)\Big\},$$

is called the LAM to the set-valued mapping F at a point $(x, u_1, u_2, v) \in gphF$, where $K^*_{gphF}(x, u_1, u_2, v)$ is the dual to the cone of tangent directions $K_{gphF}(x, u_1, u_2, v)$. Here $K^* := \{z^* : \langle \overline{z}, z^* \rangle \geq 0, \; \forall \, \overline{z} \in K\}$ denotes the dual cone to the cone K, as usual.

In the paper, we deal with the following second-order polyhedral delay discrete model with state constraints

$$\text{minimize} \sum_{t=1}^{T-1} f(x_t,t), \tag{2.1}$$

$$(PD_d) \qquad x_{t+2} \in F(x_t, x_{t+1}, x_{t-h}),\ t = 0,1,\ldots,T-2,$$

$$x_t = \alpha_t,\ t = -h,-h+1,\ldots,-1,\ x_0 = \beta_0,$$

$$x_t \in P_t,\ t = 1,2,\ldots,T,\ x_T \in \Omega, \tag{2.2}$$

$$F(x,v_1,v_2) = \{v_3 : A_0x + A_1v_1 + A_2v_2 - Bv_3 \le d\}.$$

Here $x_t \in \mathbb{R}^n$, $f(\cdot,t) : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is a polyhedral function that is its epigraph epi $f(\cdot,t)$ is polyhedral set in $\mathbb{R}^{n+1}$ and $F : \mathbb{R}^{3n} \rightrightarrows \mathbb{R}^n$ is a polyhedral set valued mapping, A_0, A_1, A_2 and B, $m \times n$ dimensional matrices, d is a m-dimensional column vector, T, h are fixed natural numbers, β_0, α_t, $t = -h, -h+1, \ldots, -1$ are fixed vectors. Additionally, $P_t \subseteq \mathbb{R}^n$ $t = 1,\ldots,T$ and $\Omega \subseteq \mathbb{R}^n$ are convex sets.

A sequence $\{x_t\}_{t=-h}^{T} = \{x_t :\ t = -h, -h+1, \ldots, T\}$ is called a feasible trajectory for the stated problem. In fact, a model of economic dynamics (PD_d) described by discrete inclusions with constant delay is considered; the functioning of some economic system take place at the discrete times $t = 0,1,\ldots,T$ and that at time t one has a resource vector $(x, u_1, u_2) \in \mathbb{R}^{3n}$ which can be transformed at time $t+1$ to one of the vectors $v \in F(x, u_1, u_2)$. Here is assumed that all possible amounts of resources $(x_t, x_{t+1}, x_{t-h}), t = 0,\ldots,T$ are connected by $x_{t+2} \in F(x_t, x_{t+1}, x_{t-h})$, $t = 0,1,\ldots,T-2$, $x_t \in P_t$, $t = 1,\ldots,T$, $x_T \in \Omega$, where $x_t = \alpha_t$, $x_0 = \beta_0$, are the vectors of initial resources. Usually, $\sum_{t=1}^{T-1} f(x_t,t)$ can be interpreted as the total expenditure.

Mostly, for this paper, we consider second-order polyhedral differential inclusion with delay

$$\text{minimize } J[x(\cdot)] = \int_0^T f(x(t),t)dt + \varphi_0(x(T)),$$

$$(PC_d) \qquad x''(t) \in F(x(t), x'(t), x(t-h)), \quad \text{a.e.} \quad t \in [0,T], \tag{2.3}$$

$$x(t) = \alpha(t),\ t \in [-h, 0), \quad x(0) = \beta,$$

$$x(t) \in P(t),\ t \in [0,T], \quad x(T) \in \Omega. \tag{2.4}$$

Here $F : \mathbb{R}^{3n} \rightrightarrows \mathbb{R}^n$ and $f(\cdot,t), \varphi_0 : \mathbb{R}^n \to \mathbb{R}$ are time dependent polyhedral set valued mapping and real-valued proper continuous functions, respectively, and $\Omega \subseteq \mathbb{R}^n$ is a convex set, $\alpha(t)$, $t \in [-h,0)$ is an absolutely continuous initial function, β is a fixed vector, $P : [0,T] \rightrightarrows \mathbb{R}^n$ is a convex set valued mapping.

It is required to find a feasible trajectory $x(t)$, $t \in [-h,T]$ minimizing the Bolza functional $J[x(\cdot)]$ over a set of feasible trajectories. Here, a feasible trajectory $x(t); t \in [-h,T]$ satisfies state constraints everywhere in $[0,T]$, endpoint constraint $x(T) \in \Omega$, the second-order polyhedral differential inclusions with delay whose second-order derivative in $[0,T]$ belongs to the standard Lebesgue space $L_1^n([0,T])$. In more detail, a feasible solution $x(\cdot)$ of (PC_d) is a mapping $x(\cdot) : [-h,T] \to \mathbb{R}^n$ satisfying $x''(t) \in F(x(t), x'(t), x(t-h))$, a.e. $t \in [0,T]$, $x(t) = \alpha(t)$, $t \in [-h,0)$, $x(t) \in P(t)$, $t \in [0,T]$, $x(0) = \beta, x(T) \in \Omega$ with $x(\cdot) \in AC([-h,T]) \cap W_{1,2}^n([0,T])$ where $AC([-h,T])$ is a space of absolutely continuous functions from $[-h,T]$ into $\mathbb{R}^n$ and $W_{1,2}^n([0,T])$ is a Banach space of absolutely continuous functions from $[0,T]$ into $\mathbb{R}^n$ together with first order derivatives for which $x''(\cdot)\in L_1^n([0,T])$. Notice that a Banach space $W_{1,2}^n([0,T])$ can be equipped with the different equivalent norms.

It should be noted that the results on the existence of solutions of second-order differential inclusions in their general setting can be found in [3, 8, 12].

3. Optimization for second-order delay discrete inclusions

In this section, by reducing the problem (PD_d) to an equivalent type of convex minimization problem, we derive the conditions of optimality for the problem (PD_d). Let us denote

$$A=\begin{pmatrix} A_0 & A_1 & A_2 & -B & 0 & \cdots & \cdots & \cdots & 0\\ 0 & A_0 & A_1 & A_2 & -B & 0 & \cdots & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & \cdots & \cdots & 0 & A_0 & A_1 & A_2 & -B \end{pmatrix},\quad \tilde{D}=\begin{pmatrix} d\\ \vdots\\ \vdots\\ d\end{pmatrix}$$

where A is the partitioned into submatrices $A_0,A_1,A_2,-B$ and $m\times n$ zero matrices 0, $\tilde{D}$ is a $m(T-2)$ dimensional column vector and A is a matrix with size $m(T-2)\times n(T+1)$. Then the problem (PD_d) can be reduced to a convex mathematical programming problem with constraints consisting of the linear inequalities.

$$\text{minimize}\quad g(w)=\sum_{t=1}^{T-1} f(x_t,t)\,, \tag{3.1}$$

$$\text{subject to}\quad w\in M\cap\Phi\cap\Big(\bigcap_{t=1}^{T}\overline{P}_t\Big)\cap\overline{\Omega},$$

where the vector $w=(x_{-h},x_{-h+1},\dots,x_T)\in\mathbb{R}^{n(h+1+T)}$ and $M=\{w: Aw\le\tilde{D}\}$,

$$\begin{aligned}\Phi &= \{w: x_t=\alpha_t,\ t=-h,-h+1,\dots,0\},\ \alpha_0=\beta_0,\\ \overline{P}_t &= \{w: x_t\in P_t\},\ t=1,\dots,N,\\ \overline{\Omega} &= \{w: x_T\in\Omega\}.\end{aligned}$$

It is easy to check that $M=\bigcap_{t=0}^{T-2}M_t$, where $M_t=\{w: A_0x_t+A_1x_{t+1}+A_2x_{t-h}-Bx_{t+2}\le d\}$, $t=0,\dots,T-2$.

Let $K_M(\tilde{w})$ be the cone of tangent directions, $\tilde{w}\in M$, and $K_M^*(\tilde{w})$ be the dual cone to the cone of tangent directions $K_M(\tilde{w})=\{\overline{w}:\ \tilde{w}+\mu\overline{w}\in M\ ,\ \mu\ge 0\}$, i.e. $K_M^*(\tilde{w})=\{w_M^*:\ \langle\overline{w},w_M^*\rangle\ge 0,\ \forall\,\overline{w}\in K_M(\tilde{w})\}$. The cone of tangent directions $K_{M_t}(\tilde{w})$, $t=0,\dots,T-2$ are polyhedral cones and so by Lemma 1.22 [18, p.23], $K_M^*(\tilde{w})=\sum_{t=0}^{T-2}K_{M_t}^*(\tilde{w})$.

Initially we prove the following lemma for the proofs of our main results.

Lemma 3.1 *For a polyhedral set M_t one has*

$$K^*_{M_t}(\tilde{w}) = \Big\{ w^*(t) : \ x^*_t(t) = -A^*_0\lambda_t, \ x^*_{t+1}(t) = -A^*_1\lambda_t, \ x^*_{t-h}(t) = -A^*_2\lambda_t, x^*_{t+2}(t) = B^*\lambda_t,$$
$$x^*_k = 0, \ k \neq t, t+1, t-h, t+2, \ \lambda_t \geq 0, \ \lambda_t \in \mathbb{R}^m,$$
$$\big\langle A_0\tilde{x}_t + A_1\tilde{x}_{t+1} + A_2\tilde{x}_{t-h} - B\tilde{x}_{t+2} - d, \ \lambda_t \big\rangle = 0, \ t = 0, \dots, T-2 \ \Big\}.$$

Proof By the definition of the cone of tangent directions, we obtain

$$K_{M_t}(\tilde{w}) = \Big\{ \overline{w} : A_0(\tilde{x}_t + \mu\overline{x}_t) + A_1(\tilde{x}_{t+1} + \mu\overline{x}_{t+1}) + A_2(\tilde{x}_{t-h} + \mu\overline{x}_{t-h}) - B(\tilde{x}_{t+2} + \mu\overline{x}_{t+2}) \leq d$$
$$\text{for a small } \mu > 0, \ t = 0, \dots, T-2 \Big\}, \tag{3.2}$$

where $\tilde{x}_{t+2} \in F(\tilde{x}_t, \tilde{x}_{t+1}, \tilde{x}_{t-h})$ is satisfied. It follows by the formula (3.2) that the inequality holds

$$A^i_0(\tilde{x}_t + \mu\overline{x}_t) + A^i_1(\tilde{x}_{t+1} + \mu\overline{x}_{t+1}) + A^i_2(\tilde{x}_{t-h} + \mu\overline{x}_{t-h}) - B_i(\tilde{x}_{t+2} + \mu\overline{x}_{t+2}) \leq d_i \ , \ t = 0, \dots, T-2$$

as

$$A^i_0\overline{x}_t + A^i_1\overline{x}_{t+1} + A^i_2\overline{x}_{t-h} - B_i\overline{x}_{t+2} \leq 0 \tag{3.3}$$

$$i \in I(\tilde{w}) = \Big\{ i : A^i_0\tilde{x}_t + A^i_1\tilde{x}_{t+1} + A^i_2\tilde{x}_{t-h} - B_i\tilde{x}_{t+2} = d_i \ , \ i = 1, \dots, m \Big\}.$$

Obviously if i is not the active indices, i.e. $i \notin I(\tilde{w})$, the inequality before (3.3) holds strongly for small μ, regardless choosing $(\overline{x}_t, \overline{x}_{t+1}, \overline{x}_{t-h}, \overline{x}_{t+2})$. Now since $\overline{x}_k$, $k \neq t, \ t+1, \ t-h, \ t+2$ are arbitrary, applying Farkas theorem 1.13 (see for example [18, p.22]) it follows from the inequality (3.3) that $w^*(t) = (x^*_0(t), x^*_1(t), \dots, x^*_T(t)) \in K^*_{M_t}(\tilde{w})$ if and only if

$$x^*_t(t) = -\sum_{i \in I(\tilde{w})} A^{i*}_0\lambda^t_i \quad , \quad x^*_{t+1}(t) = -\sum_{i \in I(\tilde{w})} A^{i*}_1\lambda^t_i,$$
$$x^*_{t-h}(t) = -\sum_{i \in I(\tilde{w})} A^{i*}_2\lambda^t_i \quad , \quad x^*_{t+2}(t) = \sum_{i \in I(\tilde{w})} B^*_i\lambda^t_i, \quad \lambda^t_i \geq 0, \tag{3.4}$$

where $A^{i*}_0, A^{i*}_1, A^{i*}_2, B^*_i$ are transposed vectors of A^i_0, A^i_1, A^i_2, B_i respectively. Consequently, taking $\lambda^t_i = 0$ for $i \notin I(\tilde{w})$ and denoting λ_t a vector with the components λ^t_i, we have the required formula of lemma. Here it is enough only remind that

$$\big\langle A_0\tilde{x}_t + A_1\tilde{x}_{t+1} + A_2\tilde{x}_{t-h} - B\tilde{x}_{t+2} - d \ , \ \lambda_t \big\rangle = 0, \ \lambda_t \geq 0 \ , \ \lambda_t \in \mathbb{R}^m, \ t = 0, \dots, T-2.$$

The proof of the lemma is completed. □

Theorem 3.2 *Let F and $f(\cdot, t)$, $t = 1, \dots, T-1$ be a polyhedral set valued mapping and a polyhedral function, respectively. Let $\{x_t\}^T_{t=-h}$ be a feasible solution of problem (PD_d) with second-order delay discrete inclusions and f be continuous at x_t, $t = 1, \dots, T-1$. Then in order for $\{\tilde{x}_t\}^T_{t=-h}$ to be an optimal solution to problem (PD_d) with the initial values $x_t = \alpha_t$, $t = -h, \dots, -1, x_0 = \beta_0$, state constraints $x_t \in P_t$, $t = 1, \dots, T$ and* *target set* Ω*, it is necessary and sufficient that there exist vectors* λ_t*,* $t = 0,\dots,T$ *not all equal to zero such that*

$$A_0^*\lambda_t + A_1^*\lambda_{t-1} + A_2^*\lambda_{t+h} - B^*\lambda_{t-2} \in K^*_{P_t}(\tilde{x}_t) - \partial f(\tilde{x}_t,t), \;\; t = 0,\dots,T-2-h,$$
$$A_0^*\lambda_t + A_1^*\lambda_{t-1} - B^*\lambda_{t-2} \in K^*_{P_t}(\tilde{x}_t) - \partial f(\tilde{x}_t,t), \;\; t = T-1-h,\dots,T-2,$$
$$\partial f(\tilde{x}_0,0) = \{0\}\,, \quad \lambda_{-1} \equiv \lambda_{-2} \equiv 0,$$
$$\big\langle A_0\tilde{x}_t + A_1\tilde{x}_{t+1} + A_2\tilde{x}_{t-h} - B\tilde{x}_{t+2} - d,\ \lambda_t\big\rangle = 0\,, \quad \lambda_t \geq 0,$$

and transversality conditions

$$-B^*\lambda_{T-3} + A_1^*\lambda_{T-2} \in K^*_{P_{T-1}}(\tilde{x}_{T-1}) - \partial f(\tilde{x}_{T-1},T-1),$$
$$-B^*\lambda_{T-2} \in K^*_{P_T\cap\Omega}(\tilde{x}_T).$$

Proof Obviously, if $\{\tilde{x}_t\}_{t=-h}^T$ is an optimal trajectory of the problem (PD_d) we say that $\tilde{w} = (\tilde{x}_{-h},\tilde{x}_{-h+1},\dots,\tilde{x}_T)$ is a solution of the convex mathematical programming problem (3.1). We can prove necessary and sufficient conditions for problem (3.1) with geometric constraints on the basis of the results regarding convex mathematical programming. Therefore, due to the continuity of $f(\cdot,t)$ at points of some feasible solution $\{\tilde{x}_t\}_{t=-h}^T$ it follows from Theorem 3.4 [18] that there exist vectors $\overline{w}^* \in \partial_w f(\tilde{w})$ and vectors $w_M^* \in K_M^*(\tilde{w}), \tilde{w}^*(t) \in K^*_{\overline{P}_t}(\tilde{w}), \hat{w}^* \in K^*_{\Phi}(\tilde{w}), \breve{w}^* \in K^*_{\overline{\Omega}}(\tilde{w})$ such that $\overline{w}^* = w_M^* + \sum\limits_{t=1}^T \tilde{w}^*(t) + \hat{w}^* + \breve{w}^*$. This means that

$$\overline{w}^* = \sum_{t=0}^{T-2} w^*(t) + \sum_{t=1}^{T} \tilde{w}^*(t) + \hat{w}^* + \breve{w}^*, \quad w^*(t) \in K^*_{M_t}(\tilde{w}). \tag{3.5}$$

Thus $\tilde{w}$ is a solution of the problem (3.1) if and only if representation (3.5) holds. It follows from the definition of the function $g(w) = \sum\limits_{t=1}^{T-1} f(x_t,t)$ the vector $\overline{w}^*$ has the form $\overline{w}^* = (0,\dots,0,\overline{x}_1^*,\dots,\overline{x}_{T-2}^*,\overline{x}_{T-1}^*,0)$ where $\overline{x}_t^* \in \partial f(x_t,t)$, $t = 1,\dots,T-1$.

By using definition of the dual cones, we can compute $K^*_{\overline{P}_t}(\tilde{w})$, $K^*_{\Phi}(\tilde{w})$ and $K^*_{\overline{\Omega}}(\tilde{w})$ as follows:

$$\begin{aligned} K^*_{\overline{P}_t}(\tilde{w}) =&\ \{\tilde{w}^*(t):\ \tilde{x}_t^*(t) \in K^*_{P_t}(\tilde{x}_t),\ \tilde{x}_k^*(t) = 0,\ k \neq t\},\ t = 1,\dots,T,\\ K^*_{\Phi}(\tilde{w}) =&\ \{\hat{w}^*:\ x_t^* = 0,\ t \neq -h,\dots,0\},\\ K^*_{\overline{\Omega}}(\tilde{w}) =&\ \{\breve{w}^*:\ x_T^* \in K^*_{\Omega}(x_T),\ x_t^* = 0,\ t < T\}. \end{aligned}$$

In more details

$$\tilde{w}^*(t) = \big(\tilde{x}^*_{-h}(t),\dots,\tilde{x}^*_T(t)\big),\ \tilde{x}_k^*(t) = 0\ ,\ k \neq t\ , t = 1,\dots,T,$$

or more convenience for the components of vector $\Big[\sum\limits_{t=1}^T \tilde{w}^*(t)\Big]_t$, we can write

$$\Big[\sum_{t=1}^T \tilde{w}^*(t)\Big]_t = \begin{cases} 0, & t = -h,\dots,0,\\ \tilde{x}_t^*(t) \in K^*_{P_t}(\tilde{x}_t), & t = 1,\dots,T. \end{cases} \tag{3.6}$$

And it is not hard to see that

$$\hat{w}^* = (\hat{x}^*_{-h}, \dots, \hat{x}^*_0, 0 \dots, 0), \breve{w}^* = (0, \dots, 0, \breve{x}^*_T),\ \breve{x}^*_T \in K^*_\Omega(\tilde{x}_T). \tag{3.7}$$

Let us denote t-th component of the vector $\sum_{t=0}^{T-2} w^*(t)$ by $\Big[\sum_{t=0}^{T-2} w^*(t)\Big]_t$, $t = -h, \dots, 1$ and from Lemma 3.1, we have

$$\Big[\sum_{t=0}^{T-2} w^*(t)\Big]_t = \begin{cases} x^*_t(t+h)\ , & t = -h, \dots, -1, \\ -A^*_0\lambda_0 - A^*_2\lambda_h\ , & t = 0, \\ -A^*_1\lambda_0 - A^*_0\lambda_1 - A^*_2\lambda_{1+h}\ , & t = 1. \end{cases} \tag{3.8}$$

On the other side, it is not difficult to infer that taking $t = 2, 3, \dots, T$, we have a different relationship to describe the components of the vector $\Big[\sum_{t=0}^{T-2} w^*(t)\Big]_t$ as follows:

$$\Big[\sum_{t=0}^{T-2} w^*(t)\Big]_t = \begin{cases} B^*\lambda_{t-2} - A^*_1\lambda_{t-1} - A^*_2\lambda_{t+h} - A^*_0\lambda_t\ , & t = 2, \dots, T-2-h, \\ B^*\lambda_{t-2} - A^*_1\lambda_{t-1} - A^*_0\lambda_t\ , & t = T-1-h, \dots, T-2, \\ B^*\lambda_{T-3} - A^*_1\lambda_{T-2}\ , & t = T-1, \\ B^*\lambda_{T-2}\ , & t = T. \end{cases} \tag{3.9}$$

Then taking into account (3.6)–(3.9) by component-wise representation, we write

$$\begin{gathered} x^*_t(t+h) + \hat{x}^*_t = 0,\ t = -h, \dots, -1 \\ -A^*_0\lambda_0 - A^*_2\lambda_h + \hat{x}^*_0 = 0, \\ -A^*_1\lambda_0 - A^*_0\lambda_1 - A^*_2\lambda_{1+h} + \tilde{x}^*_1(1) = \overline{x}^*_1, \end{gathered} \tag{3.10}$$

and by analogy, easily can be checked that

$$\begin{gathered} -A^*_0\lambda_t - A^*_1\lambda_{t-1} - A^*_2\lambda_{t+h} + B^*\lambda_{t-2} + \tilde{x}^*_t(t) = \overline{x}^*_t\ ,\ t = 2, \dots, T-2-h, \\ -A^*_0\lambda_t - A^*_1\lambda_{t-1} + B^*\lambda_{t-2} + \tilde{x}^*_t(t) = \overline{x}^*_t\ ,\ t = T-1-h, \dots, T-2, \\ -A^*_1\lambda_{T-2} + B^*\lambda_{T-3} + \tilde{x}^*_{T-1}(T-1) = \overline{x}^*_{T-1}\ , \end{gathered} \tag{3.11}$$

$$B^*\lambda_{T-2} + \tilde{x}^*_T(T) + \breve{x}^*_T = 0. \tag{3.12}$$

Since $\tilde{x}^*_t(t) \in K^*_{P_t}(\tilde{x}_t)$ and $\overline{x}^*_t \in \partial f(x_t, t)$, we can rewrite the relationship (3.11) as follows:

$$\begin{gathered} A^*_0\lambda_t + A^*_1\lambda_{t-1} + A^*_2\lambda_{t+h} - B^*\lambda_{t-2} \in K^*_{P_t}(\tilde{x}_t) - \partial f(\tilde{x}_t, t),\ \ t = 2, \dots, T-2-h, \\ A^*_0\lambda_t + A^*_1\lambda_{t-1} - B^*\lambda_{t-2} \in K^*_{P_t}(\tilde{x}_t) - \partial f(\tilde{x}_t, t),\ \ t = T-1-h, \dots, T-2. \end{gathered}$$

Moreover by setting $\tilde{x}^*_0(0) = \hat{x}^*_0$ and $\lambda_{-1} \equiv \lambda_{-2} \equiv 0$, the formula (3.11) can be extended to the case $t = 0, 1$. Thus by virtue of the previous inclusions, we have the delay discrete Euler–Lagrange type inclusions of theorem for $t = 0, \dots, T-2-h$ and $t = T-1-h, \dots, T-2$, respectively. By similar way for $t = T-1$ and $t = T$ the formula (3.12) can be rewritten as follows:

$$\begin{gathered} A^*_1\lambda_{T-2} - B^*\lambda_{T-3} \in K^*_{P_{T-1}}(\tilde{x}_{T-1}) - \partial f(\tilde{x}_{T-1}, T-1), \\ -B^*\lambda_{T-2} \in K^*_{P_T}(\tilde{x}_T) + K^*_\Omega(\tilde{x}_T) = K^*_{P_T \cap \Omega}(\tilde{x}_T). \end{gathered}$$

The proof of theorem is completed. □

4. Necessary and sufficient conditions of optimality for delay discrete approximation problem

Let us choose δ as a step on the t-axis and let $x(t) \equiv x_\delta(t)$ be a grid function on $[0,T]$. Let introduce the following first and second-order difference operators

$$\begin{aligned}\Delta x(t) &= \frac{x(t+\delta)-x(t)}{\delta},\\ \Delta^2 x(t) &= \frac{x(t+2\delta)-2x(t+\delta)+x(t)}{\delta^2}.\end{aligned}$$

It is clear that we can we associate with the continuous problem (PC_d) the following delay discrete approximation problem

$$\text{minimize} \sum_{t=0}^{T-2\delta} \delta f(x(t),t) + \varphi_0\big(x(T-\delta)\big), \tag{4.1}$$

$$\Delta^2 x(t) \in F(x(t), \Delta x(t), x(t-h))\ ,\ t=0,\delta,\ldots,T-2\delta, \tag{4.2}$$

$$x(t)=\alpha(t)\ ,\ t=-h,-h+\delta,\ldots,-\delta, \quad x(0)=\beta,$$

$$x(t) \in P(t)\ , t=\delta,\ldots,T\ ,\ x(T)\in\Omega. \tag{4.3}$$

To formulate the necessary and sufficient optimality conditions for problem (4.1)–(4.3), we reduce this problem to the form of the (PD_d) problem. Note that the delay discrete approximation inclusions (4.2) can be rewritten as follows:

$$\Big(\delta^2 A_0 - \delta A_1 - B\Big)x(t) + \Big(\delta A_1 + 2B\Big)x(t+\delta) + \delta^2 A_2 x(t-h) - Bx(t+2\delta) \le \delta^2 d\ ,\ t=0,\delta,\ldots,T-2\delta.$$

The following theorem plays a crucial role in further results based on the second-order delay differential inclusions.

Theorem 4.1 *Let F be a polyhedral set valued mapping, $f(\cdot,t)$, $t=1,\ldots,T-1$ be a proper polyhedral function and continuous at the points of some feasible trajectory $\{x(t)\}, t=0,h,\ldots,T$. Then for optimality of the trajectory $\{\tilde{x}(t)\}$ in the problem (4.1)-(4.3) with second-order delay discrete approximation, it is necessary and sufficient that there exist vectors $\lambda(t)$, $t=0,\ldots,T$ not all equal to zero satisfying the second-order delay approximate Euler–Lagrange type inclusions $(i),(ii)$*

$$(i)\ \ -B^*\Delta^2\lambda(t-2\delta) + A_0^*\lambda(t) - A_1^*\Delta\lambda(t-\delta) + A_2^*\lambda(t+h) \in K^*_{P(t)}(\tilde{x}(t)) - \partial f(\tilde{x}(t),t)\ ,$$

$$t=0,\ldots,T-2\delta-h,\ \partial f(\tilde{x}(0),0)=\{0\},$$

$$(ii)\ \ -B^*\Delta^2\lambda(t-2\delta) + A_0^*\lambda(t) - A_1^*\Delta\lambda(t-\delta) \in K^*_{P(t)}(\tilde{x}(t)) - \partial f(\tilde{x}(t),t)\ ,$$

$$t=T-\delta-h,\ldots,T-2\delta,\ \lambda(t)\ge 0,$$

$$\Big\langle A_0\tilde{x}(t) + A_1\Delta\tilde{x}(t) + A_2\tilde{x}(t-h) - B\Delta^2\tilde{x}(t) - d\ ,\ \lambda(t)\Big\rangle = 0,$$

and transversality conditions

$$B^*\Delta\lambda(T-3\delta) + A_1^*\lambda(T-2\delta) \in K^*_{P(T-\delta)}(\tilde{x}(T-\delta)) - \partial\varphi_0(\tilde{x}(T-\delta)),$$

$$-B^*\lambda(T-2\delta) \in K^*_{P(T)\cap\Omega}(\tilde{x}(T)).$$

Proof By using Theorem 3.2 that if $\{\tilde{x}(t)\} := \{\tilde{x}(t) : t = 0, \delta, \dots, T\}$ is an optimal trajectory in the (4.1)–(4.3) then there exists $\lambda(t)$ not all equal zero such that

$$\Big(\delta^2 A_0^* - \delta A_1^* - B^*\Big)\lambda(t) + \Big(\delta A_1^* + 2B^*\Big)\lambda(t-\delta) + \delta^2 A_2^* \lambda(t+h) - B^*\lambda(t-2\delta)$$

$$\in K^*_{P(t)}(\tilde{x}(t)) - \partial f(\tilde{x}(t), t)\ , t = 2, \dots, T - 2\delta - h, \tag{4.4}$$

$$\Big(\delta^2 A_0^* - \delta A_1^* - B^*\Big)\lambda(t) + \Big(\delta A_1^* + 2B^*\Big)\lambda(t-\delta) - B^*\lambda(t-2\delta)$$

$$\in K^*_{P(t)}(\tilde{x}(t)) - \partial f(\tilde{x}(t), t)\ , t = T - \delta - h, \dots, T - 2\delta, \tag{4.5}$$

$$\partial f(\tilde{x}(0), 0) = \{0\}\ , \quad \lambda(t) \geq 0,$$

$$\Big\langle (\delta^2 A_0 - \delta A_1 - B)\tilde{x}(t) + (\delta A_1 + 2B)\tilde{x}(t+\delta) + \delta^2 A_2 \tilde{x}(t-h) - B\tilde{x}(t+2\delta) - \delta^2 d,\ \lambda(t)\Big\rangle = 0\ . \tag{4.6}$$

The inclusion (4.4) can be rewritten as follows:

$$-B^*\lambda(t-2\delta) - B^*\lambda(t) + 2B^*\lambda(t-\delta) + (\delta^2 A_0^* - \delta A_1^*)\lambda(t) + \delta A_1^*\lambda(t-\delta)$$
$$+\delta^2 P_2^*\lambda(t+h) \in K^*_{P(t)}(\tilde{x}(t)) - \partial f(\tilde{x}(t), t)\ , t = 2, \dots, T - 2\delta - h,$$

or more convenient form

$$-B^*\Big(\lambda(t-2\delta) - 2\lambda(t-\delta) + \lambda(t)\Big) + \delta^2 A_0^*\lambda(t) + A_1^*\delta\Big(\lambda(t-\delta) - \lambda(t)\Big)$$
$$+\delta^2 A_2^*\lambda(t+h) \in K^*_{P(t)}(\tilde{x}(t)) - \partial f(\tilde{x}(t), t)\ , t = 2, \dots, T - 2\delta - h.$$

Dividing both sides of that relation by δ^2 (here $\delta\lambda(t)$ are denoted again by $\lambda(t)$), we have Euler–Lagrange polyhedral delay discrete inclusions (i) of theorem

$$-B^*\Delta^2\lambda(t-2\delta) + A_0^*\lambda(t) - A_1^*\Delta\lambda(t-\delta) + A_2^*\lambda(t+h) \in K^*_{P(t)}(\tilde{x}(t)) - \partial f(\tilde{x}(t), t),\ t = 2, \dots, T - 2\delta - h.$$

In a similar way, we can find delay discrete inclusions (ii) of theorem

$$-B^*\Delta^2\lambda(t-2\delta) + A_0^*\lambda(t) - A_1^*\Delta\lambda(t-\delta) \in K^*_{P(t)}(\tilde{x}(t)) - \partial f(\tilde{x}(t), t)\ , t = T - \delta - h, \dots, T - 2\delta.$$

By analogy, Equation (4.6) can be rewritten as follows:

$$\Big\langle A_0\tilde{x}(t) + A_1\Delta\tilde{x}(t) + A_2\tilde{x}(t-h) - B\Delta^2\tilde{x}(t) - d\ ,\ \lambda(t)\Big\rangle = 0.$$

By applying Theorem 3.2 the transversality condition of problem

$$-B^*\lambda(T-3\delta) + (\delta A_1^* + 2B^*)\lambda(T-2\delta) \in K^*_{P(T-\delta)}(\tilde{x}(T-\delta)) - \partial\varphi_0(\tilde{x}(T-\delta)),$$
$$-B^*\lambda(T-2\delta) \in K^*_{P(T)\cap\Omega}(\tilde{x}(T)).$$

Dividing both sides by δ, we obtain that

$$B^*\Delta\lambda(T-3\delta) + A_1^*\lambda(T-2\delta) \in K^*_{P(T-\delta)}(\tilde{x}(T-\delta)) - \partial\varphi_0(\tilde{x}(T-\delta)).$$

We have the desired result. □

5. Sufficient conditions of optimality for second-order polyhedral delay differential inclusions

Formulation of sufficient optimality conditions for the (PC_d) problem with differential inclusions for second-order delay is basically based on results in Section 4. By passing to the formal limit in conditions of Theorem 4.1 as $\delta \to 0$, these conditions are sufficient for the optimality of $\tilde{x}(t),\ t \in [0,T]$ in the problem (PC_d):

$$(a)\ -B^*\frac{d^2\lambda(t)}{dt^2} + A_0^*\lambda(t) - A_1^*\frac{d\lambda(t)}{dt} + A_2^*\lambda(t+h) \in K^*_{P(t)}(\tilde{x}(t)) - \partial f(\tilde{x}(t),t),$$

$$\text{a.e. } t \in [0, T-h),\ B^*\lambda(0) = 0,$$

$$(b)\ -B^*\frac{d^2\lambda(t)}{dt^2} + A_0^*\lambda(t) - A_1^*\frac{d\lambda(t)}{dt} \in K^*_{P(t)}(\tilde{x}(t)) - \partial f(\tilde{x}(t),t),\ \text{a.e. } t \in [T-h, T],$$

$$(c)\ \Big\langle A_0\tilde{x}(t) + A_1\frac{d\tilde{x}(t)}{dt} + A_2\tilde{x}(t-h) - B\frac{d^2\tilde{x}(t)}{dt^2} - d\ ,\ \lambda(t)\Big\rangle = 0\ ,\ \text{a.e. } t \in [0,T],$$

and transversality conditions at point $t = T$

$$(d)\ A_1^*\lambda(T) + B^*\frac{d\lambda(T)}{dt} \in K^*_{P(T)}(\tilde{x}(T)) - \partial\varphi_0\big(\tilde{x}(T)\big)\ ,\ B^*\lambda(T) = 0.$$

Theorem 5.1 *Let f be continuous and polyhedral function with respect to x and F be a polyhedral set valued mapping. Then for optimality of the arc $\tilde{x}(t)$ to the problem (PC_d) with second-order delay differential inclusions, it is sufficient that there exist $\lambda(t)$, $t \in [0,T]$ not all equal to zero satisfying the second-order delay differential Euler–Lagrange type inclusions $(a),(b)$, the equation (c) and transversality conditions (d).*

Proof By definition of subdifferential for all feasible solutions, we rewrite (a) and (b) in the form

$$f(x(t),t) - f(\tilde{x}(t),t) \geq \Big\langle B^*\frac{d^2\lambda(t)}{dt^2} - A_0^*\lambda(t) + A_1^*\frac{d\lambda(t)}{dt} - A_2^*\lambda(t+h) + \tilde{x}^*(t)\ , x(t) - \tilde{x}(t)\Big\rangle,\ \text{a.e. } t \in [0, T-h), \tag{5.1}$$

and

$$f(x(t),t) - f(\tilde{x}(t),t) \geq \Big\langle B^*\frac{d^2\lambda(t)}{dt^2} - A_0^*\lambda(t) + A_1^*\frac{d\lambda(t)}{dt} + \tilde{x}^*(t)\ , x(t) - \tilde{x}(t)\Big\rangle,\ \text{a.e. } t \in [T-h, T], \tag{5.2}$$

where $\tilde{x}^*(t) \in K^*_{P(t)}(\tilde{x}(t))$. The right hand side of the inequality (5.1) can be rewritten as follows:

$$\Big\langle B^*\frac{d^2\lambda(t)}{dt^2} - A_0^*\lambda(t) + A_1^*\frac{d\lambda(t)}{dt} - A_2^*\lambda(t+h) + \tilde{x}^*(t),\ x(t) - \tilde{x}(t)\Big\rangle$$

$$= \Big\langle B^*\frac{d^2\lambda(t)}{dt^2},\ x(t) - \tilde{x}(t)\Big\rangle - \Big\langle \lambda(t), A_0\big(x(t) - \tilde{x}(t)\big)\Big\rangle + \Big\langle \frac{d\lambda(t)}{dt},\ A_1\big(x(t) - \tilde{x}(t)\big)\Big\rangle$$

$$-\Big\langle \lambda(t+h),\ A_2\big(x(t) - \tilde{x}(t)\big)\Big\rangle + \Big\langle \tilde{x}^*(t), x(t) - \tilde{x}(t)\Big\rangle. \tag{5.3}$$

For all feasible solutions $x(\cdot)$ and for $\lambda(t) \geq 0, t \in [0,T]$ we have

$$\Big\langle A_0 x(t) + A_1 \frac{dx(t)}{dt} + A_2 x(t-h), \lambda(t) \Big\rangle \leq \Big\langle B \frac{d^2 x(t)}{dt^2} + d, \lambda(t) \Big\rangle.$$

On the other hand using the condition (c) of theorem, we can write

$$\Big\langle A_0 \tilde{x}(t) + A_1 \frac{d\tilde{x}(t)}{dt} + A_2 \tilde{x}(t-h), \lambda(t) \Big\rangle = \Big\langle B \frac{d^2 \tilde{x}(t)}{dt^2} + d, \lambda(t) \Big\rangle.$$

Then by subtracting the last two relations, we conclude that

$$\Big\langle A_0 \big(x(t) - \tilde{x}(t)\big) + A_1 \frac{d\big(x(t)-\tilde{x}(t)\big)}{dt} + A_2 \big(x(t-h) - \tilde{x}(t-h)\big),\ \lambda(t) \Big\rangle$$
$$\leq \Big\langle \frac{d^2\big(x(t)-\tilde{x}(t)\big)}{dt^2}\ ,\ B^*\lambda(t) \Big\rangle. \tag{5.4}$$

Then from (5.3) and (5.4), we deduce that

$$\Big\langle B^* \frac{d^2\lambda(t)}{dt^2} - A_0^*\lambda(t) + A_1^* \frac{d\lambda(t)}{dt} - A_2^*\lambda(t+h) + \tilde{x}^*(t), x(t) - \tilde{x}(t) \Big\rangle$$
$$\geq \Big\langle B^* \frac{d^2\lambda(t)}{dt^2}, x(t) - \tilde{x}(t) \Big\rangle - \Big\langle \frac{d^2\big(x(t)-\tilde{x}(t)\big)}{dt^2}, B^*\lambda(t) \Big\rangle + \Big\langle A_1 \frac{d\big(x(t)-\tilde{x}(t)\big)}{dt}, \lambda(t) \Big\rangle$$
$$+ \Big\langle A_2 \big(x(t-h) - \tilde{x}(t-h)\big), \lambda(t) \Big\rangle + \Big\langle \frac{d\lambda(t)}{dt}, A_1\big(x(t) - \tilde{x}(t)\big) \Big\rangle$$
$$- \Big\langle \lambda(t+h), A_2\big(x(t) - \tilde{x}(t)\big) \Big\rangle + \Big\langle \tilde{x}^*(t)\ ,\ x(t) - \tilde{x}(t) \Big\rangle.$$

By suitably rewriting this inequality we have

$$\Big\langle B^* \frac{d^2\lambda(t)}{dt^2} - A_0^*\lambda(t) + A_1^* \frac{d\lambda(t)}{dt} - A_2^*\lambda(t+h) + \tilde{x}^*(t), x(t) - \tilde{x}(t) \Big\rangle$$
$$\geq \Big\langle B^* \frac{d^2\lambda(t)}{dt^2}, x(t) - \tilde{x}(t) \Big\rangle - \Big\langle \frac{d^2\big(x(t)-\tilde{x}(t)\big)}{dt^2}, B^*\lambda(t) \Big\rangle + \frac{d}{dt}\Big[\Big\langle A_1\big(x(t) - \tilde{x}(t)\big), \lambda(t) \Big\rangle\Big]$$
$$+ \Big\langle A_2 \big(x(t-h) - \tilde{x}(t-h)\big), \lambda(t) \Big\rangle - \Big\langle \lambda(t+h), A_2\big(x(t) - \tilde{x}(t)\big) \Big\rangle + \Big\langle \tilde{x}^*(t)\ ,\ x(t) - \tilde{x}(t) \Big\rangle. \tag{5.5}$$

Then since $\tilde{x}^*(t) \in K^*_{P(t)}(\tilde{x}(t))$ we have $\langle \tilde{x}^*(t), x(t) - \tilde{x}(t) \rangle \geq 0,\ \forall\, x(t) \in K_{P(t)}(\tilde{x}(t))$, therefore from (5.1) and (5.5), we conclude that

$$f(x(t),t) - f(\tilde{x}(t),t) \geq \Big\langle B^* \frac{d^2\lambda(t)}{dt^2}, x(t) - \tilde{x}(t) \Big\rangle - \Big\langle \frac{d^2\big(x(t)-\tilde{x}(t)\big)}{dt^2}, B^*\lambda(t) \Big\rangle$$
$$+ \frac{d}{dt}\Big[\Big\langle A_1^*\lambda(t), x(t) - \tilde{x}(t) \Big\rangle\Big] + \Big\langle A_2^*\lambda(t), x(t-h) - \tilde{x}(t-h) \Big\rangle$$
$$- \Big\langle A_2^*\lambda(t+h), x(t) - \tilde{x}(t) \Big\rangle\ ,\ \text{a.e } t \in [0, T-h). \tag{5.6}$$

Integrating the inequality (5.6) over the interval $[0, T-h)$ and taking into account that $x(0) = \tilde{x}(0) = \beta$ are feasible, we obtain

$$\int_0^{T-h} \Big(f(x(t),t) - f(\tilde{x}(t),t)\Big)dt \geq \int_0^{T-h} \Big[\Big\langle B^* \frac{d^2\lambda(t)}{dt^2}, x(t) - \tilde{x}(t)\Big\rangle$$

$$-\Big\langle \frac{d^2\big(x(t) - \tilde{x}(t)\big)}{dt^2}, B^*\lambda(t)\Big\rangle\Big]dt + \Big\langle A_1^*\lambda(T-h), x(T-h) - \tilde{x}(T-h)\Big\rangle$$

$$+\int_0^{T-h}\Big[\Big\langle A_2^*\lambda(t), x(t-h) - \tilde{x}(t-h)\Big\rangle\Big]dt - \int_0^{T-h}\Big[\Big\langle A_2^*\lambda(t+h), x(t) - \tilde{x}(t)\Big\rangle\Big]dt. \tag{5.7}$$

By similar way beginning from (5.2), we conclude

$$\int_{T-h}^{T} \Big(f(x(t),t) - f(\tilde{x}(t),t)\Big)dt \geq \int_{T-h}^{T} \Big[\Big\langle B^* \frac{d^2\lambda(t)}{dt^2}, x(t) - \tilde{x}(t)\Big\rangle - \Big\langle \frac{d^2\big(x(t) - \tilde{x}(t)\big)}{dt^2}, B^*\lambda(t)\Big\rangle\Big]dt$$

$$+\Big\langle A_1^*\lambda(T), x(T) - \tilde{x}(T)\Big\rangle - \Big\langle A_1^*\lambda(T-h), x(T-h) - \tilde{x}(T-h)\Big\rangle + \int_{T-h}^{T}\Big[\Big\langle A_2^*\lambda(t), x(t-h) - \tilde{x}(t-h)\Big\rangle\Big]dt. \tag{5.8}$$

Summing the inequalities (5.7) and (5.8),

$$\int_0^{T} \Big(f(x(t),t) - f(\tilde{x}(t),t)\Big)dt \geq \int_0^{T} \Big[\Big\langle B^* \frac{d^2\lambda(t)}{dt^2}, x(t) - \tilde{x}(t)\Big\rangle - \Big\langle \frac{d^2\big(x(t) - \tilde{x}(t)\big)}{dt^2}, B^*\lambda(t)\Big\rangle\Big]dt$$

$$+\int_0^{T}\Big[\Big\langle A_2^*\lambda(t), x(t-h) - \tilde{x}(t-h)\Big\rangle\Big]dt - \int_0^{T-h}\Big[\Big\langle A_2^*\lambda(t+h), x(t) - \tilde{x}(t)\Big\rangle\Big]dt + \Big\langle A_1^*\lambda(T), x(T) - \tilde{x}(T)\Big\rangle. \tag{5.9}$$

By recalling $x(t) = \alpha(t)$, $t \in [-h, 0)$, it is easy to compute the following integral

$$\int_0^{T}\Big[\Big\langle A_2^*\lambda(t), x(t-h) - \tilde{x}(t-h)\Big\rangle\Big]dt - \int_0^{T-h}\Big[\Big\langle A_2^*\lambda(t+h), x(t) - \tilde{x}(t)\Big\rangle\Big]dt$$

$$= \int_{-h}^{0}\Big[\Big\langle A_2^*\lambda(t+h), x(t) - \tilde{x}(t)\Big\rangle\Big]dt = 0.$$

Then we have the inequality (5.9) as follows:

$$\int_0^{T} \Big(f(x(t),t) - f(\tilde{x}(t),t)\Big)dt \geq \int_0^{T} \Big[\Big\langle B^* \frac{d^2\lambda(t)}{dt^2}, x(t) - \tilde{x}(t)\Big\rangle$$

$$-\Big\langle \frac{d^2\big(x(t) - \tilde{x}(t)\big)}{dt^2}, B^*\lambda(t)\Big\rangle\Big]dt + \Big\langle A_1^*\lambda(T), x(T) - \tilde{x}(T)\Big\rangle. \tag{5.10}$$

Now we transform the expression in the square parentheses on the right hand side of (5.10):

$$\Big\langle B^* \frac{d^2\lambda(t)}{dt^2}, x(t) - \tilde{x}(t)\Big\rangle - \Big\langle \frac{d^2\big(x(t) - \tilde{x}(t)\big)}{dt^2}, B^*\lambda(t)\Big\rangle$$

$$= \frac{d}{dt}\Big\langle B^*\frac{d\lambda(t)}{dt}, x(t)-\tilde{x}(t)\Big\rangle - \frac{d}{dt}\Big\langle \frac{d(x(t)-\tilde{x}(t))}{dt}, B^*\lambda(t)\Big\rangle. \tag{5.11}$$

Taking into account $x(\cdot), \tilde{x}(\cdot)$ are feasible and the condition $B^*\lambda(0)=0$ of theorem, we have

$$\int_0^T \Big(f(x(t),t)-f(\tilde{x}(t),t)\Big)dt \geq \Big\langle B^*\frac{d\lambda(T)}{dt} + A_1^*\lambda(T), x(T)-\tilde{x}(T)\Big\rangle - \Big\langle B^*\lambda(T)\ ,\ x'(T)-\tilde{x}'(T)\Big\rangle. \tag{5.12}$$

On the other hand by transversality condition of theorem, we can write

$$\varphi_0\big(x(T)\big)-\varphi_0\big(\tilde{x}(T)\big) \geq \Big\langle -B^*\frac{d\lambda(T)}{dt} - A_1^*\lambda(T)\ ,\ x(T)-\tilde{x}(T)\Big\rangle \tag{5.13}$$

where $\forall\, x(T)\in K_{P(T)}(\tilde{x}(T))$ and $B^*\lambda(T)=0$, then by adding (5.12) and (5.13), we deduce

$$\int_0^T f(x(t),t)dt + \varphi_0(x(T)) \geq \int_0^T f(\tilde{x}(t),t)dt + \varphi_0(\tilde{x}(T))$$

i.e. $J(x(\cdot)) - J(\tilde{x}(\cdot)) \geq 0$ for all feasible solutions $x(t)$ and so $\tilde{x}(t)$ is optimal. □

Corollary 5.2 *Suppose that for the problem (PC_d) with nondelay second-order differential inclusions, a polyhedral set valued mapping G depends on x, x' and the condition Theorem 5.1 are satisfied. Accordingly, nondelay second-order polyhedral differential problem with state constraints is as follows:*

$$\begin{aligned} & \textit{minimize } J[x(\cdot)] = \int_0^T f(x(t),t)dt + \varphi_0(x(T)), \\ (PC1) \qquad & x''(t) \in G(x(t),x'(t)), \quad a.e. \quad t\in[0,T], \\ & x(t)\in P(t)\ , t\in[0,T], \quad x(0)=\beta, \quad x(T)\in\Omega. \end{aligned} \tag{5.14}$$

Then the second-order Euler–Lagrange type differential inclusions and transversality conditions of Theorem 5.1 consist of the following:

$$(i)\ A_0^*\lambda(t) - A_1^*\frac{d\lambda(t)}{dt} - B^*\frac{d^2\lambda(t)}{dt^2} \in K^*_{P(t)}(\tilde{x}(t)) - \partial f(\tilde{x}(t),t),\ a.e.\ t\in[0,T],$$

$$(ii)\ \Big\langle A_0\tilde{x}(t) + A_1\frac{d\tilde{x}(t)}{dt} - B\frac{d^2\tilde{x}(t)}{dt^2} - d\ ,\ \lambda(t)\Big\rangle = 0\ ,\ a.e.\ t\in[0,T],$$

$$\lambda(t)\geq 0,\ B^*\lambda(0)=0,$$

$$(iii)\ A_1^*\lambda(T) + B^*\frac{d\lambda(T)}{dt} \in K^*_{P(T)}(\tilde{x}(T)) - \partial\varphi_0\big(\tilde{x}(T)\big)\ ,\ B^*\lambda(T)=0.$$

Proof Indeed in this case $G(x,v_1)=\{v_2 : A_0x + A_1v_1 - Bv_2 \leq d\}$, then the proof of the corollary immediately follows from the conditions of Theorem 5.1. □

Suppose that we have a boundary value problem $(PC1)$ with nonstate conditions:

$$\text{minimize } J[x(\cdot)] = \int_0^T f(x(t),t)dt + \varphi_0(x(T)),$$
$$(PC2) \qquad x''(t) \in G(x(t), x'(t)), \quad \text{a.e.} \quad t \in [0,T],$$
$$x(0) = \beta_0,\; x(T) = \beta_T. \tag{5.15}$$

The problem is to find an arc $\tilde{x}(t)$ of the problem $(PC2)$ for the second-order differential inclusions satisfying (5.14) almost everywhere (a.e.) on $[0,T]$ and the conditions (5.15) that minimizes the Bolza functional $J(x(\cdot))$.

Corollary 5.3 *In order for trajectory $\tilde{x}(t)$, $t \in [0,T]$, lying interior to domG to be an optimal solution of the second-order polyhedral differential inclusions of problem $(PC2)$, it is sufficient that there exists an absolutely continuous function $\lambda(t)$ satisfying the following Euler–Lagrange type differential inclusions almost everywhere*

$$(e)\; B^* \frac{d^2\lambda(t)}{dt^2} \in A_0^*\lambda(t) - A_1^* \frac{d\lambda(t)}{dt} + \partial f(\tilde{x}(t),t),\; a.e.\; t \in [0,T],$$

$$(f)\; \Big\langle A_0\tilde{x}(t) + A_1 \frac{d\tilde{x}(t)}{dt} - B\frac{d^2\tilde{x}(t)}{dt^2} - d\;,\; \lambda(t) \Big\rangle = 0\;,\; a.e.\; t \in [0,T],$$

$$\lambda(t) \geq 0, \quad B^*\lambda(0) = 0,$$

and transversality conditions at point $t = T$

$$(g)\; -A_1^*\lambda(T) - B^*\frac{d\lambda(T)}{dt} \in \partial\varphi_0\big(\tilde{x}(T)\big)\;,\; B^*\lambda(T) = 0.$$

Proof We rewrite the inclusion (e) in the following form

$$f(x(t),t) - f(\tilde{x}(t),t) \geq \Big\langle B^*\frac{d^2\lambda(t)}{dt^2} - A_0^*\lambda(t) + A_1^*\frac{d\lambda(t)}{dt}\;, x(t) - \tilde{x}(t) \Big\rangle,\; \text{a.e. } t \in [0,T].$$

In the proof of Theorem 5.1, we deduce the previous inequality in a more convenient way

$$f(x(t),t) - f(\tilde{x}(t),t) \geq \Big\langle B^*\frac{d^2\lambda(t)}{dt^2}, x(t) - \tilde{x}(t) \Big\rangle - \Big\langle \frac{d^2\big(x(t) - \tilde{x}(t)\big)}{dt^2}, B^*\lambda(t) \Big\rangle$$
$$+ \frac{d}{dt}\Big[\Big\langle A_1^*\lambda(t), x(t) - \tilde{x}(t) \Big\rangle\Big],\; \text{a.e } t \in [0,T]. \tag{5.16}$$

Then integrating the inequality (5.16) over the interval $[0,T]$ and keeping in mind that $x(\cdot)$, $\tilde{x}(\cdot)$ are feasible, we get

$$\int_0^T \Big(f(x(t),t) - f(\tilde{x}(t),t)\Big)dt \geq \int_0^T \Big[\Big\langle B^*\frac{d^2\lambda(t)}{dt^2}, x(t) - \tilde{x}(t) \Big\rangle$$

$$-\Big\langle \frac{d^2\big(x(t) - \tilde{x}(t)\big)}{dt^2}, B^*\lambda(t) \Big\rangle\Big]dt + \Big\langle A_1^*\lambda(T), x(T) - \tilde{x}(T) \Big\rangle. \tag{5.17}$$

By definition of subdifferential for all feasible solutions, we write the transversality condition (g) in the form

$$\varphi_0\big(x(T)\big) - \varphi_0\big(\tilde{x}(T)\big) \geq \Big\langle -B^* \frac{d\lambda(T)}{dt} - A_1^* \lambda(T)\ ,\ x(T) - \tilde{x}(T) \Big\rangle. \tag{5.18}$$

Furthermore considering the formula (5.11) in the square parentheses on the right hand side of (5.17) and summing up the inequality (5.18), we obtain that

$$\int_0^T f(x(t),t)dt + \varphi_0(x(T)) \geq \int_0^T f(\tilde{x}(t),t)dt + \varphi_0(\tilde{x}(T))$$

i.e. $J(x(\cdot)) - J(\tilde{x}(\cdot)) \geq 0$ for all feasible solutions $x(t)$. It means that $\tilde{x}(t)$ is the optimal arc.

□

Example 5.4 *Let us consider an example on the problem of so-called linear optimal control problem for second-order delay differential equations:*

$$\begin{aligned} &\textit{minimize}\ J(x(\cdot)) = \int_0^T f(x(t),t)dt + \varphi_0(x(T)) \\ &x''(t) = C_0 x(t) + C_1 x'(t) + C_2 x(t-h) + Du(t),\ t \in [0,T], \\ &x(t) = \alpha(t),\ t \in [-h,0), \\ &x(0) = \beta,\ x(T) \in \Omega \end{aligned}$$

where $f(\cdot,t)$ and φ_0 are continuously differentiable functions, C_i, $i = 0,1,2$ and D are $n \times n$ and $n \times r$ matrices, respectively, $u(t) \in U$, U is a polyhedral subset of $\mathbb{R}^r$. The problem is to find a controlling parameter $\tilde{u}(t) \in U$ such that the arc $\widetilde{x}(t)$ corresponding to it minimizes $J(x(\cdot))$. We transform this problem of the following problem with second-order delay differential inclusions of the form:

$$\begin{aligned} &\textit{minimize}\ J(x(\cdot)) = \int_0^T f(x(t),t)dt + \varphi_0(x(T)) \\ (PL_d) \qquad &x''(t) \in F\big(x(t), x'(t), x(t-h)\big) \quad a.e. \quad t \in [0,T], \\ &x(t) = \alpha(t),\ t \in [-h,0),\ x(0) = \beta,\ x(T) \in \Omega, \\ &F(x, v_1, v_2) = C_0 x + C_1 v_1 + C_2 v_2 + Du,\ u \in U. \end{aligned}$$

Corollary 5.5 *The arc $\widetilde{x}(t)$ corresponding to the controlling parameter $\widetilde{u}(t)$ minimizes $J(x(\cdot))$ in the problem (PL_d) if there exists an absolutely continuous function $x^*(t)$ satisfying second-order adjoint delay differential inclusion(equation), the transversality and Weierstrass–Pontryagin conditions:*

$$\frac{d^2 x^*(t)}{dt^2} + C_0^* x^*(t) - C_1^* \frac{dx^*(t)}{dt} + C_2^* x^*(t+h) = f'(\tilde{x}(t),t)\ ,\ a.e.\ t \in [0, T-h),$$

$$\frac{d^2 x^*(t)}{dt^2} + C_0^* x^*(t) - C_1^* \frac{dx^*(t)}{dt} = f'(\tilde{x}(t),t)\ ,\ a.e.\ t \in [T-h, T],$$

$$\frac{dx^*(T)}{dt} - C_1^* x^*(T) \in K_\Omega^*(\tilde{x}(T)) - \varphi_0'(\tilde{x}(T)),\ x^*(0) = 0,\ x^*(T) = 0,$$

$$\Big\langle D\widetilde{u}(t),\ x^*(t)\Big\rangle = \sup_{u \in U} \Big\langle Du,\ x^*(t)\Big\rangle,\ t \in [0,T].$$

Proof In this problem, we are proceeding on the basic of Theorem 5.1. Thus taking into account that $F(x, v_1, v_2) = C_0 x + C_1 v_1 + C_2 v_2 + Du,\ u \in U$ in the problem (PC_d) and by elementary computations, we find that

$$F^*\big(v_3^*; (x, v_1, v_2)\big) = \begin{cases} \big(C_0^* v_3^*\ ,\ C_1^* v_3^*\ ,\ C_2^* v_3^*\big)\ , & -D^* v_3^* \in K_U^*(u), \\ \emptyset\ , & -D^* v_3^* \notin K_U^*(u), \end{cases} \tag{5.19}$$

where $v_3 = C_0 x + C_1 v_1 + C_2 v_2 + Du,\ u \in U$, $C_i^* (i = 0, 1, 2)$ and D^* are transposed matrices. Then taking into account conditions of Theorem 5.1, we conclude

$$-\frac{d^2 x^*(t)}{dt^2} - C_0^* x^*(t) + C_1^* \frac{dx^*(t)}{dt} - C_2^* x^*(t+h) = -f'(\tilde{x}(t), t)\ ,\ \text{a.e. } t \in [0, T-h),$$

$$-\frac{d^2 x^*(t)}{dt^2} - C_0^* x^*(t) + C_1^* \frac{dx^*(t)}{dt} = -f'(\tilde{x}(t), t)\ ,\ \text{a.e. } t \in [T-h, T],$$

$$\frac{dx^*(T)}{dt} - C_1^* x^*(T) \in K_\Omega^*(\tilde{x}(T)) - \varphi_0'(\tilde{x}(T)),\ x^*(0) = 0,\ x^*(T) = 0.$$

Besides $-D^* v_3^* \in K_U^*(u)$ means that the Weierstrass–Pontryagin maximum condition

$$\Big\langle D\widetilde{u}(t),\ x^*(t)\Big\rangle = \sup_{u \in U} \Big\langle Du,\ x^*(t)\Big\rangle$$

satisfied. We have the needed result.

□

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