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On the growth of maximum modulus of rational functions with prescribed poles

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Abstract: In this paper we prove a sharp growth estimate for rational functions with prescribed poles and restricted zeros in the Chebyshev norm on the unit disk in the complex domain. In particular we extend a polynomial inequality due to Dubinin (2007) to rational functions which also improves a result of Govil and Mohapatra (1998).

Key words: Rational functions, polynomials, inequalities, growth estimates, Schwarz lemma

1. Introduction and statement of results

Let $\mathcal{P}_n$ denote the class of all complex algebraic polynomials $p(z) := \sum_{j=0}^{n} c_j z^j$ of degree atmost n and $p'(z)$ be the derivative of $p(z)$. Let D_k^- represent the set of all points which lie inside $T_k := \{z : |z| = k\}$ and D_k^+ be the set of points which lie outside T_k. We write,

$$\mathcal{R}_n = \mathcal{R}_n(a_1, a_2, \dots, a_n) := \left\{ \frac{p(z)}{w(z)} : \quad p \in \mathcal{P}_n \right\},$$

where

$$w(z) := \prod_{j=1}^{n} (z - a_j), \ a_j \in D_1^+, \ j = 1, 2, \cdots, n.$$

And for a function f defined on T_1 in the complex domain $\mathbb{C}$,

$$\| f \| := \sup_{z \in T_1} |f(z)| \quad ; \quad f^*(z) := z^n \overline{f\left(\frac{1}{\overline{z}}\right)}, \quad \text{if} \quad f \in \mathcal{P}_n \quad ;$$

$$f^*(z) = B(z) \overline{f\left(\frac{1}{\overline{z}}\right)}, \quad \text{if} \quad f \in \mathcal{R}_n.$$

Thus $\mathcal{R}_n$ is the set of all rational functions with poles $a_1, a_2, \dots, a_n$ at most and with finite limit at ∞. We observe that the Blaschke product $B(z) \in \mathcal{R}_n$, where

$$B(z) := \prod_{j=1}^{n} \left(\frac{1 - \overline{a_j} z}{z - a_j} \right) = \frac{w^*(z)}{w(z)}. \tag{1.1}$$

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If $p \in \mathcal{P}_n$, then

$$\parallel p' \parallel \leq n \parallel p \parallel \tag{1.2}$$

and for $R \geq 1$

$$\parallel p(R\ .) \parallel \leq R^n \parallel p \parallel, \tag{1.3}$$

where $\parallel p(R\ .) \parallel := \sup_{z\in T_1} |p(Rz)| = \sup_{z \in T_R} |p(z)|$.
Inequality (1.2) is an immediate consequence of Bernstein's theorem on the derivative of a trigonometric polynomial (for reference see [3] and also [10, 13]), whereas inequality (1.3) is a simple deduction from the maximum modulus principle (for reference see [12]). In both (1.2) and (1.3) equality holds only for $p(z) = e^{i\varphi} z^n$, $\varphi \in R$, that is, if and only if $p(z)$ has all its zeros at the origin.
Inequalities (1.2) and (1.3) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in D_1^-. In fact, if $p \in \mathcal{P}_n$ does not vanish in D_1^-, then (1.2) and (1.3) can be respectively replaced by

$$\parallel p' \parallel \leq \frac{n}{2} \parallel p \parallel \tag{1.4}$$

and

$$\parallel p(R\ .) \parallel \leq \frac{R^n + 1}{2} \parallel p \parallel. \tag{1.5}$$

Inequality (1.4) was conjectured by Erdös and later verified by Lax [8], whereas Ankeny and Rivlin [1] used (1.4) to prove inequality (1.5). In both (1.4) and (1.5) equality holds for $p(z) = \alpha z^n + \beta$, where $\alpha, \beta \in T_1$. The inequalities (1.2) −−(1.5) were respectively extended to rational functions $r \in \mathcal{R}_n$ with prescribed poles to read as follows:

$$\parallel r' \parallel \leq |B'(z)| \parallel r \parallel, \tag{1.6}$$

$$\parallel r(R\ .) \parallel \leq |B(Rz)| \parallel r \parallel, \tag{1.7}$$

$$\parallel r' \parallel \leq \frac{|B'(z)|}{2} \parallel r \parallel, \tag{1.8}$$

$$\parallel r(R\ .) \parallel \leq \frac{|B(Rz)| + 1}{2} \parallel r \parallel, \qquad \text{for } z \in T_1. \tag{1.9}$$

Inequalities (1.6) and (1.8) are due to Li, Mohapatra and Rodriguez [9], whereas inequality (1.7) is due to Walsh [14] and inequality (1.9) is due to Govil and Mohapatra [5] (see also Aziz and Rather [2]). Recently Dubinin [4] improved inequality (1.5) by obtaining the following result:

Theorem A If $p(z) := c_0 + c_1 z + \ldots + c_n z^n, n \geq 2$ is a polynomial without zeros in D_1^-, then for $R \geq 1$,

$$\parallel p(R\ .) \parallel \leq \frac{(R^n + 1)(|c_0| + |c_n| R)}{(1 + R)(|c_0| + |c_n|)} \parallel p \parallel.$$

In this paper by using a generalized form of Schwarz lemma, we first prove the following extension of Theorem A to rational functions with prescribed poles. Our result besides improving inequality (1.9) yields Theorem A as a special case. In fact we prove:

Theorem 1.1 *Suppose* $r \in \mathcal{R}_n$ *is such that* $r(z) = \frac{p(z)}{w(z)}$, *where* $p(z) := \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* $n \geq 2$ *without zeros in* D_1^-. *Then for* $z \in T_1 \cup D_1^+$

$$|r(z)| \leq \frac{|c_n||z| + |c_0|}{(1+|z|)(|c_n|+|c_0|)}(|B(z)|+1) \parallel r \parallel, \tag{1.10}$$

where $B(z)$ *is defined in* (1.1).
The result is sharp and equality holds for $r(z) = B(z) + \lambda$, *where* $\lambda \in T_1$ *is chosen suitably.*

If we take $z = Re^{i\theta}, R \geq 1,\ 0 \leq \theta < 2\pi$, then from Theorem 1.1 we have :

Corollary 1.2 *Suppose* $r \in \mathcal{R}_n$ *and all the zeros of* r *lie in* $T_1 \cup D_1^+$. *Then for* $z \in T_1$ *and* $R \geq 1$,

$$\parallel r(R\ .) \parallel \leq \frac{|c_n|R + |c_0|}{(R+1)(|c_n|+|c_0|)}(|B(Rz)|+1) \parallel r \parallel .$$

Equality holds for $r(z) = B(z) + \lambda$, *where* $\lambda \in T_1$ *is choosen suitably.*

Since all the zeros of $p(z) := \sum_{j=0}^{n} c_j z^j$ lie in $T_1 \cup D_1^+$, therefore $|c_0| \geq |c_n|$ and it can be easily verified that for $R \geq 1$

$$\frac{|c_n|R + |c_0|}{(R+1)(|c_n|+|c_0|)} \leq \frac{1}{2}.$$

This shows that Corollary (1.2) is a refinement of a result due to Govil and Mohapatra [5] (see also Aziz and Rather [2, Theorem 2]).

Remark 1.3 *Taking* $a_j = \alpha > 1,\ j = 1, 2, \ldots, n$, *so that* $r \in \mathcal{R}_n$ *has a pole of order* n *at* α, *we have from* (1.10) *for* $z \in T_1 \cup D_1^+$,

$$\left|\frac{p(z)}{(z-\alpha)^n}\right| \leq \frac{|c_n||z| + |c_0|}{(1+|z|)(|c_n|+|c_0|)}\left\{\left|\left(\frac{1-\alpha z}{z-\alpha}\right)^n\right| + 1\right\} \sup_{z \in T_1}\left|\frac{p(z)}{(z-\alpha)^n}\right|. \tag{1.11}$$

If $\sup\left|\frac{p(z)}{(z-\alpha)^n}\right|$ *on* T_1 *is attained at* $\zeta \in T_1$, *then*

$$\sup_{z \in T_1}\left|\frac{p(z)}{(z-\alpha)^n}\right| = \left|\frac{p(\zeta)}{(\zeta-\alpha)^n}\right| \leq \frac{\parallel p \parallel}{|(\zeta-\alpha)^n|}.$$

Therefore for $z \in T_1 \cup D_1^+$, *we get from* (1.11) *for every* $\alpha > 1$

$$|p(z)| \leq \frac{|c_n||z| + |c_0|}{(1+|z|)(|c_n|+|c_0|)}\left\{\left|\left(\frac{1-\alpha z}{\zeta-\alpha}\right)^n\right| + \left|\left(\frac{z-\alpha}{\zeta-\alpha}\right)^n\right|\right\} \parallel p \parallel .$$

Letting $|\alpha| \to \infty$, *we get*

$$|p(z)| \leq \frac{|c_n||z| + |c_0|}{(1+|z|)(|c_n|+|c_0|)}(|z^n|+1) \parallel p \parallel \qquad \text{for} \quad z \in T_1 \cup D_1^+. \tag{1.12}$$

In particular, if we take $z = Re^{i\theta},\ 0 \le \theta < 2\pi\ , R \ge 1,$ *so that* $z \in T_1 \cup D_1^+$*, then from* (1.12) *we get*

$$\| p(R\ .) \| \le \frac{|c_n|R + |c_0|}{(1+R)(|c_n| + |c_0|)}(R^n + 1) \| p \| \,.$$

This is the same as Theorem A, a result earlier proved by Dubinin [4].

2. Lemmas and proof of the theorem

For the proof of Theorem 1.1 we need the following lemmas. The first lemma is due to Govil et al. [6].

Lemma 2.1 *If* f *is analytic in* $T_1 \cup D_1^-, f(0) = a,\ f'(0) = b,\ |f(z)| < 1$ *for* $z \in D_1^-$*, then*

$$|f(z)| \le \frac{(1-|a|)|z|^2 + |b||z| + |a|(1-|a|)}{|a|(1-|a|)|z|^2 + |b||z| + (1-|a|)}. \tag{2.1}$$

From Lemma (2.1), one can easily deduce the following generalised form of Schwarz lemma (for reference see also [11]).

Lemma 2.2 *If* f *is analytic in* $T_1 \cup D_1^-,\ f(0) = 0$ *and* $|f(z)| \le 1$ *for* $z \in T_1 \cup D_1^-$*, then*

$$|f(z)| \le |z| \frac{|z| + |f'(0)|}{1 + |f'(0)||z|} \qquad for \quad z \in D_1^-. \tag{2.2}$$

Proof of Theorem 1.1. We have $|r(z)| < |\beta| \ \| r \|$ for every β with $|\beta| > 1$ and $z \in T_1$. Hence as an application of Rouche's theorem $F(z) = r(z) + \beta \| r \|$ does not vanish in $T_1 \cup D_1^-$. If $F^*(z) = B(z)\overline{F\left(\frac{1}{\overline{z}}\right)}$, then we have $|F^*(z)| = |F(z)|$ for $z \in T_1$. Further it can be easily verified that

$$F^*(z) = r^*(z) + \overline{\beta} B(z) \| r \|, \tag{2.3}$$

where

$$\begin{aligned} r^*(z) &= B(z)\overline{r\left(\frac{1}{\overline{z}}\right)} \\ &= \prod_{j=1}^{n}\left(\frac{1-\overline{a_j}z}{z-a_j}\right)\frac{\overline{p\left(\frac{1}{\overline{z}}\right)}}{\overline{w\left(\frac{1}{\overline{z}}\right)}} \\ &= \prod_{j=1}^{n}\left(\frac{1-\overline{a_j}z}{z-a_j}\right)\frac{z^n\overline{p\left(\frac{1}{\overline{z}}\right)}}{\prod_{j=1}^{n}(1-\overline{a_j}z)} \\ &= \frac{p^*(z)}{w(z)}. \end{aligned} \tag{2.4}$$

Using (2.4) and the fact that $B(z)w(z) = \prod_{j=1}^{n}(1-\overline{a_j}z) = w^*(z),$ we get from (2.3)

$$\begin{aligned} F^*(z) &= \frac{p^*(z)}{w(z)} + \overline{\beta} B(z) \| r \| \\ &= \frac{p^*(z) + \overline{\beta} w^*(z) \| r \|}{w(z)}. \end{aligned} \tag{2.5}$$

Since the zeros of $w(z)$ lie in D_1^+, therefore F^* is analytic in $T_1 \cup D_1^-$ and also $F \neq 0$ in $T_1 \cup D_1^-$. Hence in particular $\frac{F^*}{F}$ is analytic in $T_1 \cup D_1^-$ and

$$\left|\frac{F^*(z)}{F(z)}\right| = |B(z)| = 1, \qquad z \in T_1.$$

By the maximum modulus principle, it follows that

$$|F^*(z)| \leq |F(z)| \qquad \text{for } z \in T_1 \cup D_1^-.$$

Replacing z by $\frac{1}{\bar{z}}$, this gives

$$|F(z)| \leq |F^*(z)| \qquad \text{for } z \in T_1 \cup D_1^+.$$

That is, for $z \in T_1 \cup D_1^+$ we have

$$|r(z) + \beta \parallel r \parallel| \leq |r^*(z) + \overline{\beta} B(z) \parallel r \parallel|. \tag{2.6}$$

Since, $r^* \in \mathcal{R}_n$, we can choose by inequality (1.7) the argument of β such that

$$|r^*(z) + \overline{\beta} B(z) \parallel r \parallel| = |\beta||B(z)| \parallel r \parallel - |r^*(z)|. \tag{2.7}$$

Using (2.7) in (2.6), we get for $z \in T_1 \cup D_1^+$

$$|r(z)| - |\beta| \parallel r \parallel \leq |\beta||B(z)| \parallel r \parallel - |r^*(z)|.$$

This implies

$$|r(z)| + |r^*(z)| \leq |\beta|(|B(z)| + 1) \parallel r \parallel.$$

Letting $|\beta| \to 1$, we get

$$|r(z)| + |r^*(z)| \leq (|B(z)| + 1) \parallel r \parallel. \tag{2.8}$$

Here we note that this result was also proved by Aziz and Rather [2, Theorem 1]. Again $r(z) = \frac{p(z)}{w(z)}$, where $p(z) := \sum\limits_{j=0}^{n} c_j z^j = c_n \prod\limits_{k=1}^{n}(z - \alpha_k)$, $\alpha_k \in T_1 \cup D_1^+$, therefore all the zeros of $p^*(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)} = \overline{c_n} \prod\limits_{k=1}^{n}(1 - \overline{a_k} z)$ are in $T_1 \cup D_1^-$. If

$$G(z) = \frac{zp^*(z)}{p(z)} = z\frac{\overline{c_n}}{c_n} \prod_{k=1}^{n} \left(\frac{1 - \overline{\alpha_k} z}{z - \alpha_k}\right),$$

then $G(z)$ is regular in $T_1 \cup D_1^-$, $G(0) = 0$ and $|G(z)| \leq 1$ for $z \in T_1 \cup D_1^-$. Hence by Lemma (2.2) we conclude that

$$|G(z)| \leq |z| \frac{|z| + |G'(0)|}{1 + |G'(0)||z|} \qquad for\ z \in D_1^-. \tag{2.9}$$

Using the fact that

$$|G'(0)| = \frac{1}{|\prod\limits_{k=1}^{n} \alpha_k|} = \left|\frac{c_n}{c_0}\right|$$

and substituting for G(z), we get from (2.9)

$$\left|\frac{p^*(z)}{p(z)}\right| \le \frac{|c_0||z|+|c_n|}{|c_0|+|c_n||z|} \qquad z \in T_1 \cup D_1^-.$$

This in particular gives,

$$\left|\frac{p(z)}{p^*(z)}\right| \le \frac{|c_n||z|+|c_0|}{|c_n|+|c_0||z|} \qquad z \in T_1 \cup D_1^+.$$

Equivalently for $z \in T_1 \cup D_1^+$, we have

$$\left|\frac{p^*(z)}{p(z)}\right| \ge \frac{|c_0||z|+|c_n|}{|c_0|+|c_n||z|}. \tag{2.10}$$

Since $r(z) = \frac{p(z)}{w(z)}$ and $r^*(z) = \frac{p^*(z)}{w(z)}$, therefore for $z \in T_1 \cup D_1^+$, we have from (2.10),

$$\left|\frac{r^*(z)}{r(z)}\right| = \left|\frac{p^*(z)}{p(z)}\right| \ge \frac{|c_0||z|+|c_n|}{|c_0|+|c_n||z|}.$$

This implies

$$\frac{|c_0||z|+|c_n|}{|c_0|+|c_n||z|}|r(z)| \le |r^*(z)|, \qquad z \in T_1 \cup D_1^+. \tag{2.11}$$

Adding $|r(z)|$ on both sides of (2.11), we get

$$\left\{\frac{|c_0||z|+|c_n|}{|c_0|+|c_n||z|}+1\right\}|r(z)| \le |r(z)|+|r^*(z)|. \tag{2.12}$$

Using (2.8) in (2.12) it follows that for $z \in T_1 \cup D_1^+$,

$$|r(z)| \le \frac{|c_0|+|c_n||z|}{(1+|z|)(|c_0|+|c_n|)}(|B(z)|+1) \parallel r \parallel .$$

This completes the proof of Theorem 1.1.

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