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Traces and inverse nodal problems for a class of delay Sturm–Liouville operators

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Abstract: In this study, we investigate the regularized sums of eigenvalues, oscillation of eigenfunctions and solutions of inverse nodal problems of discontinuous Sturm–Liouville operators with a delayed argument and with a finite number of transmission conditions. With this aim, we obtain asymptotic formulas for eigenvalues, eigenfunctions and nodal points of the problem. Moreover, some numerical examples are given to illustrate the results. The problem differs from the other discontinuous Sturm–Liouville problems with retarded argument in that it contains a spectral parameter in boundary conditions. If we take the delayed argument $\Delta \equiv 0$, the coefficients $\alpha_i^+ = \beta_i^+ = 0$ $(i = 1, 2)$ in boundary conditions and the transmission coefficients $\delta_i = 1$ $(i = \overline{1, m-1})$ the results obtained below coincide with corresponding results in the classical Sturm–-Liouville operator.

Key words: Differential equation with delayed argument, transmission conditions, regularized trace, nodal points, inverse problem

1. Introduction

Sturm–Liouville problems with transmission conditions (also known as interface conditons, discontinuity conditions, impulse effects) arise in many applications. Amongst the applications are thermal conduction in a thin laminated plate made up of layers of different materials and diffraction problems. The main goal of this paper is to extent and generalize some approaches and results of this kind of boundary value problems to similar types of problems but with spectral parameter dependent boundary conditions. With this aim, we calculate regularized traces and solve inverse nodal problems of a class of Sturm–Liouville operators with delayed argument and with a finite number of transmission conditions.

We consider the following boundary value problem which consists of the differential equation

$$y''(t) + q(t)y(t - \Delta(t)) + \mu^2 y(t) = 0 \tag{1.1}$$

on $\Omega = \cup_{j=0}^{m} \Omega_j$ $\left(\Omega_0 = [0, \theta_1),\ \Omega_i = (\theta_i, \theta_{i+1})\ \left(i = \overline{1, m-1}\right),\ \Omega_m = (\theta_m, \pi]\right)$, spectral parameter dependent boundary conditions

$$\alpha_1^- y(0) - \alpha_2^- y'(0) + \mu\left(\alpha_1^+ y(0) - \alpha_2^+ y'(0)\right) = 0, \tag{1.2}$$

$$\beta_1^- y(\pi) - \beta_2^- y'(\pi) + \mu\left(\beta_1^+ y(\pi) - \beta_2^+ y'(\pi)\right) = 0 \tag{1.3}$$

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and transmission conditions

$$y\left(\theta_i-\right)-\delta_i y(\theta_i+)=0, \tag{1.4}$$

$$y'\left(\theta_i-\right)-\delta_i y'(\theta_i+)=0, \tag{1.5}$$

where the real-valued function $q\left(t\right)$ is continuous in Ω and has finite limits

$$q\left(\theta_i\pm\right)=\lim_{t\to\theta_i\pm} q\left(t\right),$$

the real-valued function $\Delta(t)\geq 0$ is continuous in Ω and has finite limits

$$\Delta\left(\theta_i\pm\right)=\lim_{t\to\theta_i\pm}\Delta\left(t\right),$$

if $t\in\Omega_1$ then $t-\Delta(t)\geq 0$; *if* $t\in\Omega_i$ then $t-\Delta(t)\geq\theta_i$ $(i=\overline{2,m})$; μ is a spectral parameter; $\alpha_j^{\pm},\beta_j^{\pm}$ $(j=1,2)$, $\delta_i\neq 0$, θ_i $(i=\overline{1,m})$ are arbitrary real numbers such that $\theta_0=0<\theta_1<\theta_2<...\theta_m<\theta_{m+1}=\pi$ and $\alpha_2^{+}\beta_2^{+}\neq 0$.

In physcial systems, the presence of a time lag is observed experimentally. Therefore, various applied problems formulated with the use of delayed arguments. Such a consideration is an improvement compared with the model of an "ideal" process which is obtained if it is assumed that there are no delays at all, that the "functioning takes place instantly (The case $\Delta\equiv 0$). The delayed argument Δ reflects rather well the real process in a number of cases, for example, when the delayed argument is connected with the transmission of an audio signal, in hydraulic shock or in other wave processes. In other cases, such an assumption describes a process approximately and more roughly. Problems with feedback controls such as the steady states of a thermostat, where a controller at one of its ends adds or removes heat, depending upon the temperature registered in another point, can be interpreted with a second-order ordinary differential equation with spectral parameter dependent boundary conditions. However, the delayed argument in the differential equation (1.1) allows us to take into consideration the delays (retardations) in time for the abovementioned applied problems.

We want to also note that some eigenvalue problems encountered in areas of data mining requires the investigation of traces of operators and matrices optimizing the certain properties of given input high-dimensional data (see [12, 14]).

The paper is organized as follows. In Section 2 notation and definitions used in the paper are stated. In the same section, we also write the general solution of the (1.1) corresponding to the initial function $\varphi(t,\mu)$ in terms of fundamental solutions of the initial value problem (1.1), (2.1)–(2.2), investigate the spectrum and find a formula for the regularized sums of eigenvalues. In Section 3 we obtain asymptotic formulas of nodal points for the boundary value problem (1.1)–(1.5) and construct the potential function using these formulas. Later on, in the same section, we give two numerical examples to illustrate and verify the results.

2. Regularized trace

Let $\varphi_1(t,\mu)$ be a solution of (1.1) on $[0,\theta_1]$, satisfying the initial conditions

$$\varphi_1\left(0,\mu\right)=\mu\alpha_2^{+}+\alpha_2^{-}\ \textit{and}\ \varphi_1'\left(0,\mu\right)=\mu\alpha_1^{+}+\alpha_1^{-}. \tag{2.1}$$

The conditions (2.1) define a unique solution of Eq. (1.1) on $\Omega_1\cup\theta_1$ (see [5, 15]).

After defining the above solution, then we will define the solution $\varphi_i(t,\mu)$ of (1.1) on $\Omega_i \cup \{\theta_i, \theta_{i+1}\}$ $\left(i=\overline{2,m}\right)$ by means of the solution $\varphi_1(t,\mu)$ using the initial conditions

$$\varphi_{i+1}(\theta_i,\mu) = \frac{\varphi_i(\theta_i,\mu)}{\delta_i} \text{ and } \varphi'_{i+1}(\theta_i,\mu) = \frac{\varphi'_i(\theta_i,\mu)}{\delta_i}. \tag{2.2}$$

The conditions (2.2) define a unique solution of (1.1) on $\Omega_i \cup \{\theta_i, \theta_{i+1}\}$ $\left(i=\overline{2,m}\right)$ (see [1–3, 6, 13, 21]).

Let us define the function $\varphi(t,\mu)$ by the equality

$$\varphi(t,\mu) = \varphi_{i+1}(t,\mu),\ t \in \Omega_i\ \left(i=\overline{0,m}\right).$$

Consequently, $\varphi(t,\mu)$ is a solution of (1.1) on Ω, which satisfies one of the boundary conditions and the transmission conditions (1.4)–(1.5). Then the following integral equations hold:

$$\varphi_1(t,\mu) = \left(\mu\alpha_2^+ + \alpha_2^-\right)\cos(\mu t) - \frac{\mu\alpha_1^+ + \alpha_1^-}{\mu}\sin(\mu t) - \frac{1}{\mu}\int_0^t q(\tau)\sin(\mu(t-\tau))\varphi_1(\tau-\Delta(\tau),\mu)\,d\tau, \tag{2.3}$$

$$\begin{aligned}\varphi_{i+1}(t,\mu) &= \frac{1}{\delta_i}\varphi_i(\theta_i,\mu)\cos(\mu(t-\theta_i)) + \frac{\varphi'_i(\theta_i,\mu)}{\mu\delta_i}\sin(\mu(t-\theta_i)) \\ &- \frac{1}{\mu}\int_{\theta_i}^t q(\tau)\sin(\mu(t-\tau))\varphi_{i+1}(\tau-\Delta(\tau),\mu)\,d\tau,\ \left(i=\overline{1,m}\right).\end{aligned} \tag{2.4}$$

If we solve the equations (2.3)–(2.4) by the method of successive approximation then we obtain the following asymptotic equalities for $|\mu| \to \infty$:

$$\begin{aligned}\varphi_1(t,\mu) &= \mu\alpha_2^+\cos(\mu t) + \alpha_2^-\cos(\mu t) - \alpha_1^+\sin(\mu t) - \frac{\alpha_2^+}{2}\int_0^t q(\tau)\sin(\mu(t-\Delta(\tau)))d\tau \\ &- \frac{\alpha_2^+}{2}\int_0^t q(\tau)\sin(\mu(t-2\tau+\Delta(\tau)))d\tau + \frac{1}{\mu}\Bigg(\alpha_1^-\sin(\mu t) - \frac{\alpha_2^-}{2}\int_0^t q(\tau)\sin(\mu(t-\Delta(\tau)))d\tau. \\ &- \frac{\alpha_2^-}{2}\int_0^t q(\tau)\sin(\mu(t-2\tau+\Delta(\tau)))d\tau + \frac{\alpha_1^+}{2}\int_0^t q(\tau)\cos(\mu(t-\Delta(\tau)))d\tau \\ &- \frac{\alpha_1^+}{2}\int_0^t q(\tau)\cos(\mu(t-2\tau+\Delta(\tau)))d\tau\Bigg) + \mathcal{O}\left(\frac{1}{\mu^2}\right),\end{aligned} \tag{2.5}$$

$$\varphi_{i+1}(t,\mu) = \frac{1}{\prod_{i=1}^m \delta_i}\left(\mu\alpha_2^+\cos(\mu t) + (\alpha_2^-\cos(\mu t) - \alpha_1^+\sin(\mu t)\right).$$

$$-\frac{\alpha_2^+}{2}\int_0^t q(\tau)\sin(\mu\,(t-\Delta\,(\tau)))d\tau-\frac{\alpha_2^+}{2}\int_0^t q(\tau)\sin(\mu\,(t-2\tau+\Delta\,(\tau)))d\tau\Bigg)+\mathcal{O}\left(\frac{1}{\mu}\right). \tag{2.6}$$

Differentiating (2.5)–(2.6) with respect to t, we get

$$\varphi_1'(t,\mu)=-\mu^2\alpha_2^+\sin(\mu t)+\mu\Bigg(-\alpha_2^-\sin(\mu t)-\alpha_1^+\cos(\mu t)-\frac{\alpha_2^+}{2}\int_0^t q(\tau)\cos(\mu\,(t-\Delta\,(\tau)))d\tau$$

$$-\frac{\alpha_2^+}{2}\int_0^t q(\tau)\cos(\mu\,(t-2\tau+\Delta\,(\tau)))d\tau\Bigg)+\alpha_1^-\cos(\mu t)$$

$$-\frac{\alpha_2^-}{2}\int_0^t q(\tau)\cos(\mu\,(t-\Delta\,(\tau)))d\tau-\frac{\alpha_2^-}{2}\int_0^t q(\tau)\cos(\mu\,(t-2\tau+\Delta\,(\tau)))d\tau$$

$$-\frac{\alpha_1^+}{2}\int_0^t q(\tau)\sin(\mu\,(t-\Delta\,(\tau)))d\tau+\frac{\alpha_1^+}{2}\int_0^t q(\tau)\sin(\mu\,(t-2\tau+\Delta\,(\tau)))d\tau+\mathcal{O}\left(\frac{1}{\mu}\right), \tag{2.7}$$

$$\varphi_{i+1}'(t,\mu)=-\frac{1}{\prod\limits_{i=1}^{m}\delta_i}\Bigg[\alpha_2^+\mu^2\sin(\mu t)+\mu\Bigg(-\alpha_2^-\sin(\mu t)-\alpha_1^+\cos(\mu t)-\frac{\alpha_2^+}{2}\int_0^t q(\tau)\cos(\mu(t-\Delta(\tau)))d\tau$$

$$-\frac{\alpha_2^+}{2}\int_0^t q(\tau)\cos(\mu(t-2\tau+\Delta(\tau)))d\tau\Bigg)\Bigg]+\mathcal{O}(1),\ i=\overline{1,m}. \tag{2.8}$$

The solution $\varphi(t,\mu)$ defined above is a nontrivial solution of (1.1) satisfying conditions (1.2) and (1.4)–(1.5). Putting $\varphi(t,\mu)$ into (1.3), we get the characteristic equation

$$\Xi(\mu)\equiv\left(\mu\beta_1^++\beta_1^-\right)\varphi(\pi,\mu)-\left(\mu\beta_2^++\beta_2^-\right)\varphi'(\pi,\mu)=0. \tag{2.9}$$

The set of eigenvalues of boundary value problem (1.1)–(1.5) coincides with the set of the squares of roots of (2.9) and eigenvalues are simple (see [15], Theorem 2.1.1]). From (2.5)–(2.9), we obtain

$$\Xi(\mu)\equiv\frac{\mu^3\alpha_2^+\beta_2^+}{\prod\limits_{i=1}^{m}\delta_i}\sin(\mu\pi)+\frac{\mu^2}{\prod\limits_{i=1}^{m}\delta_i}\Big[\Big(\alpha_2^+\beta_1^++\alpha_1^+\beta_2^+$$

$$+\frac{\alpha_2^+\beta_2^+}{2}\int_0^\pi q(\tau)\cos(\mu\Delta(\tau))d\tau+\frac{\alpha_2^+\beta_2^+}{2}\int_0^\pi q(\tau)\cos(\mu\,(2\tau-\Delta(\tau)))d\tau\Big)\cos(\mu\pi)$$

$$+\Bigg(\alpha_2^-\beta_2^++\alpha_2^+\beta_2^-+\frac{\alpha_2^+\beta_2^+}{2}\int_0^\pi q(\tau)\sin(\mu\Delta(\tau))d\tau+\frac{\alpha_2^+\beta_2^+}{2}\int_0^\pi q(\tau)\sin(\mu\,(2\tau-\Delta(\tau)))d\tau\Bigg)\sin(\mu\pi)\Bigg]+\mathcal{O}(\mu)$$

which is deduced to

$$\Xi(\mu)=\frac{\mu^3\alpha_2^+\beta_2^+}{\prod\limits_{i=1}^{m}\delta_i}\sin(\mu\pi)+\frac{\mu^2}{\prod\limits_{i=1}^{m}\delta_i}\Big[\big(\alpha_2^+\beta_1^++\alpha_1^+\beta_2^+$$

$$+\alpha_2^+\beta_2^+\left(U^+(\mu)+V^+(\mu)\right)\big)\cos(\mu\pi)+\big(\alpha_2^-\beta_2^++\alpha_2^+\beta_2^-+\alpha_2^+\beta_2^+\left(U^-(\mu)+V^-(\mu)\right)\big)\sin(\mu\pi)\Big]+\mathcal{O}(\mu). \quad (2.10)$$

Here,

$$U^+(\mu)=\frac{1}{2}\int_0^\pi q(\tau)\cos(\mu\Delta(\tau))\,d\tau,\qquad U^-(\mu)=\frac{1}{2}\int_0^\pi q(\tau)\sin(\mu\Delta(\tau))\,d\tau,$$

$$V^+(\mu)=\frac{1}{2}\int_0^\pi q(\tau)\cos(\mu(2\tau-\Delta(\tau)))\,d\tau,\qquad V^-(\mu)=\frac{1}{2}\int_0^\pi q(\tau)\sin(\mu(2\tau-\Delta(\tau)))\,d\tau.$$

Define

$$\Xi_0(\mu)\equiv\frac{\mu^3\alpha_2^+\beta_2^+}{\prod\limits_{i=1}^{m}\delta_i}\sin(\mu\pi), \quad (2.11)$$

and denote by $\mu_{\pm n}^0, n\in\mathbb{Z}$, the zeros of the function $\Xi_0(\mu)$, except that zero is multiplicity 4; then $\mu_{\pm 0}^0=\mu_{\pm 1}^0=0$ and

$$\mu_n^0=\begin{cases} n-1, & n\geq 1,\\ n+1, & n\leq -1.\end{cases}$$

Denote by C_n the circle of radius, $0<\varepsilon<\frac{1}{2}$, centered at the origin μ_n^0 and by Γ_{N_0} the counterclockwise square contours with four vertices

$$\begin{aligned} A &= (N_0-1+\varepsilon)(1-i), \quad B=(N_0-1+\varepsilon)(1+i),\\ C &= (N_0-1+\varepsilon)(-1+i), \quad D=(N_0-1+\varepsilon)(-1-i),\end{aligned}$$

where $i=\sqrt{-1}$ and N_0 is a natural number. Obviously, if $\mu\in C_n$ or $\mu\in\Gamma_{N_0}$, then $|\Xi_0(\mu)|\geq M|\mu|\,e^{|\Im\mu|\pi}$ $(M>0)$ by using a similar method in [26]. Thus, on $\mu\in C_n$ or $\mu\in\Gamma_{N_0}$, from (2.10) and (2.11), we have

$$\frac{\Xi(\mu)}{\Xi_0(\mu)}=1+\frac{1}{\mu}\left[\left(\frac{\beta_1^+}{\beta_2^+}+\frac{\alpha_1^+}{\alpha_2^+}+U^+(\mu)+V^+(\mu)\right)\cot(\mu\pi)+\frac{\alpha_2^-}{\alpha_2^+}+\frac{\beta_2^-}{\beta_2^+}+U^-(\mu)+V^-(\mu)\right]+\mathcal{O}\left(\frac{1}{\mu^2}\right).$$

Expanding $\ln\frac{\Xi(\mu)}{\Xi_0(\mu)}$ by the Maclaurin formula, we find that

$$\ln\frac{\Xi(\mu)}{\Xi_0(\mu)}=\frac{1}{\mu}\left[\left(\frac{\beta_1^+}{\beta_2^+}+\frac{\alpha_1^+}{\alpha_2^+}+U^+(\mu)+V^+(\mu)\right)\cot(\mu\pi)+\frac{\alpha_2^-}{\alpha_2^+}+\frac{\beta_2^-}{\beta_2^+}+U^-(\mu)+V^-(\mu)\right]$$

$$-\frac{1}{2\mu^2}\left[\left(\frac{\beta_1^+}{\beta_2^+}+\frac{\alpha_1^+}{\alpha_2^+}+U^+(\mu)+V^+(\mu)\right)^2\cot^2(\mu\pi)+\left(\frac{\alpha_2^-}{\alpha_2^+}+\frac{\beta_2^-}{\beta_2^+}+U^-(\mu)+V^-(\mu)\right)^2\right.$$

$$+2\left(\frac{\beta_1^+}{\beta_2^+}+\frac{\alpha_1^+}{\alpha_2^+}+U^+(\mu)+V^+(\mu)\right)\left(\frac{\alpha_2^-}{\alpha_2^+}+\frac{\beta_2^-}{\beta_2^+}+U^-(\mu)+V^-(\mu)\right)\cot(\mu\pi)\bigg]+\mathcal{O}\left(\frac{1}{\mu^3}\right).$$

Using the well-known Rouche Theorem, we get that $\Xi(\mu)$ has the same number of zeros inside Γ_{N_0} as $\Xi_0(\mu)$ (see [24]). It is easy to prove that the spectrum of problem (1.1)–(1.5) is

$$\mu_n \sim \mu_n^0+\mathcal{O}\left(\frac{1}{n}\right) \text{ as } |n|\to\infty.$$

Next, we present the more exact asymptotic distribution of the spectrum. Using the residue theorem, we have

$$\mu_n-\mu_n^0=-\frac{1}{2\pi i}\oint_{C_n}\ln\frac{\Xi(\mu)}{\Xi_0(\mu)}d\mu$$

$$=-\frac{1}{2\pi i}\oint_{C_n}\left(\frac{\beta_1^+}{\beta_2^+}+\frac{\alpha_1^+}{\alpha_2^+}+U^+(\mu)+V^+(\mu)\right)\frac{\cot(\mu\pi)}{\mu}d\mu$$

$$-\frac{1}{2\pi i}\oint_{C_n}\left(\frac{\alpha_2^-}{\alpha_2^+}+\frac{\beta_2^-}{\beta_2^+}+U^-(\mu)+V^-(\mu)\right)\frac{1}{\mu}d\mu+\mathcal{O}\left(\frac{1}{n^2}\right)$$

$$=-\frac{1}{\mu_n^0\pi}\left(\frac{\beta_1^+}{\beta_2^+}+\frac{\alpha_1^+}{\alpha_2^+}+U^+(n)+V^+(n)\right)+\mathcal{O}\left(\frac{1}{n^2}\right).$$

Thus we have proven the following theorem.

Theorem 2.1 *The spectrum of the problem (1.1)–(1.5) has the following asymptotic distribution for sufficiently large* $|n|$:

$$\mu_n=\mu_n^0-\frac{1}{\mu_n^0\pi}\left(\frac{\beta_1^+}{\beta_2^+}+\frac{\alpha_1^+}{\alpha_2^+}+U^+\left(\mu_n^0\right)+V^+\left(\mu_n^0\right)\right)+\mathcal{O}\left(\frac{1}{\left(\mu_n^0\right)^2}\right). \tag{2.12}$$

Now, following [16, 24], we will obtain a regularized trace formula for the problem (1.1)–(1.5).

The asymptotic formula (2.12) for the eigenvalues implies that for all sufficiently large N_0, the numbers μ_n with $|n|\le N_0$ are inside Γ_{N_0} and the numbers μ_n with $|n|>N_0$ are outside Γ_{N_0}. It follows that

$$\sum_{\Gamma_n}\left(\mu_n^2-\left(\mu_n^0\right)^2\right)=\mu_{-0}^2+\mu_0^2+\sum_{0\neq n=-N_0}^{N_0}\left(\mu_n^2-\left(\mu_n^0\right)^2\right)=-\frac{1}{\pi i}\oint_{\Gamma_n}\mu\ln\frac{\Xi(\mu)}{\Xi_0(\mu)}d\mu$$

$$=-\frac{1}{\pi i}\oint_{\Gamma_n}\left(\frac{\beta_1^+}{\beta_2^+}+\frac{\alpha_1^+}{\alpha_2^+}+U^+(\mu)+V^+(\mu)\right)\cot(\mu\pi)\,d\mu-\frac{1}{\pi i}\oint_{\Gamma_n}\left(\frac{\alpha_2^-}{\alpha_2^+}+\frac{\beta_2^-}{\beta_2^+}+U^-(\mu)+V^-(\mu)\right)d\mu$$

$$+\frac{1}{2\pi i}\oint_{\Gamma_n}\left(\frac{\beta_1^+}{\beta_2^+}+\frac{\alpha_1^+}{\alpha_2^+}+U^+(\mu)+V^+(\mu)\right)^2\frac{\cot^2(\mu\pi)}{\mu}d\mu+\frac{1}{2\pi i}\oint_{\Gamma_n}\left(\frac{\alpha_2^-}{\alpha_2^+}+\frac{\beta_2^-}{\beta_2^+}+U^-(\mu)+V^-(\mu)\right)^2\frac{1}{\mu}d\mu$$

$$+\frac{1}{\pi i}\oint_{\Gamma_n}\left(\frac{\beta_1^+}{\beta_2^+}+\frac{\alpha_1^+}{\alpha_2^+}+U^+(\mu)+V^+(\mu)\right)\left(\frac{\alpha_2^-}{\alpha_2^+}+\frac{\beta_2^-}{\beta_2^+}+U^-(\mu)+V^-(\mu)\right)\frac{\cot(\mu\pi)}{\mu}d\mu+\mathcal{O}\left(\frac{1}{N_0}\right),$$

by calculations, which implies that

$$\mu_{-0}^2+\mu_0^2+\sum_{0\neq n=-N_0}^{N_0}\left(\mu_n^2-\left(\mu_n^0\right)^2+\frac{4}{\pi}\left(\frac{\beta_1^+}{\beta_2^+}+\frac{\alpha_1^+}{\alpha_2^+}+U^+(n)+V^+(n)\right)\right)$$

$$=-\frac{2}{\pi}\left(\frac{\beta_1^+}{\beta_2^+}-\frac{\alpha_1^+}{\alpha_2^+}+U^+(0)+V^+(0)\right)-\left(\frac{\beta_1^+}{\beta_2^+}+\frac{\alpha_1^+}{\alpha_2^+}+U^+(0)+V^+(0)\right)^2$$

$$+\left(\frac{\alpha_2^-}{\alpha_2^+}+\frac{\beta_2^-}{\beta_2^+}+U^-(0)+V^-(0)\right)^2+\mathcal{O}\left(\frac{1}{N_0}\right). \tag{2.13}$$

Passing to the limit as $N_0\to\infty$ in (2.13), we have

$$\mu_{-0}^2+\mu_0^2+\sum_{0\neq n=-\infty}^{+\infty}\left(\mu_n^2-\left(\mu_n^0\right)^2+\frac{4}{\pi}\left(\frac{\beta_1^+}{\beta_2^+}+\frac{\alpha_1^+}{\alpha_2^+}+U^+\left(\mu_n^0\right)+V^+\left(\mu_n^0\right)\right)\right)$$

$$=-\frac{2}{\pi}\left(\frac{\beta_1^+}{\beta_2^+}+\frac{\alpha_1^+}{\alpha_2^+}+U^+(0)+V^+(0)\right)-\left(\frac{\beta_1^+}{\beta_2^+}+\frac{\alpha_1^+}{\alpha_2^+}+U^+(0)+V^+(0)\right)^2+\left(\frac{\alpha_2^-}{\alpha_2^+}+\frac{\beta_2^-}{\beta_2^+}\right)^2. \tag{2.14}$$

The series on the left side of (2.14) is called the regularized trace of the problem (1.1)–(1.5). Thus we state the following theorem.

Theorem 2.2 *For the regularized trace of the problem (1.1)–(1.5) we have the Gelfand–Levitan type formula (2.14).*

We want to note that the trace formulas for different types of boundary value problems with delayed argument obtained in [4, 11, 16, 19, 20, 24] and approximate calculation of the eigenvalues of the problem (1.1)–(1.5) can also be obtained via formula (2.14) (see [8, 9, 17]).

3. Inverse problem

Inverse nodal problems for differential operators with or without delayed argument were investigated by a number of authors (see [10, 16, 18–20, 22, 23, 25, 27] and the references therein). In this chapter, following [24], we deal with inverse spectral analysis of the problem (1.1)–(1.5) using the nodal points (zeros) of its eigenfunctions.

Let us rewrite the equation (2.3) as

$$\varphi_1(t,\mu)=\left(\mu\alpha_2^++\alpha_2^-\right)\cos(\mu t)-\frac{1}{\mu}\left(\mu\alpha_1^++\alpha_1^-\right)\sin(\mu t)$$

$$-\frac{\alpha_2^+}{2}\int_0^t q(\tau)\big(\sin(\mu\,(t-2\tau+\Delta(\tau)))+\sin(\mu\,(t-\Delta(\tau)))\big)d\tau$$

$$-\frac{\alpha_2^-}{2\mu}\int_0^t q(\tau)\big(\sin(\mu\,(t-2\tau+\Delta(\tau)))+\sin(\mu\,(t-\Delta(\tau)))\big)d\tau$$

$$-\frac{\alpha_1^+}{2\mu}\int_0^t q(\tau)\big(\cos(\mu\,(t-2\tau+\Delta(\tau)))-\cos(\mu\,(t-\Delta(\tau)))\big)d\tau+\mathcal{O}\left(\frac{1}{\mu^2}\right),$$

and using the fact that

$$\int_0^t q(\tau)\sin(\mu\,(2\tau-\Delta(\tau)))d\tau=\int_0^t q(\tau)\cos(\mu\,(2\tau-\Delta(\tau)))d\tau=\mathcal{O}\left(\frac{1}{\mu}\right),$$

(see [15], Lemma 2.3.3]) it yields that

$$\varphi_1(t,\mu)=\mu\alpha_2^+\cos(\mu t)+\alpha_2^-\cos(\mu t)-\alpha_1^+\sin(\mu t)-\frac{\alpha_1^-}{\mu}\sin(\mu t)$$

$$-\frac{\alpha_2^+\cos(\mu t)}{2}\int_0^t q(\tau)\sin(\mu\Delta(\tau))d\tau-\frac{\alpha_2^+\sin(\mu t)}{2}\int_0^t q(\tau)\cos(\mu\Delta(\tau))d\tau$$

$$-\frac{\alpha_2^-\cos(\mu t)}{2\mu}\int_0^t q(\tau)\sin(\mu\Delta(\tau))d\tau-\frac{\alpha_2^-\sin(\mu t)}{2\mu}\int_0^t q(\tau)\cos(\mu\Delta(\tau))d\tau$$

$$+\frac{\alpha_1^+\cos(\mu t)}{2\mu}\int_0^t q(\tau)\sin(\mu\Delta(\tau))d\tau+\frac{\alpha_1^+\sin(\mu t)}{2\mu}\int_0^t q(\tau)\cos(\mu\Delta(\tau))d\tau+\mathcal{O}\left(\frac{1}{\mu^2}\right),$$

$$=\mu_n\alpha_2^+\cos(\mu_n t)+\alpha_2^-\cos(\mu_n t)-\alpha_1^+\sin(\mu_n t)$$

$$-\frac{\alpha_1^-}{\mu_n}\sin(\mu_n t)-\frac{\alpha_2^+}{2}\int_0^t q(\tau)\sin(\mu_n\,(t-\Delta(\tau)))d\tau-\frac{\alpha_2^-}{2\mu_n}\int_0^t q(\tau)\sin(\mu_n\,(t-\Delta(\tau)))d\tau$$

$$+\frac{\alpha_1^+}{2\mu_n}\int_0^t q(\tau)\cos(\mu_n\,(t-\Delta(\tau)))d\tau+\mathcal{O}\left(\frac{1}{\mu^2}\right)$$

$$=\mu_n\alpha_2^+\cos(\mu_n t)+\alpha_2^-\cos(\mu_n t)-\alpha_1^+\sin(\mu_n t)-\frac{\alpha_1^-}{\mu_n}\sin(\mu_n t)$$

$$-\frac{\alpha_2^+\sin(\mu_n t)}{2}\int_0^t q(\tau)\cos\left(\mu_n\Delta(\tau)\right)d\tau-\frac{\alpha_2^+\cos(\mu_n t)}{2}\int_0^t q(\tau)\sin\left(\mu_n\Delta(\tau)\right)d\tau$$

$$-\frac{\alpha_2^- \sin(\mu_n t)}{2\mu_n}\int_0^t q(\tau)\cos\left(\mu_n \Delta(\tau)\right)d\tau - \frac{\alpha_2^- \cos(\mu_n t)}{2\mu_n}\int_0^t q(\tau)\sin\left(\mu_n \Delta(\tau)\right)d\tau$$

$$+\frac{\alpha_1^+ \sin(\mu_n t)}{2\mu_n}\int_0^t q(\tau)\sin\left(\mu_n \Delta(\tau)\right)d\tau + \frac{\alpha_1^+ \cos(\mu_n t)}{2\mu_n}\int_0^t q(\tau)\cos\left(\mu_n \Delta(\tau)\right)d\tau + \mathcal{O}\left(\frac{1}{\mu_n^2}\right).$$

Let us assume that t_n^j are the nodal points of the eigenfunction $\varphi(t,\mu_n)$. Taking $\sin(\mu_n t) \neq 0$ into account for sufficiently large n, we get

$$T(\mu_n, t)\cot(\mu_n t) = \alpha_1^+ + \frac{\alpha_1^-}{\mu_n} + \frac{\alpha_2^+}{2}\int_0^t q(\tau)\cos\left(\mu_n \Delta(\tau)\right)d\tau$$

$$+\frac{\alpha_2^-}{2\mu_n}\int_0^t q(\tau)\cos\left(\mu_n \Delta(\tau)\right)d\tau - \frac{\alpha_1^+}{2\mu_n}\int_0^t q(\tau)\sin\left(\mu_n \Delta(\tau)\right)d\tau + \mathcal{O}\left(\frac{1}{\mu_n^2}\right).$$

Here

$$T(\mu_n, t) = \mu_n \alpha_2^+ + \alpha_2^- - \frac{\alpha_2^+}{2}\int_0^t q(\tau)\sin\left(\mu_n \Delta(\tau)\right)d\tau$$

$$-\frac{\alpha_2^-}{2\mu_n}\int_0^t q(\tau)\sin\left(\mu_n \Delta(\tau)\right)d\tau + \frac{\alpha_1^+}{2\mu_n}\int_0^t q(\tau)\cos\left(\mu_n \Delta(\tau)\right)d\tau$$

and it follows easily that

$$\tan\left(\mu_n t + \frac{\pi}{2}\right) = \frac{\alpha_1^+}{T(\mu_n, t)} + \frac{\alpha_2^+}{2\mu_n T(\mu_n, t)}\int_0^t q(\tau)\cos\left(\mu_n \Delta(\tau)\right)d\tau + \mathcal{O}\left(\frac{1}{\mu_n^3}\right). \tag{3.1}$$

Thus, solving Equation (3.1), one obtains

$$t_n^j = \frac{\left(j-\frac{1}{2}\right)\pi}{\mu_n} + \frac{\alpha_1^+}{\mu_n T\left(\mu_n, t_n^j\right)} + \frac{\alpha_2^+}{2\mu_n T\left(\mu_n, t_n^j\right)}\int_0^{t_n^j} q(\tau)\cos\left(\mu_n \Delta(\tau)\right)d\tau + \mathcal{O}\left(\frac{1}{\mu_n^3}\right). \tag{3.2}$$

Note that

$$\mu_n^{-1} = \frac{1}{\mu_n^0} - \frac{\left(\frac{\beta_1^+}{\beta_2^+} + \frac{\alpha_1^+}{\alpha_2^+} + U^+(n)\right)}{\left(\mu_n^0\right)^3 \pi} + \mathcal{O}\left(\frac{1}{n^4}\right). \tag{3.3}$$

Substituting (3.3) into (3.2) we have

$$t_n^j = \left(j - \frac{1}{2}\right)\pi\left(\frac{1}{\mu_n^0} - \frac{\left(\frac{\beta_1^+}{\beta_2^+} + \frac{\alpha_1^+}{\alpha_2^+} + U^+\left(\mu_n^0\right)\right)}{\left(\mu_n^0\right)^3 \pi}\right)$$

$$+\frac{\alpha_1^+}{\mu_n^0 T_0(n)}+\frac{\alpha_2^+}{2\mu_n^0 T_0(n)}\int_0^{\frac{j\pi}{n}} q(\tau)\cos\left(\mu_n^0\Delta(\tau)\right)d\tau+\mathcal{O}\left(\frac{1}{\left(\mu_n^0\right)^3}\right), \qquad j=\overline{1,\left[\frac{n}{2}\right]}. \tag{3.4}$$

Here $T_0(n)=T\left(\mu_n^0,\frac{j\pi}{n}\right)$. Analogically, from (2.4), we get

$$\left(\prod_{i=1}^m \delta_i\right)\varphi_2(t,\mu_n)=\mu_n\alpha_2^+\cos(\mu_n t)+\alpha_2^-\cos(\mu_n t)-\alpha_1^+\sin(\mu_n t)$$

$$-\frac{\alpha_2^+}{2}\int_0^t q(\tau)\sin(\mu_n(t-\Delta(\tau)))d\tau+\mathcal{O}\left(\frac{1}{\mu_n}\right)$$

and

$$\left(\prod_{i=1}^m \delta_i\right)\varphi_2(t,\mu_n)=\mu_n\alpha_2^+\cos(\mu_n t)+\alpha_2^-\cos(\mu_n t)-\alpha_1^+\sin(\mu_n t)$$

$$+\frac{\alpha_2^+\cos(\mu_n t)}{2}\int_0^t q(\tau)\sin(\mu_n\Delta(\tau))d\tau-\frac{\alpha_2^+\sin(\mu_n t)}{2}\int_0^t q(\tau)\cos(\mu_n\Delta(\tau))d\tau+\mathcal{O}\left(\frac{1}{\mu_n}\right).$$

Thus, for nodal points of $\varphi_2(t,\mu_n)$, we have the following equality:

$$0=\alpha_2^+\cos(\mu_n t)+\frac{\alpha_2^-}{\mu_n}\cos(\mu_n t)-\frac{\alpha_1^+}{\mu_n}\sin(\mu_n t)$$

$$+\frac{\alpha_2^+\cos(\mu_n t)}{2\mu_n}\int_0^t q(\tau)\sin(\mu_n\Delta(\tau))d\tau-\frac{\alpha_2^+\sin(\mu_n t)}{2\mu_n}\int_0^t q(\tau)\cos(\mu_n\Delta(\tau))d\tau+\mathcal{O}\left(\frac{1}{\mu_n^2}\right).$$

Again, taking $\sin(\mu_n t)\neq 0$ into account for sufficiently large n, we have

$$0=\alpha_2^+\cot(\mu_n t)+\frac{\alpha_2^-}{\mu_n}\cot(\mu_n t)-\frac{\alpha_1^+}{\mu_n}$$

$$+\frac{\alpha_2^+\cot(\mu_n t)}{2\mu_n}\int_0^t q(\tau)\sin(\mu_n\Delta(\tau))d\tau-\frac{\alpha_2^+}{2\mu_n}\int_0^t q(\tau)\cos(\mu_n\Delta(\tau))d\tau+\mathcal{O}\left(\frac{1}{\mu_n^2}\right)$$

and

$$\tan\left(\mu_n t+\frac{\pi}{2}\right)=\frac{\alpha_1^+}{\alpha_2^+\mu_n}+\frac{1}{2\mu_n}\int_0^t q(\tau)\cos(\mu_n\Delta(\tau))d\tau+\mathcal{O}\left(\frac{1}{\mu_n^2}\right). \tag{3.5}$$

Thus, solving Equation (3.5), one obtains

$$t_n^j=\frac{\left(j-\frac{1}{2}\right)\pi}{\mu_n}+\frac{\alpha_1^+}{\alpha_2^+\mu_n^2}+\frac{1}{2\mu_n^2}\int_0^{t_n^j} q(\tau)\cos(\mu_n\Delta(\tau))d\tau+\mathcal{O}\left(\frac{1}{\mu_n^3}\right). \tag{3.6}$$

Note that

$$\mu_n^{-2} = \frac{1}{\left(\mu_n^0\right)^2} + \mathcal{O}\left(\frac{1}{n^4}\right). \tag{3.7}$$

Substituting (3.7) into (3.6) we have

$$t_n^j = \left(j - \frac{1}{2}\right)\pi \left(\frac{1}{\mu_n^0} - \frac{\left(\frac{\beta_1^+}{\beta_2^+} + \frac{\alpha_1^+}{\alpha_2^+} + U^+\left(\mu_n^0\right)\right)}{\left(\mu_n^0\right)^3 \pi}\right)$$

$$+\frac{1}{\left(\mu_n^0\right)^2}\left(\frac{\alpha_1^+}{\alpha_2^+} + \frac{1}{2}\int\limits_0^{\frac{j\pi}{n}} q(\tau)\cos\left(\mu_n^0 \Delta(\tau)\right) d\tau\right) + \mathcal{O}\left(\frac{1}{\left(\mu_n^0\right)^3}\right), \qquad j = \overline{\left[\frac{n}{2}\right] + 1, n}. \tag{3.8}$$

Thus we have proven the following theorem:

Theorem 3.1 *Let n be sufficiently large. Then we have the formulas (3.4) and (3.8) for the nodal points of the problem (1.1)–(1.5).*

We see that there exists an integer N_0 such that for all $n > N_0$ the eigenfunction $\varphi(t, \mu_n)$ of the problem has exactly n simple nodes in the interval $(0, \pi)$. The set $\Lambda = \left\{t_n^j\right\}$ is called the nodal set of the problem (1.1)–(1.5). We also define the function $j_n(t)$ to be the largest index j such that $0 \le t_n^j \le t$. Thus, $j = j_n(t)$ if and only if $t \in \left[t_n^j, t_n^{j+1}\right)$.

Theorem 3.2 *For each $t \in [0, \pi]$, let $\left\{t_n^j\right\} \subset \Lambda$ be chosen such that $\lim_{n\to\infty} t_n^j = t$. Then the following finite limit exists and corresponding equality holds:*

$$\lim_{n\to\infty} \left(\mu_n^0\right)^2 \left(t_n^j - \frac{\left(j - \frac{1}{2}\right)\pi}{\mu_n^0}\right) = f(t), \tag{3.9}$$

where

$$f(t) = \begin{cases} \frac{\left(\frac{\beta_1^+}{\beta_2^+} + \frac{\alpha_1^+}{\alpha_2^+} + U^+(0)\right)t}{\pi} - \frac{\alpha_1^+}{\alpha_2^+} - \frac{1}{2}\int\limits_0^t q(\tau)\, d\tau, & \Delta(\tau) = 0, \\ \frac{\left(\frac{\beta_1^+}{\beta_2^+} + \frac{\alpha_1^+}{\alpha_2^+}\right)t}{\pi} - \frac{\alpha_1^+}{\alpha_2^+}, & \Delta(\tau) \neq 0. \end{cases} \tag{3.10}$$

Proof Using the formulas (3.4) and (3.8) for nodal points and the fact that $\lim_{n\to\infty} t_n^j = t$, it follows that as $n \to \infty$ the limits of left-hand side in (3.9) exists and (3.10) holds. Thus, the proof is completed. □

Now, we can construct the potential function $q(t)$ via following theorem:

Theorem 3.3 *Let $\Lambda^0 = \left\{t_{n_k}^j\right\}$ and assume that $\Lambda^0 \subset \Lambda$ be a subset of nodal points which satisfy $\left\{t_{n_k}^j\right\}$ is dense in $(0, \pi)$. For each $t \in [0, \pi]$ choose a sequence $\left\{t_n^j\right\} \subset \Lambda^0$ such that $\lim_{n\to\infty} t_n^j = t$. If $\Delta(\tau) = 0$, then the function $q(t)$ can be written as*

$$q(t) = \frac{2}{\pi}\left(U^+(0) + f(\pi) - f(0)\right) - 2f'(t). \tag{3.11}$$

Here, $f(t)$ is defined by (3.10).

Now, we give a few numerical examples to verify and illustrate the results:

In each example we take $n = 40$.

Example 1. Consider the following boundary value problem:

$$y''(t) + ty(\frac{t}{2}) + \mu^2 y(t) = 0,\ t \in \Omega = [0,1) \cup (1,\pi]\,; \tag{3.12}$$

$$y(0) + 8y'(0) + \mu y'(0) = 0, \tag{3.13}$$

$$y\,(\pi) + \frac{1}{10}y'\,(\pi) + \mu y'\,(\pi) = 0, \tag{3.14}$$

$$y\,(1-) = y(1+), \tag{3.15}$$

$$y'\,(1-) = y'(1+). \tag{3.16}$$

Here $\Delta(t) = \frac{t}{2}$, $\alpha_1^+ = 0$, $\alpha_1^- = 1$, $\alpha_2^- = -8$, $\alpha_2^+ = -1$, $\beta_1^+ = 0$, $\beta_2^+ = -1$, $\beta_1^- = 1$, $\beta_2^- = -0.1$ and $\delta_1 = 1$. Denote this problem by L_1. We obtain $U^+\,(0) = V^+\,(0) = 4.93480220054$, $U^+\,(39) = 0.00065746219$ and $V^+\,(39) = 0.02670511654$. Thus, for the regularized trace (tr=trace) of L_1, we have $trL_1 = 58.3910062461$. Now, let us take $\Delta = 0$ in (3.12). Namely, we consider the differential equation

$$y''(t) + q(t)y(t) + \mu^2 y(t) = 0,\ t \in \Omega$$

together with the boundary conditions (3.13)–(3.14) and transmission conditions (3.15)–(3.16). Thus, from Theorems 3.1 and 3.2, for the solution of inverse problem we find $q\,(t) = t$ in Ω.

Example 2. Consider the following Sturm–Liouville problem:

$$y''(t) + q(t)y(t) + \mu^2 y(t) = 0,\ t \in \Omega = [0,1.5) \cup (1.5,2) \cup (2,\pi]$$

$$\begin{aligned}(3\mu + 2)\,y(0) - (7\mu + 4)\,y'\,(0) &= 0,\\ (\mu - 5)\,y\,(\pi) - (\mu + 0.3)\,y'\,(\pi) &= 0,\end{aligned}$$

$$y\,(1.5-) - 2y(1.5+) = 0,$$

$$y'\,(1.5-) - 2y'(1.5+) = 0,$$

$$y\,(2-) - 8y(2+) = 0,$$

$$y'\,(2-) - 8y'(2+) = 0.$$

Here , $\Delta(t) = 0$, $\alpha_1^+ = 3$, $\alpha_1^- = 2$, $\alpha_2^- = 4$, $\alpha_2^+ = 7$, $\beta_1^+ = 1$, $\beta_1^- = -5$, $\beta_2^+ = 1$, $\beta_2^- = 0.3$, $\delta_1 = 2$ and $\delta_2 = 8$. Denote this problem by L_2. Since $U^+\,(39) = 11.0703463164$ and $V^+\,(39) = 0.00181958345$, it follows that $trL_2 = -569.751286593$. Consequently, from Theorems 3.1 and 3.2, for the solution of inverse problem we have $q\,(t) = e^t$ in Ω.

4. Conclusion

In this paper, we investigate the spectrum [see (2.12)], calculate the regularized sums of eigenvalues [see (2.14)], obtain asymptotic formulas of nodal points [see (3.4) and (3.8)] for a Sturm–Liouville problem with delayed argument and with a finite number of transmission conditions. Furthermore, after the parameters $\alpha_i^{\pm}$, $\beta_i^{\pm}$ $(i = 1, 2)$ in the boundary conditions are determined we construct the potential function $q(x)$ via formula (3.11). Namely we solve an inverse problem for this kind of boundary value problems. The considered problem differs from the classical Sturm–Liouville problems with delayed argument in that it contains a spectral parameter in boundary conditions. Main results of this study are given by Theorem 2.1, Theorem 2.2, Theorem 3.1 and Theorem 3.3. If we take $\Delta \equiv 0$ and the coefficients $\alpha_i^{+} = \beta_i^{+} = 0$ $(i = 1, 2)$ in boundary conditions the formulas obtained in this study coincide with those obtained in [2, 3, 13, 18, 23]. In addition to this, if we take the transmission coefficients $\delta_i = 1$ $(i = \overline{1, m-1})$ then we get the classical case and formulas obtained in this study coincide with those obtained in [7–10, 16, 26]. We also note that regularized trace formulas, asymptotics of nodal points and solution of an inverse problem of the same differential equation but with nonlocal boundary conditions can also be investigated.

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