Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2021) 45: 281 – 287
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doi:10.3906/mat-2001-3

# Quasi-idempotent ranks of the proper ideals in finite symmetric inverse semigroups

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**Received:** 02.01.2020 • **Accepted/Published Online:** 23.11.2020 • **Final Version:** 21.01.2021

**Abstract:** Let $I_n$ and $S_n$ be the symmetric inverse semigroup and the symmetric group on a finite chain $X_n = \{1,\dots,n\}$, respectively. Also, let $I_{n,r} = \{\alpha \in I_n : |\mathrm{im}(\alpha)| \leq r\}$ for $1 \leq r \leq n-1$. For any $\alpha \in I_n$, if $\alpha \neq \alpha^2 = \alpha^4$ then $\alpha$ is called a quasi-idempotent. In this paper, we show that the quasi-idempotent rank of $I_{n,r}$ (both as a semigroup and as an inverse semigroup) is $\binom{n}{2}$ if $r = 2$, and $\binom{n}{r} + 1$ if $r \geq 3$. The quasi-idempotent rank of $I_{n,1}$ is $n$ (as a semigroup) and $n-1$ (as an inverse semigroup).

**Key words:** Symmetric inverse semigroup, symmetric group, quasi-idempotent, rank

## 1. Introduction

Let $I_n$ and $S_n$ be the symmetric inverse semigroup and the symmetric group on a finite chain $X_n = \{1,\dots,n\}$, respectively. Throughout this paper, we assume that $n \geq 2$ unless otherwise stated. It is well known that $I_n$ is an inverse semigroup, and that every finite inverse semigroup $S$ is embeddable in $I_n$. Hence, the importance of $I_n$ in inverse semigroup theory is similar to the importance of $S_n$ in group theory. Although the semigroup $I_n$ has been extensively studied (see, for example, [1, 5, 10, 14]), there are still many interesting problems concerning $I_n$ to be investigated.

For any $\alpha \in I_n$, for a suitable $n$, if $\alpha^2 = \alpha$ then $\alpha$ is called an idempotent; and if $\alpha \neq \alpha^2 = \alpha^4$ then $\alpha$ is called a quasi-idempotent. Throughout this paper we use the notation $Q(U)$ to denote the set of all quasi-idempotents in any subset $U$ of any semigroup.

Let $S$ be a semigroup and let $\emptyset \neq A \subseteq S$. The smallest subsemigroup of $S$ containing $A$ is called the subsemigroup generated by $A$ and denoted by $\langle A \rangle$. Clearly $\langle A \rangle$ is the set of all finite products of elements of $A$. If there exists a nonempty subset $A$ of $S$ such that $S = \langle A \rangle$, then $A$ is called a generating set of $S$. Also, the rank of a finitely generated semigroup $S$ is defined by

$$\mathrm{rank}(S) = \min\{\, |A| : \langle A \rangle = S\}. \tag{1.1}$$

There are studies of various ranks of semigroups, such as idempotent-rank, $(m,r)$ rank and nilpotent rank, as well as of minimal generating sets of elements of a given kind (see, for example, [2, 3, 6, 8, 13, 17, 18]). In particular, if there exists a generating set $A$ of $S$ consisting entirely of quasi-idempotents, then $A$ is called

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2010 *AMS Mathematics Subject Classification:* 20M20

a quasi-idempotent generating set of $S$, and the quasi-idempotent rank of $S$ is defined by

$$\mathrm{qrank}(S) = \min\{\, |A| : \langle A \rangle = S,\ A \subseteq Q(S)\}. \tag{1.2}$$

Let $S$ be an inverse semigroup and let $\emptyset \neq A \subseteq S$. The smallest inverse subsemigroup of $S$ containing $A$ is called the subsemigroup generated by $A$ as an inverse semigroup. To avoid confusion, as in [11], we shall use the notation $\langle\langle A \rangle\rangle$ for this inverse subsemigroup. It is clear that $\langle\langle A \rangle\rangle$ is the set of all finite products of elements of $A$ and their inverses. If there exists a nonempty subset $A$ of $S$ such that $S = \langle\langle A \rangle\rangle$, then $A$ is called a generating set of $S$ as an inverse semigroup. Also, the inverse rank of a finitely generated semigroup $S$ is defined by

$$\mathrm{rank}^*(S) = \min\{\, |A| : \langle\langle A \rangle\rangle = S\}. \tag{1.3}$$

Similarly, if there exists a generating set $A$ of $S$ as an inverse semigroup such that $A$ consists of quasi-idempotents, then $A$ is called a quasi-idempotent generating set of $S$ as an inverse semigroup, and if $S$ is finitely generated, then the quasi-idempotent inverse rank of $S$ is defined by

$$\mathrm{qrank}^*(S) = \min\{\, |A| : \langle\langle A \rangle\rangle = S,\ A \subseteq Q(S)\}. \tag{1.4}$$

For every transformation $\alpha \in I_n$, we denote by $\mathrm{dom}(\alpha)$ and $\mathrm{im}(\alpha)$ the *domain* and the *image* of $\alpha$, respectively. Let $a_1, \ldots, a_k$, where $k \geq 1$, be distinct elements of $X_n$. Then $\sigma \in I_n$ is called a *cycle* of length $k$ (or a $k-$cycle), and denoted by $\sigma = (a_1 \ldots a_k)$, if $a_i\sigma = a_{i+1}$ $(1 \leq i \leq k-1)$, $a_k\sigma = a_1$, and $x\sigma$ is undefined if $x \notin \{a_1, \ldots, a_k\}$. Similarly, $\lambda \in I_n$ is called a *chain* of length $k$ (or a $k-$chain), and denoted by $\lambda = [a_1 \ldots a_k]$, if $a_i\lambda = a_{i+1}$ $(1 \leq i \leq k-1)$, and $x\lambda$ is undefined if $x \notin \{a_1, \ldots, a_{k-1}\}$. Note that $[a_1]$ is the zero element of $I_n$. We say that $\alpha, \beta \in I_n$ are disjoint if $(\mathrm{dom}(\alpha) \cup \mathrm{im}(\alpha)) \cap (\mathrm{dom}(\beta) \cup \mathrm{im}(\beta)) = \emptyset$. Every nonzero $\alpha \in I_n$ can be written uniquely (up to the order) as a join (set theoretical union) of disjoint cycles and chains [14]. We write the join of $\alpha$ and $\beta$ as $\alpha\beta$ (only if $\alpha$ and $\beta$ are disjoint), and the product as $\alpha \circ \beta$.

If $\alpha \in I_n$ with $\mathrm{dom}(\alpha) = A$ and $\mathrm{im}(\alpha) = B$, we will write $\alpha : A \mapsto B$. Moreover, if $\mathrm{dom}(\alpha)$ is specified, we will skip 1-cycles in the notation of $\alpha$. With some abuse of the definition of a cycle, if $\alpha : A \mapsto A$ and $\alpha = (a_1 \ldots a_k)$, with $k \geq 1$ and 1-cycles omitted (except one 1-cycle if $k = 1$), we will refer to $\alpha$ as a $k$-cycle on $A$. For example, the map $\alpha : \{1,2,3,4,5\} \mapsto \{1,2,3,4,6\}$ defined by tabular form

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 4 & 6 & - & - & - \end{pmatrix} \in I_{8,5}$$

can be written as $\alpha = (12)[56]$ if we know that $\mathrm{dom}(\alpha) = \{1,2,3,4,5\}$.

It is well known that $S_2 = \langle (12) \rangle$, $S_3 = \langle (13),(23) \rangle = \langle (12),(123) \rangle$, and $S_n = \langle (12),(12\ldots n) \rangle$ for $n \geq 4$; $I_2 = \langle (12),(1) \rangle$ and $I_n = \langle (12),(12\ldots n),(1)(2)\ldots(n-1) \rangle$ for $n \geq 3$. Furthermore,

$$\mathrm{rank}(S_n) = \begin{cases} 1 & \text{for } n = 2 \\ 2 & \text{for } n \geq 3 \end{cases}, \text{ and } \mathrm{rank}(I_n) = \begin{cases} 2 & \text{for } n = 2 \\ 3 & \text{for } n \geq 3 \end{cases}. \tag{1.5}$$

Umar showed in [15] that rank and quasi-idempotent rank of $L^-(n,n-1)$ are both equal to $\dfrac{n(n+1)}{2}$, where $L^-(n,r)$, $1 \leq r \leq n-1$ is the subsemigroup of $I_n$ consisting of all decreasing partial one-to-one maps $\alpha$ (including the empty or zero map) for which $|\mathrm{im}(\alpha)| \leq r$. Also, Umar [16] calculated the quasi-idempotent rank of $L^-(n,r)$ for $1 \leq r \leq n-1$.

For any element $a$ of a semigroup $S$, the smallest left ideal of $S$ containing $a$ is $Sa \cup \{a\}$, which is denoted by $S^1a$. We shall call it the principal left ideal of $S$ generated by $a$. Principal right ideal of $S$ generated by $a$, $aS^1$, can be defined similarly. An equivalence $\mathcal{L}$ on $S$ is defined by the rule that $a\mathcal{L}b$ if and only if $S^1a = S^1b$, and an equivalence $\mathcal{R}$ on $S$ is defined by the rule that $a\mathcal{R}b$ if and only if $aS^1 = bS^1$. It is well known that $\mathcal{L}\circ\mathcal{R} = \mathcal{R}\circ\mathcal{L}$ (see, for example, [12]). Also, an equivalence $\mathcal{D}$ on $S$ is defined by $\mathcal{D} = \mathcal{L}\circ\mathcal{R}$, and an equivalence $\mathcal{H}$ on $S$ is defined by $\mathcal{H} = \mathcal{L}\cap\mathcal{R}$. An element $a$ of a semigroup $S$ is called regular if there exists $x \in S$ such that $axa = a$. If all elements of a semigroup $S$ are regular then $S$ is called regular. It follows from [12, Proposition 2.3.1 ] that, for any $\mathcal{D}$-class $D$, either every element of $D$ is regular or no element of $D$ is regular. If all elements of $D$ are regular then we call the $\mathcal{D}$-class $D$ regular. For more information see, for example, [12].

Now let

$$I_{n,r} = \{\alpha \in I_n : |\mathrm{im}(\alpha)| \le r\} \tag{1.6}$$

for $1 \le r \le n-1$. Clearly $I_{n,r}$ is an ideal of $I_n$ for each $1 \le r \le n-1$, and notice that $I_{n,n-1} = I_n \setminus S_n$. Moreover, it is easy to see that Green's equivalences $\mathcal{L},\mathcal{R},\mathcal{D}$, and $\mathcal{H}$ on $I_{n,r}$ are characterized by

$$\begin{aligned}
(\alpha,\beta)\in\mathcal{L} &\quad\Leftrightarrow\quad \mathrm{im}(\alpha)=\mathrm{im}(\beta)\\
(\alpha,\beta)\in\mathcal{R} &\quad\Leftrightarrow\quad \mathrm{dom}(\alpha)=\mathrm{dom}(\beta)\\
(\alpha,\beta)\in\mathcal{D} &\quad\Leftrightarrow\quad |\mathrm{im}(\alpha)|=|\mathrm{im}(\beta)|\\
(\alpha,\beta)\in\mathcal{H} &\quad\Leftrightarrow\quad \mathrm{dom}(\alpha)=\mathrm{dom}(\beta) \text{ and } \mathrm{im}(\alpha)=\mathrm{im}(\beta)
\end{aligned}$$

for any $\alpha,\beta \in I_{n,r}$. Hence, there exist $r+1$ $\mathcal{D}$-classes in $I_{n,r}$ as follows:

$$D_k = \{\alpha \in I_{n,r} : |\mathrm{im}(\alpha)| = k\} \quad \text{for } 0 \le k \le r. \tag{1.7}$$

All $\mathcal{D}$-classes of $I_{n,r}$ are regular, since $I_{n,r}$ is a regular semigroup. For $\alpha \in I_{n,r}$ with $\mathrm{dom}(\alpha) = A$ and $\mathrm{im}(\alpha) = B$ we will denote the $\mathcal{H}$-class of $\alpha$ by $H_{A,B}$. The rank and inverse rank of $I_{n,r}$ have been determined. By [11, Theorem 3.7] and [7, Theorem 2.2],

$$\mathrm{rank}^*(I_{n,r}) = \begin{cases} n-1 & \text{for } r=1\\ \binom{n}{2} & \text{for } r=2\\ \binom{n}{r}+1 & \text{for } 3\le r\le n-1 \end{cases}. \tag{1.8}$$

Nearly twenty years after the introduction of $\mathrm{rank}^*(I_{n,r})$ into the literature, Zhao and Fernandes showed in [18, Theorem 4.2] that

$$\mathrm{rank}(I_{n,r}) = \begin{cases} n & \text{for } r=1\\ \binom{n}{2} & \text{for } r=2\\ \binom{n}{r}+1 & \text{for } 3\le r\le n-1 \end{cases}. \tag{1.9}$$

Note that the rank of $I_{n,r}$ coincides with its inverse rank for $2 \le r \le n-1$. However, $\mathrm{rank}(I_{n,1}) = n$ while $\mathrm{rank}^*(I_{n,1}) = n-1$. In the next section, we show that $\mathrm{qrank}(I_{n,r}) = \mathrm{rank}(I_{n,r})$ and $\mathrm{qrank}^*(I_{n,r}) = \mathrm{rank}^*(I_{n,r})$ for each $1 \le r \le n-1$.

## 2. Quasi-idempotent rank of $I_{n,r}$

It is clear that $\alpha \in I_n$ is a quasi-idempotent if and only if it is a join of some (perhaps none) 1-cycles and at least one 2-cycle or 2-chain.

**Lemma 2.1** *Let $n \geq 4$ and $3 \leq r \leq n-1$. Order $1,\ldots,r$ arbitrarily as $a_1,\ldots,a_r$, and let $a_{r+1} = r+1$. Consider the subsets $A_1,\ldots,A_{r+1}$ of $\{a_1,\ldots,a_{r+1}\}$ defined by $A_i = \{a_1,\ldots,a_{r+1}\} \setminus \{a_{r+2-i}\}$, $1 \leq i \leq r+1$. Then there are quasi-idempotents $\alpha_1,\ldots,\alpha_r,\beta_1,\beta_2 \in I_n$ such that $\alpha_i : A_i \mapsto A_{i+1}$, $1 \leq i \leq r$, $\beta_1,\beta_2 : A_{r+1} \mapsto A_1$, $\alpha_1 \circ \ldots \circ \alpha_r \circ \beta_1 = (a_1 \cdots a_r)$, and $\alpha_1 \circ \ldots \circ \alpha_r \circ \beta_2 = (a_1 \cdots a_{r-2} a_r)$.*

**Proof** Recall that we skip 1-cycles in $\alpha \in I_n$ when $\mathrm{dom}(\alpha)$ is specified. Let $\alpha_i = [a_{r+1-i}\, a_{r+2-i}]$, $1 \leq i \leq r$, $\beta_1 = [a_{r+1}\, a_1]$, and $\beta_2 = [a_{r+1}\, a_1](a_{r-1}\, a_r)$. Then the result is clear. □

**Lemma 2.2** *Let $n \geq 4$ and $3 \leq r \leq n-2$. Let $A_1 = \{1,\ldots,r\}$. Order the $r$-elements subsets of $X_n$ that are not included in $\{1,\ldots,r+1\}$, $A_{r+2},\ldots,A_{m_r}$, where $m_r = \binom{n}{r}$, with $A_{m_r} = \{1,\ldots,r-1,n\}$. Then there are quasi-idempotents $\gamma,\alpha_{r+2},\ldots,\alpha_{m_r-1},\delta \in I_n$ such that $\gamma : A_1 \mapsto A_{r+2}$, $\alpha_i : A_i \mapsto A_{i+1}$, $r+2 \leq i \leq m_r - 1$, $\delta : A_{m_r} \mapsto A_1$, and $\langle \gamma,\alpha_{r+2},\ldots,\alpha_{m_r-1},\delta \rangle$ contains a 2-cycle, or an $(r-1)$-cycle, or an $r$-cycle on $A_1$.*

**Proof** Define $\gamma,\alpha_{r+2},\ldots,\alpha_{m_r-1}$ arbitrarily, and let $\eta = \gamma \circ \alpha_{r+2} \circ \cdots \circ \alpha_{m_r-1} \circ [n\, r]$, where $\mathrm{dom}([n\, r]) = A_{m_r}$. If $\eta$ is the identity on $A_1$, take $\delta = [n\, r](1\, 2)$.

Otherwise, let $(c_1^1 \ldots c_{i_1}^1),\ldots,(c_1^k \ldots c_{i_k}^k)$, $k \geq 1$ be the cycles in $\eta$ of length at least 2. If $r$ occurs in any of these cycles, we may assume that $r = c_1^1$. Define $\theta : A_{m_r} \mapsto A_1$ by $\theta : [n\, 2]$ if $k = 1$, and $\theta = [n\, r][c_{i_1}^1\, c_1^2] \cdots [c_{i_{k-1}}^{k-1}\, c_1^k]$, if $k \geq 2$. Then $\gamma \circ \alpha_{r+2} \circ \cdots \circ \alpha_{m_r-1} \circ \theta$ is an $l$-cycle on $A_1$. If $l \in \{2, r-1, r\}$, take $\delta = \theta$.

If $l = r-2$, then for $\delta = \theta(c_{i_k}^k\, d)$, where $d$ is a fixed point of $\gamma \circ \alpha_{r+2} \circ \cdots \circ \alpha_{m_r-1} \circ \theta$ with $d \neq r$, $\gamma \circ \alpha_{r+2} \circ \cdots \circ \alpha_{m_r-1} \circ \delta$ is an $(r-1)$-cycle on $A_1$.

Suppose $l \leq r-3$ and there are distinct fixed points $d_1, d_2, d_3$ of $\gamma \circ \alpha_{r+2} \circ \cdots \circ \alpha_{m_r-1} \circ \theta$ such that $r \notin \{d_1, d_2, d_3\}$. If $l$ is odd, then for $\delta = \theta(d_1\, d_2)$, we have $(d_1\, d_2) = (\gamma \circ \alpha_{r+2} \circ \cdots \circ \alpha_{m_r-1} \circ \delta)^l \in \langle \gamma,\alpha_{r+2},\ldots,\alpha_{m_r-1},\delta \rangle$. If $l$ is even, then for $\delta = \theta(c_{i_k}^k\, d_3)(d_1\, d_2)$, we have $(d_1\, d_2) = (\gamma \circ \alpha_{r+2} \circ \cdots \circ \alpha_{m_r-1} \circ \delta)^{l+1} \in \langle \gamma,\alpha_{r+2},\ldots,\alpha_{m_r-1},\delta \rangle$.

Suppose $l \leq r-3$ and there are only two distinct fixed points $d_1, d_2$ of $\gamma \circ \alpha_{r+2} \circ \cdots \circ \alpha_{m_r-1} \circ \theta$ such that $r \notin \{d_1, d_2\}$. Then $l = r-3$ and $r$ is a fixed point of $\gamma \circ \alpha_{r+2} \circ \cdots \circ \alpha_{m_r-1} \circ \theta$. If $i_1 = \cdots = i_k = 2$, then taking $\delta = [n\, r](c_1^1\, c_2^1) \cdots (c_1^k\, c_2^k)(d_1\, d_2)$, we obtain $\gamma \circ \alpha_{r+2} \circ \cdots \circ \alpha_{m_r-1} \circ \delta = (d_1\, d_2)$. Otherwise, we may assume that $i_1 \geq 3$. Then taking $\delta = \theta(c_1^1\, d_1)(c_2^1\, d_2)$, we obtain $\gamma \circ \alpha_{r+2} \circ \cdots \circ \alpha_{m_r-1} \circ \delta$ to be an $(r-1)$-cycle. □

**Lemma 2.3** *Let $S$ be a finite semigroup, and let $D$ be a regular $\mathcal{D}$-class of $S$. Let $G$ be a group $\mathcal{H}$-class of $D$, and let $W$ be a subset of $S$ that intersects each $\mathcal{H}$-class of $D$. Then $D \subseteq \langle G, W \rangle$.*

**Proof** This is Lemma 4.9 in [18]. □

**Lemma 2.4** *Let $n \geq 2$ and $1 \leq r \leq n-1$. Then the ideal $I_{n,r}$ is generated by its elements of rank $r$.*

**Proof** This is Lemma 4.7 in [18].

**Lemma 2.5** *Let* $n \geq 2$ *and* $1 \leq r \leq n-1$. *Suppose* $\alpha_1, \ldots, \alpha_k \in I_n$ *are elements of rank* $r$ *such that there exists a group* $\mathcal{H}$*-class* $G$ *of* $D_r$ *such that* $G \subseteq \langle \alpha_1, \ldots, \alpha_k \rangle$, *and* $\langle \alpha_1, \ldots, \alpha_k \rangle$ *intersects each* $\mathcal{H}$*-class of* $D_r$. *Then* $\langle \alpha_1, \ldots, \alpha_k \rangle = I_{n,r}$.

**Proof** The result follows immediately from Lemmas 2.3 and 2.4. □

**Lemma 2.6** *For* $n \geq 2$, *let* $A_i = \{i\}$ *for* $1 \leq i \leq n$, *let* $\alpha_i : A_i \mapsto A_{i+1}$ *be the quasi-idempotent* $\alpha_i = [i\, i+1]$ *for* $1 \leq i \leq n-1$, *and let* $\alpha_n : A_n \mapsto A_1$ *be the quasi-idempotent* $\alpha_n = [n\, 1]$. *Then* $I_{n,1} = \langle \alpha_1, \ldots, \alpha_n \rangle$.

**Proof** For any $\alpha \in D_1$ there exist $1 \leq i, j \leq n$ such that $\mathrm{dom}(\alpha) = \{i\}$ and $\mathrm{im}(\alpha) = \{j\}$, so the result follows from the equality $\alpha = \alpha_i \circ \cdots \circ \alpha_n \circ \alpha_1 \circ \cdots \circ \alpha_{j-1}$ if $j \leq i$ and $\alpha = \alpha_i \circ \alpha_{i+1} \circ \cdots \circ \alpha_{j-1}$ if $j > i$. □

**Lemma 2.7** *For* $n \geq 3$, *let* $A_{i,j} = \{i, j\}$ *for* $1 \leq i < j \leq n$, *and let* $\alpha_{i,j} : A_{i,j} \mapsto A_{i,j+1}$ *for* $1 \leq i < j \leq n-1$, $\alpha_{i,n} : A_{i,n} \mapsto A_{i+1,i+2}$ *for* $1 \leq i \leq n-2$, *and* $\alpha_{n-1,n} : A_{n-1,n} \mapsto A_{1,2}$ *be the quasi-idempotents*

$$\begin{array}{lcll} \alpha_{i,j} & = & [j \quad j+1] & \textit{for } 1 \leq i < j \leq n-1, \\ \alpha_{i,n} & = & [i \quad i+1][n \quad i+2] & \textit{for } 1 \leq i \leq n-3, \\ \alpha_{n-2,n} & = & [n-2 \quad n-1] & \textit{and} \\ \alpha_{n-1,n} & = & [n-1 \quad 2][n \quad 1]. & \end{array}$$

*Then* $I_{n,2} = \langle \alpha_{1,2}, \alpha_{1,3}, \ldots, \alpha_{1,n}, \alpha_{2,3}, \ldots, \alpha_{2,n}, \ldots, \alpha_{n-1,n} \rangle$.

**Proof** For convenience, order the sets $A_{1,2}, A_{1,3}, \ldots, A_{1,n}, A_{2,3}, \ldots, A_{2,n}, \ldots, A_{n-1,n}$, respectively, by $A_1, A_2, \ldots, A_{m_2}$, and order the elements $\alpha_{1,2}, \alpha_{1,3}, \ldots, \alpha_{1,n}, \alpha_{2,3}, \ldots, \alpha_{2,n}, \ldots, \alpha_{n-1,n}$, respectively, by $\alpha_1, \alpha_2, \ldots, \alpha_{m_2}$ where $m_2 = \binom{n}{2}$.

It is easy to see that $\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_{m_2} = (12)$. Then, $(1\,2) \in \langle \alpha_1, \alpha_2, \ldots, \alpha_{m_2} \rangle$, so $H_{A_1,A_1} \subseteq \langle \alpha_1, \alpha_2, \ldots, \alpha_{m_2} \rangle$. Moreover, for any $\alpha \in H_{A_i,A_j}$ $(1 \leq i, j \leq m_2)$, there exists $\overline{\alpha} \in H_{A_1,A_1}$ such that

$$\alpha = \alpha_i \circ \cdots \circ \alpha_{m_2} \circ \overline{\alpha} \circ \alpha_1 \circ \cdots \circ \alpha_{j-1}.$$

Hence, $I_{n,2} = \langle \alpha_1, \alpha_2, \ldots, \alpha_{m_2} \rangle$, by Lemma 2.5. □

**Theorem 2.8** *For* $n \geq 2$ *and* $1 \leq r \leq n-1$,

$$\mathrm{qrank}(I_{n,r}) = \mathrm{rank}(I_{n,r}) = \begin{cases} n & \textit{for } r = 1 \\ \binom{n}{2} & \textit{for } r = 2 \\ \binom{n}{r} + 1 & \textit{for } 3 \leq r \leq n-1 \end{cases}.$$

**Proof** The cases $r = 1$ and $r = 2$ follow from Lemmas 2.6 and 2.7, (1.9), and the fact that $\mathrm{rank}(I_{n,r}) \leq \mathrm{qrank}(I_{n,r})$. Let $n \geq 4$ and $3 \leq r \leq n-1$.

Suppose $r = n-1$. Let $A_1, \ldots, A_{r+1}$ be the $r$-element subsets of $X_n$ as in Lemma 2.1, with $a_i = i$, and let $\alpha_1, \ldots, \alpha_r, \beta_1 \in I_n$ be the quasi-idempotents as in Lemma 2.1. Then $(1 \ldots r) \in \langle \alpha_1, \ldots, \alpha_r, \beta_1 \rangle$, and $\langle \alpha_1, \ldots, \alpha_r, \beta_1 \rangle$ intersects each $\mathcal{H}$-class of $D_r$. Consider $(1\,2)$ as a 2-cycle on $A_1$. Then $H_{A_1,A_1} \subseteq \langle \alpha_1, \ldots, \alpha_r, \beta_1, (1\,2) \rangle$. Hence, by Lemma 2.5, $\langle \alpha_1, \ldots, \alpha_r, \beta_1, (1\,2) \rangle = I_{n,r}$, so $\mathrm{qrank}(I_{n,r}) \leq r + 2 = \binom{n}{r} + 1$.

Suppose $r \leq n-2$. Let $A_1, A_{r+2}, \ldots, A_{m_r}$, where $m_r = \binom{n}{r}$, be the $r$-element subsets of $X_n$ as in Lemma 2.2, and let $\gamma, \alpha_{r+2}, \ldots, \alpha_{m_r-1}, \delta \in I_n$ be the quasi-idempotents as in Lemma 2.2. Note that the set $A_1$ in Lemma 2.1 is the same as the $A_1$ in Lemma 2.2, namely $\{1, \ldots, r\}$. Then $\langle \gamma, \alpha_{r+2}, \ldots, \alpha_{m_r-1}, \delta \rangle$ contains a 2-cycle, or an $(r-1)$-cycle, or an $r$-cycle on $A_1$. We now have three cases to consider.

Suppose $(b_1 \ldots b_r) \in \langle \gamma, \alpha_{r+2}, \ldots, \alpha_{m_r-1}, \delta \rangle$, where $(b_1 \ldots b_r)$ is an $r$-cycle on $A_1$. Let $A_1, \ldots, A_{r+1}$ be the $r$-element subsets of $X_n$ as in Lemma 2.1, with $a_1 = b_1, \ldots, a_{r-2} = b_{r-2}$ and $a_r = b_{r-1}$, and let $\alpha_1, \ldots, \alpha_r, \beta_2 \in I_n$ be the quasi-idempotents as in Lemma 2.1. Then $(b_1 \ldots b_{r-1}) \in \langle \alpha_1, \ldots, \alpha_r, \beta_2 \rangle$, so $H_{A_1,A_1} \subseteq \langle \alpha_1, \ldots, \alpha_r, \beta_2, \gamma, \alpha_{r+2}, \ldots, \alpha_{m_r-1}, \delta \rangle$. Moreover, $\langle \alpha_1, \ldots, \alpha_r, \beta_2, \gamma, \alpha_{r+2}, \ldots, \alpha_{m_r-1}, \delta \rangle$ intersects each $\mathcal{H}$-class of $D_r$. Hence, by Lemma 2.5, $\langle \alpha_1, \ldots, \alpha_r, \beta_2, \gamma, \alpha_{r+2}, \ldots, \alpha_{m_r-1}, \delta \rangle = I_{n,r}$, so $\operatorname{qrank}(I_{n,r}) \leq m_r - 2 + 3 = \binom{n}{r} + 1$.

Suppose $(b_1 \ldots b_{r-1}) \in \langle \gamma, \alpha_{r+2}, \ldots, \alpha_{m_r-1}, \delta \rangle$, where $(b_1 \ldots b_{r-1})$ is an $(r-1)$-cycle on $A_1$. Let $A_1, \ldots, A_{r+1}$ be the $r$-element subsets of $X_n$ as in Lemma 2.1, with $a_1 = b_1, \ldots a_{r-1} = b_{r-1}$ and $\{b_r\} = A_1 \setminus \{b_1, \ldots, b_{r-1}\}$, and let $\alpha_1, \ldots, \alpha_r, \beta_1 \in I_n$ be the quasi-idempotents as in Lemma 2.1. Then $(b_1 \ldots b_r) \in \langle \alpha_1, \ldots, \alpha_r, \beta_1 \rangle$, so $H_{A_1,A_1} \subseteq \langle \alpha_1, \ldots, \alpha_r, \beta_1, \gamma, \alpha_{r+2}, \ldots, \alpha_{m_r-1}, \delta \rangle$. Moreover, $\langle \alpha_1, \ldots, \alpha_r, \beta_1, \gamma, \alpha_{r+2}, \ldots, \alpha_{m_r-1}, \delta \rangle$ intersects each $\mathcal{H}$-class of $D_r$. Hence, by Lemma 2.5, $\langle \alpha_1, \ldots, \alpha_r, \beta_1, \gamma, \alpha_{r+2}, \ldots, \alpha_{m_r-1}, \delta \rangle = I_{n,r}$, so $\operatorname{qrank}(I_{n,r}) \leq m_r - 2 + 3 = \binom{n}{r} + 1$.

Suppose $(b_1\, b_2) \in \langle \gamma, \alpha_{r+2}, \ldots, \alpha_{m_r-1}, \delta \rangle$, where $(b_1\, b_2)$ is a 2-cycle on $A_1$. Let $A_1, \ldots, A_{r+1}$ be the $r$-element subsets of $X_n$ as in Lemma 2.1, with $a_1 = b_1, a_2 = b_2$ and $\{a_3, \ldots, a_r\} = A_1 \setminus \{b_1, b_2\}$, and let $\alpha_1, \ldots, \alpha_r, \beta_1 \in I_n$ be the quasi-idempotents as in Lemma 2.1. Then $(b_1 \ldots b_r) \in \langle \alpha_1, \ldots, \alpha_r, \beta_1 \rangle$, so $H_{A_1,A_1} \subseteq \langle \alpha_1, \ldots, \alpha_r, \beta_1, \gamma, \alpha_{r+2}, \ldots, \alpha_{m_r-1}, \delta \rangle$. Moreover, $\langle \alpha_1, \ldots, \alpha_r, \beta_1, \gamma, \alpha_{r+2}, \ldots, \alpha_{m_r-1}, \delta \rangle$ intersects each $\mathcal{H}$-class of $D_r$. Hence, by Lemma 2.5, $\langle \alpha_1, \ldots, \alpha_r, \beta_1, \gamma, \alpha_{r+2}, \ldots, \alpha_{m_r-1}, \delta \rangle = I_{n,r}$, so $\operatorname{qrank}(I_{n,r}) \leq m_r - 2 + 3 = \binom{n}{r} + 1$.

Therefore, the result follows immediately from the fact that $\binom{n}{r} + 1 = \operatorname{rank}(I_{n,r}) \leq \operatorname{qrank}(I_{n,r}) \leq \binom{n}{r} + 1$. □

**Theorem 2.9** *For $n \geq 2$ and $1 \leq r \leq n-1$,*

$$\operatorname{qrank}^*(I_{n,r}) = \operatorname{rank}^*(I_{n,r}) = \begin{cases} n-1 & \text{for } r = 1 \\ \binom{n}{2} & \text{for } r = 2 \\ \binom{n}{r} + 1 & \text{for } 3 \leq r \leq n-1 \end{cases}.$$

**Proof** The case $r = 1$ is easy to prove directly. Let $n \geq 3$ and $2 \leq r \leq n-1$. Then the result follows immediately from the facts that $\operatorname{rank}^*(I_{n,r}) \leq \operatorname{qrank}^*(I_{n,r}) \leq \operatorname{qrank}(I_{n,r})$ and $\operatorname{rank}^*(I_{n,r}) = \operatorname{qrank}(I_{n,r})$ (by (1.8) and Theorem 2.8). □

**Acknowledgments**

My sincere thanks are due to Prof. Dr. Hayrullah Ayık and the referees for their helpful suggestions and encouragement that really improved the paper.

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