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Generalized Littlewood-Paley functions on product spaces

Hussain AL-QASSEM[1,*], **Leslie CHENG**[2], **Yibiao PAN**[3]
[1]Department of Mathematics and Physics, Qatar University, Doha, Qatar
[2]Department of Mathematics, Bryn Mawr College, Bryn Mawr, PA, USA
[3]Department of Mathematics, University of Pittsburgh, Pittsburgh, PA, USA

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Abstract: We are interested in investigating the L^p boundedness of the product of generalized Littlewood–Paley functions $S_{\Phi}^{(\lambda)}(f)$ arising from kernels satisfying only size and cancellation conditions. We obtain L^p estimates of $S_{\Phi}^{(\lambda)}(f)$ for a sharp range of p and under optimal conditions on Φ. Using these estimates and an extrapolation argument, we obtain some new and improved results on generalized Littlewood–Paley functions on product spaces. As a consequence of our main results, we get two results, one of which answers a question posed by D. Fan and H. Wu and the other one answers a question raised by Y. Wu and H. Wu. In addition, one of our lemmas on Triebel–Lizorkin spaces answers a question posed by Y. Wu and H. Wu.

Key words: Littlewood–Paley functions, Triebel–Lizorkin spaces, Orlicz spaces, block spaces, extrapolation, L^p boundedness

1. Introduction

Throughout this paper, we let ξ' denote $\xi/|\xi|$ for $\xi \in \mathbf{R}^n \backslash \{0\}$ and p' denote the exponent conjugate to p, that is $1/p + 1/p' = 1$.

For a function $\Phi \in L^1(\mathbf{R}^n \times \mathbf{R}^m)$ which satisfies

$$\int_{\mathbf{R}^n} \Phi(x,\cdot)dx = \int_{\mathbf{R}^m} \Phi(\cdot,y)dy = 0, \tag{1.1}$$

we define the product generalized Littlewood–Paley function $S_{\Phi}^{(\lambda)} f(x,y)$ on $\mathbf{R}^n \times \mathbf{R}^m$ by

$$S_{\Phi}^{(\lambda)}(f)(x,y) = \left(\int_0^\infty \int_0^\infty |\Phi_{t,s} * f(x,y)|^\lambda \frac{dtds}{ts}\right)^{1/\lambda},$$

where $1 < \lambda < \infty$, $\Phi_{t,s}(x,y) = t^{-n}s^{-m}\Phi(x/t, y/s)$ and $f \in S(\mathbf{R}^n \times \mathbf{R}^m)$, the space of Schwartz functions.

The product Littlewood–Paley square g-function of $f \in S(\mathbf{R}^n \times \mathbf{R}^m)$ is defined by

$$g(f)(x,y) = \left(\int_0^\infty \int_0^\infty |\Phi_{t,s} * f(x,y)|^2 \frac{dtds}{ts}\right)^{1/2}.$$

*Correspondence: husseink@qu.edu.qa
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Clearly, $g(f) = S_\Phi^{(2)}(f)$, which is often called a square function or a multiple Littlewood–Paley function. Littlewood–Paley functions have a long history, and they are one of the central parts of harmonic analysis. The use of the class of functions $S_\Phi^{(2)}(f)$ appears in the study of singular integrals on product domains, the product Hardy spaces as well as other function spaces. The generalized Paley–Littlewood functions, in the product and nonproduct case, have attracted the attention of many authors in recent years. Readers are referred to [20–22, 25, 26, 35] for their significance and historical developments.

One of the most important examples of this class of functions $S_\Phi^{(\lambda)}(f)(x,y)$ (for $\lambda = 2$) is obtained by letting

$$\Phi(x,y) = |x|^{-n+1}\,|y|^{-m+1}\,\Omega(x',y')\chi_{(0,1]}(|x|)\chi_{(0,1]}(|y|)$$

for $x \in \mathbf{R}^n \setminus \{0\}, y \in \mathbf{R}^m \setminus \{0\}$, where $\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega(x,\cdot)d\sigma(x) = \int_{\mathbf{S}^{m-1}} \Omega(\cdot,y)d\sigma(y) = 0. \tag{1.2}$$

Then $S_\Phi^{(\lambda)}(f) = \mu_\Omega^{(\lambda)}(f)$, the generalized Marcinkiewicz integral on $\mathbf{R}^n \times \mathbf{R}^m$, is given by

$$\mu_\Omega^{(\lambda)}(f)(x,y) = \left(\int_0^\infty \int_0^\infty \left|\int_{|v|\le s}\int_{|u|\le t} f\left(x-u,y-v\right)\frac{\Omega(u',v')}{|u|^{n-1}\,|v|^{m-1}}dudv\right|^\lambda \frac{dtds}{t^3 s^3}\right)^{1/\lambda}.$$

If $\lambda = 2,$ we write $\mu_\Omega = \mu_\Omega^{(\lambda)}$. Here $\mu_\Omega(f)(x,y)$ is the classical Marcinkiewicz integral on $\mathbf{R}^n \times \mathbf{R}^m$.

One of the main issues of concern in this paper is investigating the L^p boundedness of $S_\Phi^{(\lambda)}$ under optimal conditions on Φ (in an appropriate sense) and for sharp ranges of p. We are very much motivated by the work of several authors in the one parameter case of $S_\Phi^{(\lambda)}$ and also by the work on generalized Marcinkiewicz integrals on product domains (See work by S. Sato, L. Cheng, J. Duoandikoetxea, D. Fan and H. Wu, Y. Wu and H. Wu, and H. Al-Qassem, L. Cheng and Y. Pan).

Before stating our results, we would like to mention that our results in this paper will mirror the recent results obtained in the one parameter setting obtained by Duoandikoetxea concerning $S_\Phi^{(\lambda)}$ (for $\lambda = 2$) in [17] and by Al-Qassem et al. concerning $S_\Phi^{(\lambda)}$ (for $\lambda > 1$) in [4]. Also, one of our results answers a question posed by Fan and Wu concerning $\mu_\Omega^{(\lambda)}$ in [19] and two of our results answer two questions posed by Wu and Wu in [35] concerning a result on $\mu_\Omega^{(\lambda)}$ and a result on Triebel-Lizorkin spaces.

We will begin by recalling some definitions. The class $L(\log L)^\alpha(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ (for $\alpha > 0$) denotes the class of all measurable functions Ω on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ which satisfy

$$\|\Omega\|_{L(\log L)^\alpha(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega(x,y)|\log^\alpha(2+|\Omega(x,y)|)d\sigma(x)d\sigma(y) < \infty.$$

Let $L(\log L)^\alpha(\mathbf{R}^n \times \mathbf{R}^m)$ (for $\alpha > 0$) denote the class of all measurable functions Ψ on $\mathbf{R}^n \times \mathbf{R}^m$ which satisfy

$$\|\Psi\|_{L(\log L)^\alpha(\mathbf{R}^n\times\mathbf{R}^m)} = \int_{\mathbf{R}^n\times\mathbf{R}^m} |\Psi(x,y)|\log^\alpha(2+|\Psi(x,y)|)dxdy < \infty.$$

Now we recall the definition of the block space $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$. This space was introduced in the one parameter case by Jiang and Lu (see [27]) in their study of the mapping properties of homogeneous singular integral operators and it is defined as follows:

Definition 1.1. *A q-block on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$ is an L^q $(1<q\leq\infty)$ function $b(x,y)$ that satisfies*

(i) $\operatorname{supp}(b)\subset I$; (ii) $\|b\|_{L^q}\leq |I|^{-1/q'}$,

where $|\cdot|$ denotes the product measure on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$ and I is an interval on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$, i.e.

$$I=\left\{x'\in\mathbf{S}^{n-1}:|x'-x_0'|<\alpha\right\}\times\left\{y'\in\mathbf{S}^{m-1}:|y'-y_0'|<\beta\right\}$$

for some $\alpha,\ \beta>0$, $x_0'\in\mathbf{S}^{n-1}$ and $y_0'\in\mathbf{S}^{m-1}$.

Definition 1.2. *The block space $B_q^{(0,\upsilon)}=B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ is defined by*

$$B_q^{(0,\upsilon)}=\left\{\Omega\in L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})\colon \Omega=\sum_{\mu=1}^{\infty}\lambda_\mu b_\mu,\ M_q^{(0,\upsilon)}\left(\{\lambda_\mu\}\right)<\infty\right\}$$

where each λ_μ is a complex number, each b_μ is a q-block supported on an interval I_μ on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$, $\upsilon>-1$, and

$$M_q^{(0,\upsilon)}\left(\{\lambda_\mu\}\right)\ =\sum_{\mu=1}^{\infty}\left|\lambda_\mu\right|\left\{1+\log^{(\upsilon+1)}(\left|I_\mu\right|^{-1})\right\}.$$

Let $\|\Omega\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}=N_q^{(0,\upsilon)}(\Omega)=\inf\{M_q^{(0,\upsilon)}\left(\{\lambda_\mu\}\right)\colon\ \Omega=\sum_{\mu=1}^{\infty}\lambda_\mu b_\mu$ and each b_μ is a q-block function supported on a cap I_μ on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}\}$. Then $\|\cdot\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}$ is a norm on the space $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$, and $(B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}),\|\cdot\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})})$ is a Banach space.

The definition of the block space $B_q^{(0,\upsilon)}(\mathbf{R}^n\times\mathbf{R}^m)$ (for $\upsilon>-1$) of functions on $\mathbf{R}^n\times\mathbf{R}^m$ is defined similarly.

Remark. For any $q>1$ and $0<\upsilon\leq 1$, the following inclusions hold and are proper:

$$\begin{aligned}
L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}) &\subset L(\log L)(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})\subset L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}),\\
\bigcup_{r>1}L^r(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}) &\subset B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})\text{ for any }-1<\upsilon\text{ and }q>1,\\
L(\log L)^{\beta}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}) &\subset L(\log L)^{\alpha}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})\text{ if }0<\alpha<\beta.
\end{aligned}$$

The question with regard to the relationship between $B_q^{(0,\upsilon-1)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ and $L(\log^+L)^{\upsilon}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ (for $\upsilon>0$) remains open.

We remark that similar relations as above hold if the classes $L(\log L)^{\alpha}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ and $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ are replaced by $L(\log L)^{\alpha}(\mathbf{R}^n\times\mathbf{R}^m)$ and $B_q^{(0,\upsilon)}(\mathbf{R}^n\times\mathbf{R}^m)$.

Now we recall the definition of the homogeneous Triebel–Lizorkin spaces $\dot{F}_{p,q}^{\overrightarrow{s}}(\mathbf{R}^n \times \mathbf{R}^m)$. Let $\varphi \in C_0^\infty(\mathbf{R}^n)$ and $\psi \in C_0^\infty(\mathbf{R}^m)$ be functions satisfying

$$\begin{aligned} \text{supp } \varphi &\subset \left\{x \in \mathbf{R}^n : \frac{1}{2} \leq |x| \leq 2\right\}, \text{supp } \psi \subset \left\{y \in \mathbf{R}^m : \frac{1}{2} \leq |y| \leq 2\right\}, \\ \varphi(x), \psi(y) &\geq c > 0 \text{ if } \frac{3}{5} \leq |x|, |y| \leq \frac{5}{3} \text{ for some constant } c. \end{aligned}$$

Let Φ and Ψ the Fourier transforms of φ and ψ, respectively. For $1 < p, q < \infty$ and $\alpha, \beta \in \mathbf{R}, \overrightarrow{s} = (\alpha, \beta)$, the homogeneous Triebel–Lizorkin space $\dot{F}_{p,q}^{\overrightarrow{s}}(\mathbf{R}^n \times \mathbf{R}^m)$ is defined to be the set of all distributions f on $\mathbf{R}^n \times \mathbf{R}^m$ such that

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} 2^{k\alpha q} 2^{j\beta q} \left| (\Phi_k \otimes \Psi_j) * f \right|^q \right)^{1/q} \right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} < \infty$$

where $\Phi_k(x) = 2^{-kn}\varphi(2^{-k}x)$ for $k \in \mathbf{Z}$ and $\Psi_j(y) = 2^{-jm}\psi(2^{-j}y)$ for $j \in \mathbf{Z}$.

The following can be found in [19]:

(1) $L^p(\mathbf{R}^n \times \mathbf{R}^m) = \dot{F}_{p,2}^{\overrightarrow{0}}(\mathbf{R}^n \times \mathbf{R}^m)$;

(2) $\left(\dot{F}_{p,q}^{\overrightarrow{s}}(\mathbf{R}^n \times \mathbf{R}^m)\right)^* = \dot{F}_{p',q'}^{-\overrightarrow{s}}(\mathbf{R}^n \times \mathbf{R}^m)$;

(3) $\dot{F}_{p,q_1}^{\overrightarrow{s}}(\mathbf{R}^n \times \mathbf{R}^m) \subseteq \dot{F}_{p,q_2}^{\overrightarrow{s}}(\mathbf{R}^n \times \mathbf{R}^m)$ if $q_1 \leq q_2$.

2. Main results

The main results of this paper are the following:

Theorem 2.1. *Let $1 < \lambda < \infty$ and let K_1 and K_2 be two compact sets in $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Suppose that Φ is a function supported in $K_1 \times K_2$ satisfying (1.1). If $\Phi \in L^q(\mathbf{R}^n \times \mathbf{R}^m)$ for some $q > 1$, then for any $p \in \left(1/\left(\min\{1, \frac{1}{\lambda} + \frac{1}{q'}\}\right), \infty\right)$ there exists a constant $C_p > 0$ such that*

$$\left\| S_\Phi^{(\lambda)}(f) \right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \leq C_p (q-1)^{-\frac{2}{\lambda}} \left\| \Phi \right\|_{L^q(\mathbf{R}^n \times \mathbf{R}^m)} \|f\|_{\dot{F}_{\lambda,p}^{\overrightarrow{0}}(\mathbf{R}^n \times \mathbf{R}^m)} \tag{2.1}$$

for $f \in \dot{F}_{\lambda,p}^{\overrightarrow{0}}(\mathbf{R}^n \times \mathbf{R}^m)$. The constant C_p may depend on λ and on the diameters of K_1 and K_2, but it is independent of q and Φ. The range of p is the best possible.

It is clear from Theorem 2.1 if $\lambda > 1$, the range of p is the entire interval $(1, \infty)$ if $\Phi \in L^q(\mathbf{R}^n \times \mathbf{R}^m)$ for some $q \geq \lambda$. However, if Φ is dominated by a function $\tilde{\Phi}$ with $\tilde{\Phi}(x, y) = \varphi(|x|, |y|)$ for some function φ, we have a better range of p if $q < \lambda$ as in the following result.

Theorem 2.2. *Suppose that $n > 1$ and $m > 1$. Let $1 < \lambda < \infty$ and let K_1 and K_2 be two compact sets in $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Let Φ be a function supported in $K_1 \times K_2$ satisfying (1.1) and $|\Phi(x, y)| \leq \tilde{\Phi}(x, y)$ for some function $\tilde{\Phi} \in L^q(\mathbf{R}^n \times \mathbf{R}^m)$ for some $q > 1$ with $\tilde{\Phi}(x, y) = \varphi(|x|, |y|)$ for some function φ defined on $(0, \infty) \times (0, \infty)$. Then if $q < \lambda$, $S_\Phi^{(\lambda)}$ is a bounded operator on $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ for any $p \in ((\lambda q \omega)/(\lambda q \omega - \lambda + q), \infty)$, where $\omega = \min\{n, m\}$. The range of p is the best possible.*

In the case $\Phi \in L^q(\mathbf{R}^n \times \mathbf{R}^m)$ for $q = 1$ we have the following:

Theorem 2.3. *Suppose that $n > 1$ and $m > 1$. Let $1 < \lambda < \infty$ and let K_1 and K_2 be two compact sets in $\mathbf{R}^n \backslash \{0\}$ and $\mathbf{R}^m \backslash \{0\}$, respectively. Let $\Phi(x,y)$ be an $L^1(\mathbf{R}^n \times \mathbf{R}^m)$ function satisfying (1.1) and that is supported in $K_1 \times K_2$ and $\Phi(x,y) = \varphi(|x|, |y|)$ for some function φ defined on $(0,\infty) \times (0,\infty)$. Then $S_\Phi^{(\lambda)}$ is bounded on $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ for any $p \in \left((\lambda\omega)', \infty\right)$, where $\omega = \min\{n,m\}$. The range of p is the best possible.*

For certain other classes of Φ we get the range of p the entire interval $(1,\infty)$ as described in the following theorem.

Theorem 2.4. *Suppose that $n > 1$ and $m > 1$. Let $1 < \lambda < \infty$ and let K_1 and K_2 two compact sets in $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Let Φ be a function supported in $K_1 \times K_2$ satisfying (1.1). Suppose that $|\Phi(x,y)| \le h_1(|x|)h_2(|y|)\Omega(x',y')$ for all $(x,y) \in \mathbf{R}^n \backslash \{0\} \times \mathbf{R}^m \backslash \{0\}$, where h_1 and h_2 are nonnegative nonincreasing functions on $(0,\infty)$ and supported in $(0,1]$ and Ω is a nonnegative function on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$. Assume that $h_1(|x|) \in L^q(\mathbf{R}^n), h_2(|y|) \in L^q(\mathbf{R}^m)$ and $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for some $q > 1$. Then for any $p \in (1,\infty)$, there exists a constant $C_p > 0$ such that*

$$\left\| S_\Phi^{(\lambda)}(f) \right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \le C_p (q-1)^{-\frac{2}{\lambda}} \|\Omega\|_{L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \|f\|_{\dot{F}_{\lambda,p}^{\vec{0}}(\mathbf{R}^n \times \mathbf{R}^m)} \tag{2.2}$$

for $f \in \dot{F}_{\lambda,p}^{\vec{0}}(\mathbf{R}^n \times \mathbf{R}^m)$. The constant C_p may depend on the diameters of K_1 and $K_2, \|h_1(|\cdot|)\|_{L^q(\mathbf{R}^n)}$, $\|h_2(|\cdot|)\|_{L^q(\mathbf{R}^m)}$ and λ, but it is independent of q and Ω.

By using an extrapolation argument and the estimates (2.1) and (2.2), we get the following results:

Theorem 2.5. *Let $1 < \lambda < \infty$ and let K_1 and K_2 two compact sets in $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Let Φ be a function supported in $K_1 \times K_2$ satisfying (1.1).*

(a) If $\Phi \in B_q^{(0,\frac{2}{\lambda}-1)}(\mathbf{R}^n \times \mathbf{R}^m)$ for some $q > 1$, then for any $p \in [\lambda, \infty)$ there exists a a constant $C_p > 0$ independent of Φ such that

$$\left\| S_\Phi^{(\lambda)}(f) \right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \le C_p \left(1 + \|\Phi\|_{B_q^{(0,\frac{2}{\lambda}-1)}(\mathbf{R}^n \times \mathbf{R}^m)} \right) \|f\|_{\dot{F}_{\lambda,p}^{\vec{0}}(\mathbf{R}^n \times \mathbf{R}^m)}. \tag{2.3}$$

(b) *If $\Phi \in L(\log L)^{2/\lambda}(\mathbf{R}^n \times \mathbf{R}^m)$, then for any $p \in [\lambda, \infty)$ there exists a constant $C_p > 0$ independent of Φ* such that

$$\left\| S_\Phi^{(\lambda)}(f) \right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \le C_p \left(1 + \|\Phi\|_{L(\log L)^{2/\lambda}(\mathbf{R}^n \times \mathbf{R}^m)} \right) \|f\|_{\dot{F}_{\lambda,p}^{\vec{0}}(\mathbf{R}^n \times \mathbf{R}^m)}. \tag{2.4}$$

Theorem 2.6. *Let $1 < \lambda < \infty$. Let K_1 and K_2 be two compact sets in $\mathbf{R}^n$ and $\mathbf{R}^m$,* respectively. *Let Φ be a function supported in $K_1 \times K_2$ satisfying (1.1). Suppose that $|\Phi(x,y)| \le h_1(|x|)h_2(|y|)\Omega(x',y')$ for all $(x,y) \in \mathbf{R}^n \backslash \{0\} \times \mathbf{R}^m \backslash \{0\}$, where Ω is a nonnegative function on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ and h_1* and *h_2 are as in Theorem 2.4. Assume that $h_1(|x|) \in L^q(\mathbf{R}^n)$, $h_2(|x|) \in L^q(\mathbf{R}^m)$ and $\Omega \in L(\log L)^{2/\lambda}\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right)$ or $\Omega \in B_q^{(0,\frac{2}{\lambda}-1)}\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right)$ for some $q > 1$. Then* for *any $p \in (1,\infty)$, there exists a constant $C_p > 0$ such that*

$$\left\| S_\Phi^{(\lambda)}(f) \right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \le C_p \|f\|_{\dot{F}_{\lambda,p}^{\vec{0}}(\mathbf{R}^n \times \mathbf{R}^m)}. \tag{2.5}$$

As an application of Theorem 2.6, we get the following result.

Theorem 2.7. *Let* $1 < \lambda < \infty$. *If* $\Omega \in L(\log L)^{2/\lambda}\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right)$ *or* $\Omega \in B_q^{(0,\frac{2}{\lambda}-1)}\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right)$ *for some* $q > 1$ *and satisfies (1.2), then* for *any* $p \in (1, \infty)$, *there exists a constant* $C_p > 0$ *such that*

$$\left\| \mu_\Omega^{(\lambda)}(f) \right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \leq C_p \, \|f\|_{\dot{F}^{\vec{0}}_{\lambda,p}(\mathbf{R}^n \times \mathbf{R}^m)} . \tag{2.6}$$

Remarks. 1) The conditions on Φ in Theorem 2.5 and conditions on Ω in Theorem 2.6 are the weakest conditions in their respective classes.

2) The result in Theorem 2.5 in the case $\Omega \in L(\log L)^{2/\lambda}\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right)$ improves substantially the main result in [19] in which they proved $\mu_\Omega^{(\lambda)}$ is bounded on L^p if $\Omega \in L(\log L)^{2/\lambda}\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right)$ for $\lambda \geq 2$ and bounded on L^p if $1 < \lambda < 2$ and $\Omega \in L(\log L)^{2/\lambda+\varepsilon}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for any $\varepsilon > 0$. In fact, our result answers the question posed by the authors in ([19], p. 102). Also, we point out the condition $\Omega \in L(\log L)^{2/\lambda}\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right)$ is the best possible in the case $\lambda = 2$ as indicated in [8].

3) The result in Theorem 2.5 in the case $\Omega \in B_q^{(0,\frac{2}{\lambda}-1)}\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right)$ for some $q > 1$) improves substantially the main result in [35] in which they proved if $1 < p, q, \lambda < \infty$ and $\Omega \in B_q^{(0,2/\lambda+2w-1)}\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right)$ for any $w \in [w(p,\lambda), 1]$, then $\mu_\Omega^{(\lambda)}$ is bounded on L^p where $w(p,\lambda) = |2/\lambda - 1| + |2/p - 1|\,(1 - |2/\lambda - 1|)$. We notice that in the special case if $p = \lambda = 2$ we have $w(p,\lambda) = 0$ and $\Omega \in B_q^{(0,0)}\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right)$. Our result answers the question posed by the authors in ([35], p. 2397). We point out the condition $\Omega \in B_q^{(0,\frac{2}{\lambda}-1)}\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right)$ for some $q > 1$ is the best possible in the case $\lambda = 2$ as proved in [1] .

4) All our results obtained in Theorems 2.1–2.6 are new.

5) In [25], the authors proved $S_\Phi^{(2)}$ is bounded on L^2 if Φ satisfies certain size conditions.

6) One of the key ingredients in the proof of our main results is Lemma 3.1 in which we have a characterization on the homogeneous Triebel–Lizorkin spaces in terms of lacunary sequences. In the nonproduct case this was obtained by [30]. This result provides an answer to an open problem left unresolved in [35].

Throughout this paper, the letter C will denote a positive constant that may vary at each occurrence but is independent of the essential variables.

3. Some lemmas

Let $\{a_k : k \in \mathbf{Z}\}$ be a lacunary sequence of positive numbers in the sense that $\frac{a_{k+1}}{a_k} \geq a > 1$ for each $k \in \mathbf{Z}$. A sequence $\{\Phi_k^{n,a} : k \in \mathbf{Z}\}$ of $C^\infty(\mathbf{R}^n)$ functions is said to be a partition of unity adapted to $\{a_k : k \in \mathbf{Z}\}$ if

$$\begin{aligned} \operatorname{supp}\widehat{\Phi_k^{n,a}} &\subset \{\xi \in \mathbf{R}^n : a_{k-1} \leq |\xi| \leq a_{k+1}\}\,(k \in \mathbf{Z}), \\ \sum_{k \in \mathbf{Z}} \widehat{\Phi_k^{n,a}}(\xi) &= 1 \;(\xi \in \mathbf{R}^n \backslash \{0\}), \end{aligned}$$

and

$$\left| \xi^\omega \partial^\omega \widehat{\Phi_k^{n,a}}(\xi) \right| \leq C_\omega$$

for any multiindex ω. Let $\alpha \in \mathbf{R}$ and $1 < p, q < \infty$. Let $\{a_k : k \in \mathbf{Z}\}$ and $\{b_j : j \in \mathbf{Z}\}$ be lacunary sequences of positive numbers with $\frac{a_{k+1}}{a_k} \geq a > 1$ and $\frac{b_{k+1}}{b_k} \geq b > 1$ $(k, j \in \mathbf{Z})$. Let $1 < p, q < \infty$, $\alpha, \beta \in \mathbf{R}$ and $\overrightarrow{s} = (\alpha, \beta)$. For a distribution on f on $\mathbf{R}^n \times \mathbf{R}^m$ we define the norm $\|f\|_{\dot{F}_{p,q}^{\overrightarrow{s}},\left\{\Phi_k^{n,a},\Phi_k^{m,b}\right\}(\mathbf{R}^n\times\mathbf{R}^m)}$ by

$$\|f\|_{\dot{F}_{p,q}^{\overrightarrow{s}},\left\{\Phi_k^{n,a},\Phi_j^{m,b}\right\}(\mathbf{R}^n\times\mathbf{R}^m)} = \left\| \left(\sum_{k,j\in\mathbf{Z}} a_k^{\alpha q} b_j^{\beta q} \left| \left(\Phi_k^{n,a} \otimes \Phi_j^{m,b} \right) * f \right|^q \right)^{1/q} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}.$$

We shall need the following result which is similar to the corresponding result in the nonproduct case in [30].

Lemma 3.1. *Let $\alpha \in \mathbf{R}$ and $1 < p, q < \infty$. Let $\{a_k : k \in \mathbf{Z}\}$ and $\{b_j : j \in \mathbf{Z}\}$ be lacunary sequences of positive numbers with $\frac{a_{k+1}}{a_k} \geq a > 1$ and $\frac{b_{k+1}}{b_k} \geq b > 1$ $(k, j \in \mathbf{Z})$. Then the norm $\|f\|_{\dot{F}_{p,q}^{\overrightarrow{s}},\left\{\Phi_k^{n,a},\Phi_k^{m,b}\right\}(\mathbf{R}^n\times\mathbf{R}^m)}$ is equivalent to the usual homogeneous Triebel–Lizorkin space norm $\|f\|_{\dot{F}_p^{\alpha,q}(\mathbf{R}^n\times\mathbf{R}^m)}$ if $\frac{a_{k+1}}{a_k} \leq d$ and $\frac{b_{j+1}}{b_j} \leq d$ $(k, j \in \mathbf{Z})$ for some $d \geq \min\{a, b\}$.*

A proof of Lemma 3.1 can be obtained by following an argument similar to what was used in [30] for the case $\alpha \neq 0$. We notice also that the same argument works for the case $\alpha = 0$ (see also the remark in [33]). We omit the details.

For $1 \leq q < \infty$, define the maximal operator $N^{(q)}$ by

$$N^{(q)}(f)(x) = \sup_{t>0} \left(\frac{1}{t^n} \int_0^t \left(\int_{\mathbf{S}^{n-1}} |f(x - r\theta)| \, d\sigma(\theta) \right)^q r^{n-1} dr \right)^{1/q}.$$

The case $q = \infty$ corresponds to the spherical maximal operator

$$N^{(\infty)}(f)(x) = \sup_{t>0} \int_{\mathbf{S}^{n-1}} |f(x - t\theta)| \, d\sigma(\theta).$$

By the results of Stein [32] and Bourgain [10] we have

Lemma 3.2. *Suppose that $n \geq 2$ and $p > n'$. Then $N^{(\infty)}$ is bounded on $L^p(\mathbf{R}^n)$.*

We point out that $N^{(1)}$ is a constant multiple of the Hardy–Littlewood maximal operator. The class of operators $N^{(q)}$ was used in [23], and the following result was proved by Duoandikoetxea [17].

Lemma 3.3. *The maximal function $N^{(q)}$ is bounded on $L^p(\mathbf{R}^n)$ if and only if $p' < nq'$.*

For $1 \leq q < \infty$, define the maximal operator $N_P^{(q)}$ on the product space $\mathbf{R}^n \times \mathbf{R}^m$ by

$$N_P^{(q)}(f)(x,y) = \sup_{t,s>0} \left(\frac{1}{t^n s^m} \int_0^s \int_0^t \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |f(x - r\theta, y - w\eta)| \, d\sigma(\theta) d\sigma(\eta) \right)^q r^{n-1} w^{m-1} dr dw \right)^{1/q}.$$

The case $q = \infty$ corresponds to the spherical maximal operator on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ given by

$$N_P^{(\infty)}(f)(x,y) = \sup_{t,s>0} \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |f(x - t\theta, y - s\eta)| \, d\sigma(\theta) d\sigma(\eta).$$

Define four maximal operators $N_1^{(\infty)}$, $N_2^{(\infty)}$, $N_1^{(1)}$ and $N_2^{(1)}$ on $\mathbf{R}^n \times \mathbf{R}^m$ by

$$\begin{aligned} N_1^{(\infty)}(f)(x,y) &= (N^{(\infty)})f(\cdot,y))(x), \\ N_2^{(\infty)}(f)(x,y) &= (N^{(\infty)})f(x,\cdot)(y), \\ N_1^{(1)}(f)(x,y) &= (N^{(1)})f(\cdot,y)(x), \end{aligned}$$

and

$$N_2^{(1)}(f)(x,y) = (N^{(1)})f(x,\cdot)(y).$$

Then

$$N_P^{(\infty)}(f)(x,y) \leq \left(N_2^{(\infty)} \circ N_1^{(\infty)}\right)(f)(x,y) \tag{3.1}$$

and

$$N_P^{(1)}(f)(x,y) \leq \left(N_2^{(1)} \circ N_1^{(1)}\right)(f)(x,y). \tag{3.2}$$

By using Lemma 3.2 and (3.1) we get

Lemma 3.4. *Suppose that* $n, m \geq 2$ *and* $p > \max\{n', m'\}$. *Then* $N_P^{(\infty)}$ *is bounded on* $L^p(\mathbf{R}^n \times \mathbf{R}^m)$.

By using Lemma 3.3 and (3.2) we get

Lemma 3.5. $N_P^{(1)}$ *is bounded on* $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ *for* $1 < p < \infty$.

Lemma 3.6. *Suppose that* $n, m \geq 2$. *Then* $N_P^{(q)}$ *is bounded on* $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ *for* $(p'/q')' > \max\{n', m'\}$.

Proof. The proof will be similar to the proof in the nonproduct case given by Duoandikoetxea in [17]. By Hölder's inequality on the integral over $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ we get

$$N_P^{(q)}(f)(x,y) \leq \left(N_P^{(\infty)}(|f|^{\alpha q'})(x,y)\right)^{1/q'} \left(N_P^{(1)}(|f|^{(1-\alpha)q})(x,y)\right)^{1/q}$$

for some $\alpha \in [0,1]$, which will be chosen later. Again by Hölder's inequality we have

$$\left\|N_P^{(q)}(f)\right\|_p \leq \left\|N_P^{(\infty)}(|f|^{(\alpha q')})\right\|_{p/(\alpha q')}^{1/q'} \left\|N_P^{(1)}(|f|^{(1-\alpha)q})\right\|_{p/((1-\alpha)q)}^{1/q}.$$

Now since $(p'/q')' > \max\{n', m'\}$, we can choose $\alpha \in (0,1)$ such that $p/((1-\alpha)q) > 1$ and $p/(\alpha q') > \max\{n', m'\}$. The proof is complete.

Let $\theta \geq 2$. For a suitable function Φ defined on $\mathbf{R}^n \times \mathbf{R}^m$, define the maximal operator M_Φ on $\mathbf{R}^n \times \mathbf{R}^m$ by

$$M_\Phi(f)(x,y) = \sup_{k,j \in \mathbf{Z}} \int_{\theta^j}^{\theta^{(j+1)}} \int_{\theta^k}^{\theta^{(k+1)}} ||\Phi_{t,s}| * f(x,y)| \frac{dtds}{ts}.$$

We shall need the following result, which will play a key role in the proof of Theorem 2.1.

Lemma 3.7. *Let* K_1 *and* K_2 *be two compact sets in* $\mathbf{R}^n$ *and* $\mathbf{R}^m$, *respectively. Let* Φ *be a function supported in* $K_1 \times K_2$. *Suppose that* $\Phi \in L^q(\mathbf{R}^n \times \mathbf{R}^m)$ *for some* $q > 1$ *and let* $\theta = 2^{q'}$. *Then for every* $p, 1 < p \leq \infty$, *there exists a positive constant* C_p *which is independent of* q *such that*

$$\|M_\Phi(f)\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \leq C_p(q-1)^{-2} \|\Phi\|_{L^q(\mathbf{R}^n \times \mathbf{R}^m)} \|f\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \tag{3.3}$$

for every $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$.

Proof. We will follow an argument similar to one in [6] and [7]. Choose and fix $\varphi \in S(\mathbf{R}^n)$ and $\psi \in S(\mathbf{R}^m)$ such that $\hat{\varphi}(\xi) = 1$ for $|\xi| \leq 1$ and $\hat{\varphi}(\xi) = 0$ for $|\xi| \geq 2, \hat{\psi}(\eta) = 1$ for $|\eta| \leq 1$ and $\hat{\psi}(\eta) = 0$ for $|\eta| \geq 2$. For each $t, s \in \mathbf{R}_+$, let $(\varphi_t)\hat{}(\xi) = \hat{\varphi}(t\xi)$ and $(\psi_s)\hat{}(\eta) = \hat{\psi}(s\eta)$. Let $\mu_{t,s} = |\Phi_{t,s}|$. Define the family of measures $\{\Upsilon_{t,s}\}_{t,s\in\mathbf{R}_+}$ and $\{\sigma_{k,j}\}_{k,j\in\mathbf{Z}}$ by

$$\begin{aligned} &\hat{\Upsilon}_{t,s}(\xi,\eta) \\ = \;& \hat{\mu}_{t,s}(\xi,\eta) - \hat{\mu}_{t,s}(0,\eta)(\varphi_t)\hat{}(\xi) - \hat{\mu}_{t,s}(\xi,0)(\psi_s)\hat{}(\eta) - \hat{\mu}_{t,s}(0,0)(\varphi_t)\hat{}(\xi)(\psi_s)\hat{}(\eta) \end{aligned} \tag{3.4}$$

and

$$\hat{\sigma}_{k,j}(\xi,\eta) = \int_{\theta^j}^{\theta^{(j+1)}} \int_{\theta^k}^{\theta^{(k+1)}} \hat{\Upsilon}_{t,s}(\xi,\eta) \frac{dtds}{ts}. \tag{3.5}$$

Now, let

$$\begin{aligned} g(f) &= \left(\sum_{k,j\in\mathbf{Z}} |\sigma_{k,j} * f|^2 \right)^{1/2}, \\ \sigma^*(f) &= \sup_{k,j\in\mathbf{Z}} \left(||\sigma_{k,j}| * f| \right), \\ \sigma^{(i)*}(f) &= \sup_{k,j\in\mathbf{Z}} \left(\left| \sigma_{k,j}^{(i)} * f \right| \right), \mathrm{i} = 1,2, \\ \widehat{\sigma_{k,j}^{(1)}}(\xi,\eta) &= \int_{\theta^j}^{\theta^{(j+1)}} \int_{\theta^k}^{\theta^{(k+1)}} \hat{\mu}_{t,s}(0,\eta) \frac{dtds}{ts}, \end{aligned}$$

and

$$\widehat{\sigma_{k,j}^{(2)}}(\xi,\eta) = \int_{\theta^j}^{\theta^{(j+1)}} \int_{\theta^k}^{\theta^{(k+1)}} \hat{\mu}_{t,s}(\xi,0) \frac{dtds}{ts}.$$

By definition,

$$\hat{\mu}_{t,s}(\xi,\eta) = \int_{\mathbf{R}^n\times\mathbf{R}^m} e^{-it\eta\cdot y} e^{-it\xi\cdot x} |\Phi(x,y)| \, dxdy.$$

Since Φ is supported in $K_1 \times K_2$, it is easy to see that

$$|\hat{\mu}_{t,s}(\xi,\eta)| \leq C \|\Phi\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)} \qquad \text{for } t,s \in \mathbf{R}_+,$$

and hence

$$|\hat{\sigma}_{k,j}(\xi,\eta)| \leq C(q-1)^{-2} \|\Phi\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}. \tag{3.6}$$

Next, if $|t\xi| \geq 2$ and $|s\eta| \geq 2$, by the choices of φ and ψ we have $\hat{\Upsilon}_{t,s}(\xi,\eta) = \hat{\mu}_{t,s}(\xi,\eta)$, and hence

$$\begin{aligned} &\hat{\sigma}_{k,j}(\xi,\eta) \\ = \;& \int_{\mathbf{R}^n\times\mathbf{R}^m} |\Phi(u,v)| \left(\int_{\theta^k}^{\theta^{(k+1)}} e^{-it\xi\cdot u} \frac{dt}{t} \right) \left(\int_{\theta^j}^{\theta^{(j+1)}} e^{-is\eta\cdot v} \frac{ds}{s} \right) dudv. \end{aligned} \tag{3.7}$$

By integration by parts we obtain

$$\left|\int_{\theta^k}^{\theta^{(k+1)}} e^{-it\xi\cdot u}\frac{dt}{t}\right| \leq C(q-1)^{-1}\min\{1, \left|\left|\theta^k\xi\right|^{-1}\right|\left|\xi'\cdot u\right|^{-1}\}$$

and hence

$$\left|\int_{\theta^k}^{\theta^{(k+1)}} e^{-it\xi\cdot u}\frac{dt}{t}\right| \leq C(q-1)^{-1}\left|\left|\theta^k\xi\right|^{-\beta}\right|\left|\xi'\cdot u\right|^{-\beta} \tag{3.8}$$

for any $0<\beta\leq 1$. Hence by (3.6)–(3.8) we have

$$\left|\hat{\sigma}_{k,j}(\xi,\eta)\right| \leq C(q-1)^{-2}\left|\theta^k\xi\right|^{-\frac{\beta}{q'}}\left|\theta^j\eta\right|^{\frac{\beta}{2q'}}\|\Phi\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}. \tag{3.9}$$

Now if $|t\xi|\geq 2$ and $|s\eta|<2$ we have

$$\begin{aligned} &\hat{\Upsilon}_{t,s}(\xi,\eta) = \hat{\mu}_{t,s}(\xi,\eta) - \hat{\mu}_{t,s}(\xi,0)(\psi_s\hat{)}(\eta) \\ =\quad & \left(\hat{\mu}_{t,s}(\xi,\eta)-\hat{\mu}_{t,s}(\xi,0)\right)(\psi_s\hat{)}(\eta) + \hat{\mu}_{t,s}(\xi,0)\left(1-(\psi_s\hat{)}(\eta)\right). \end{aligned} \tag{3.10}$$

Thus by (3.6), (3.8) and (3.10) we have

$$\left|\hat{\sigma}_{k,j}(\xi,\eta)\right| \leq C(q-1)^{-2}\left|\theta^k\xi\right|^{-\frac{\beta}{q'}}\left|\theta^j\eta\right|^{\frac{\beta}{2q'}}\|\Phi\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)} \tag{3.11}$$

for some positive constants C and β. Similarly if $|t\xi|<2$ and $|s\eta|\geq 2$ we have

$$\left|\hat{\sigma}_{k,j}(\xi,\eta)\right| \leq C(q-1)^{-2}\left|\theta^k\xi\right|^{\frac{\beta}{q'}}\left|\theta^j\eta\right|^{-\frac{\beta}{2q'}}\|\Phi\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)} \tag{3.12}$$

for some positive constants C and β.

Finally, if $|t\xi|<2$ and $|s\eta|<2$ we have

$$\begin{aligned} &\hat{\Upsilon}_{t,s}(\xi,\eta) = \hat{\mu}_{t,s}(\xi,\eta) - \hat{\mu}_{t,s}(0,\eta) - \hat{\mu}_{t,s}(\xi,0) - \hat{\mu}_{t,s}(0,0) + \\ =\quad & \left(\hat{\mu}_{t,s}(\xi,\eta)-\hat{\mu}_{t,s}(0,\eta)\right)\left(1-(\psi_s\hat{)}(\eta)\right) \\ & + \left(\hat{\mu}_{t,s}(\xi,0)-\hat{\mu}_{t,s}(0,0)\right)\left(1-(\varphi_t\hat{)}(\xi)\right). \end{aligned}$$

By the last inequality we have

$$\left|\hat{\sigma}_{k,j}(\xi,\eta)\right| \leq C(q-1)^{-2}\left|\theta^k\xi\right|^{\frac{\beta}{q'}}\left|\theta^j\eta\right|^{\frac{\beta}{2q'}}\|\Phi\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)} \tag{3.13}$$

for some positive constants C and β.

By (3.4) we have

$$\begin{aligned} M_\Phi f(x,y) \quad \leq \quad & g\left(f\right)(x,y) + C\left(\left(M_{\mathbf{R}^n}\otimes id_{\mathbf{R}^m}\right)\left(\sigma^{(1)*}f(x,y)\right)\right) \\ & + \left(\left(id_{\mathbf{R}^n}\otimes M_{\mathbf{R}^m}\right)\left(\sigma^{(2)*}f(x,y)\right)\right) + C(q-1)^{-2}\left(M_{\mathbf{R}^n}\otimes M_{\mathbf{R}^m}\right) \end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\sigma^* f(x,y) \quad &\leq \quad g\,(f)\,(x,y) + 2C\left((M_{\mathbf{R}^n} \otimes id_{\mathbf{R}^m})\,(\sigma^{(1)*} f(x,y))\right) \\ &+ \left((id_{\mathbf{R}^n} \otimes M_{\mathbf{R}^m}\,) \left(\sigma^{(2)*} f(x,y)\right)\right) + C(q-1)^{-2}\,(M_{\mathbf{R}^n} \otimes M_{\mathbf{R}^m}\,)\,, \qquad (3.15)\end{aligned}$$

where $M_{\mathbf{R}^d}$ is the classical Hardy-Littlewood maximal function on $\mathbf{R}^d$.

Now,

$$\begin{aligned}&\left|\sigma_{k,j}^{(1)} * f(x,y)\right| \\ \leq \quad & C(q-1)^{-1} \int_{\theta^j}^{\theta^{(j+1)}} \int_{\mathbf{R}^m} |\ f_x\,(y-v)| \left|\widetilde{\Phi}_s\,(v)\right| dv \frac{ds}{s} \\ = \quad & C(q-1)^{-1}\ \zeta^*(f_x),\end{aligned}$$

where

$$f_x(y) = f(x,y),\ \widetilde{\Phi}_s\,(v) = \int_{\mathbf{R}^m} |\Phi_{t,s}\,(u,v)|\,du = \int_{\mathbf{R}^m} |\Phi_{1,s}\,(u,v)|\,du$$

and

$$\zeta^*(f_x) = \int_{\theta^j}^{\theta^{(j+1)}} |f_x| * \left|\widetilde{\Phi}_s\right| \frac{ds}{s}.$$

It is easy to verify that $\widetilde{\Phi}$ is a function on $\mathbf{R}^m$ which is of compact support, $\widetilde{\Phi} \in L^1(\mathbf{R}^m)$ and $\widetilde{\Phi} \in L^q(\mathbf{R}^m)$. Then by following the proof of the L^p boundedness of the corresponding maximal function in the one-parameter setting in [29], we get

$$\|\zeta^*(f_x)\|_{L^p(\mathbf{R}^m)} \leq C_p(q-1)^{-1}\ \left\|\widetilde{\Phi}\right\|_{L^q(\mathbf{R}^m)} \|f_x\|_{L^p(\mathbf{R}^m)}$$

for $1 < p < \infty$ and $f \in L^p\,(\mathbf{R}^m)\,,$ which in turn implies

$$\left\|\sigma^{(1)*}(f)\right\|_p \leq C_p(q-1)^{-2}\,\|\Phi\|_{L^q(\mathbf{R}^n \times \mathbf{R}^m)}\,\|f\|_p \qquad (3.16)$$

for $1 < p < \infty$ and $f \in L^p\,(\mathbf{R}^n \times \mathbf{R}^m)\,.$ By the same method employed in proving (3.16), we have

$$\left\|\sigma^{(2)*}(f)\right\|_p \leq C_p(q-1)^{-2}\,\|\Phi\|_{L^q(\mathbf{R}^n \times \mathbf{R}^m)}\,\|f\|_p \text{ for } 1 < p < \infty \text{ and } f \in L^p\,(\mathbf{R}^n \times \mathbf{R}^m)\,. \qquad (3.17)$$

By (3.6), (3.9), (3.11)–(3.13) and Plancherel's theorem, we obtain

$$\|g\,(f)\|_2 \leq C(q-1)^{-2}\,\|\Phi\|_{L^q(\mathbf{R}^n \times \mathbf{R}^m)}\,\|f\|_2\,. \qquad (3.18)$$

By the L^p boundedness of the Hardy–Littlewood maximal function and (3.16)–(3.18) we get

$$\|\sigma^*\,(f)\|_2 \leq C(q-1)^{-2}\,\|\Phi\|_{L^q(\mathbf{R}^n \times \mathbf{R}^m)}\,\|f\|_2\,. \qquad (3.19)$$

Now, by (3.6), (3.19), applying the proof of the lemma ([16], p. 189) with $p_0 = 4$ and $q = 2$, and using the trivial estimate $\|\sigma_{k,j}\| \leq (q-1)^{-2} \|\Phi\|_{L^q(\mathbf{R}^n \times \mathbf{R}^m)}$, we get

$$\begin{aligned} &\left\| \left(\sum_{k,j\in\mathbf{Z}} |\sigma_{k,j} * g_{k,j}|^2 \right)^{1/2} \right\|_{p_0} \\ \leq \quad & C_{p_0}(q-1)^{-2} \|\Phi\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)} \left\| \left(\sum_{k,j\in\mathbf{Z}} |g_{k,j}|^2 \right)^{1/2} \right\|_{p_0} \end{aligned} \tag{3.20}$$

for arbitrary functions $\{g_{k,j}\}_{k,j\in\mathbf{Z}}$ on $\mathbf{R}^n \times \mathbf{R}^m$. By (3.6), (3.9), (3.11)–(3.13) and applying Lemma 11 in [5], we get

$$\|g(f)\|_p \leq C_p(q-1)^{-2} \|\Phi\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)} \|f\|_p \tag{3.21}$$

for all p satisfying $p \in (4/3, 4)$ and $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$. By replacing $p = 2$ with $p = 4/3+\varepsilon$ $(\varepsilon \to 0^+)$ in (3.19) and repeating the preceding arguments, we get (3.20) for every p satisfying $p \in (8/7, 8)$ and $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$. By continuing this process, we ultimately get

$$\|g(f)\|_p \leq C_p(q-1)^{-2} \|\Phi\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)} \|f\|_p \tag{3.22}$$

for all $p \in (1, \infty)$ and $L^p(\mathbf{R}^n \times \mathbf{R}^m)$. Therefore, by (3.22), and (3.14)–(3.15), we obtain (3.3) to complete the proof of the lemma.

Lemma 3.8. *Let K_1 and K_2 be two compact sets in $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Let Φ be a function supported in $K_1 \times K_2$. Suppose that $\Phi \in L^q(\mathbf{R}^n \times \mathbf{R}^m)$ for some $q > 1$ and $\theta = 2^{q'}$. Let λ be a real number with $\lambda > 1$. Then for $1/\left(\min\{1, \frac{1}{\lambda} + \frac{1}{q'}\}\right) < p < \infty$ there exists a positive constant C_p which is independent of q such that the following inequality*

$$\begin{aligned} &\left\| \left(\sum_{k,j\in\mathbf{Z}} \int_{\theta^j}^{\theta^{(j+1)}} \int_{\theta^k}^{\theta^{(k+1)}} |\Phi_{t,s} * g_{k,j}|^\lambda \frac{dtds}{ts} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \\ \leq \quad & C_p(q-1)^{-\frac{2}{\lambda}} \|\Phi\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)} \left\| \left(\sum_{k,j\in\mathbf{Z}} |g_{k,j}|^\lambda \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \end{aligned} \tag{3.23}$$

holds for arbitrary functions $\{g_{k,j}(\cdot,\cdot)\}_{k,j\in\mathbf{Z}}$ on $\mathbf{R}^n \times \mathbf{R}^m$.

Proof. The proof of this lemma will be similar to the proof of Lemma 2.5 in [4]. For completeness and rigor, we present its proof here. We need to consider two cases:

Case 1. $p \geq \lambda$. This is further divided into two subcases.

Case 1 (i): $p > \lambda$. By duality there exists a nonnegative function b in $L^{(p/\lambda)'}(\mathbf{R}^n \times \mathbf{R}^m)$ with

$\|b\|_{(p/\lambda)'} \leq 1$ such that

$$\begin{aligned}
&\left\| \left(\sum_{k,j\in\mathbf{Z}} \int_{\theta^j}^{\theta^{(j+1)}} \int_{\theta^k}^{\theta^{(k+1)}} \left|\Phi_{t,s} * g_{k,j}\right|^{\lambda} \frac{dtds}{ts} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}^{\lambda} \\
&= \int_{\mathbf{R}^n\times\mathbf{R}^m} \sum_{k,j\in\mathbf{Z}} \int_{\theta^j}^{\theta^{(j+1)}} \int_{\theta^k}^{\theta^{(k+1)}} \left|\Phi_{t,s} * g_{k,j}(x,y)\right|^{\lambda} b(x,y) \frac{dtds}{ts} dxdy. \qquad (3.24)
\end{aligned}$$

By Hölder's inequality we get

$$\begin{aligned}
&\left|\Phi_{t,s} * g_{k,j}(x,y)\right|^{\lambda} \\
&\leq C \left(\|\Phi\|_{L^1(\mathbf{R}^n\times\mathbf{R}^m)} \right)^{(\lambda/\lambda')} \left(\int_{\mathbf{R}^n\times\mathbf{R}^m} \left|\Phi_{t,s}(u,v)\right| \left|g_{k,j}(x-u,y-v)\right|^{\lambda} dudv \right).
\end{aligned}$$

Thus

$$\begin{aligned}
&\left\| \left(\sum_{k,j\in\mathbf{Z}} \int_{\theta^j}^{\theta^{(j+1)}} \int_{\theta^k}^{\theta^{(k+1)}} \left|\Phi_{t,s} * g_{k,j}\right|^{\lambda} dt/t \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}^{\lambda} \\
&\leq C \left(\|\Phi\|_{L^1(\mathbf{R}^n\times\mathbf{R}^m)} \right)^{(\lambda/\lambda')} \int_{\mathbf{R}^n\times\mathbf{R}^m} \sum_{k,j\in\mathbf{Z}} \int_{\theta^j}^{\theta^{(j+1)}} \int_{\theta^k}^{\theta^{(k+1)}} \int_{\mathbf{R}^n\times\mathbf{R}^m} \left|\Phi_{t,s}(u,v)\right| \times \\
&\quad \left|g_{k,j}(x,y)\right|^{\lambda} b(x+u,y+v) \frac{dtds}{ts} dudvdxdy. \qquad (3.25)
\end{aligned}$$

By the last inequality

$$\begin{aligned}
&\left\| \left(\sum_{k,j\in\mathbf{Z}} \int_{\theta^j}^{\theta^{(j+1)}} \int_{\theta^k}^{\theta^{(k+1)}} \left|\Phi_{t,s} * g_{k,j}\right|^{\lambda} \frac{dtds}{ts} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}^{\lambda} \\
&\leq C \left(\|\Phi\|_{L^1(\mathbf{R}^n\times\mathbf{R}^m)} \right)^{(\lambda/\lambda')} \int_{\mathbf{R}^n\times\mathbf{R}^m} \left(\sum_{k,j\in\mathbf{Z}} \left|g_{k,j}(x,y)\right|^{\lambda} \right) M_{\Phi}\tilde{b}(-x,-y) dxdy, \qquad (3.26)
\end{aligned}$$

where $\tilde{b}(x,y) = b(-x,-y)$. Thus, by Lemma 3.7, (3.26) and Hölder's inequality, we get (3.23) for $\lambda < p < \infty$.

Case 1 (ii): $p = \lambda$. We notice that

$$\begin{aligned}
&\left\| \left(\sum_{k,j\in\mathbf{Z}} \int_{\theta^j}^{\theta^{(j+1)}} \int_{\theta^k}^{\theta^{(k+1)}} \left|\Phi_{t,s} * g_{k,j}\right|^{\lambda} \frac{dtds}{ts} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}^{\lambda} \\
&= \sum_{k,j\in\mathbf{Z}} \int_{\mathbf{R}^n\times\mathbf{R}^m} \int_{\theta^j}^{\theta^{(j+1)}} \int_{\theta^k}^{\theta^{(k+1)}} \left|\Phi_{t,s} * g_{k,j}(x,y)\right|^{\lambda} \frac{dtds}{ts} dxdy.
\end{aligned}$$

By Fubini's theorem, Hölder's inequality and the support of Φ we have

$$\begin{aligned}
&\left\|\left(\sum_{k,j\in\mathbf{Z}}\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}}\left|\Phi_{t,s}*g_{k,j}\right|^{\lambda}\frac{dtds}{ts}\right)^{1/\lambda}\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}^{\lambda} \\
\leq\ & C\left(\|\Phi\|_{L^1(\mathbf{R}^n\times\mathbf{R}^m)}\right)^{(\lambda/\lambda')}\sum_{k,j\in\mathbf{Z}}\int_{\mathbf{R}^n\times\mathbf{R}^m}\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}} \\
&\times\left(\int_{\mathbf{R}^n\times\mathbf{R}^m}\left|g_{k,j}(x-u,y-v)\right|^{\lambda}\left|\Phi_{t,s}(u,v)\right|dudv\right)\frac{dtds}{ts}dxdy \\
\leq\ & C(q-1)^{-2}\left(\|\Phi\|_{L^1(\mathbf{R}^n\times\mathbf{R}^m)}\right)^{(\lambda/\lambda'+1)}\int_{\mathbf{R}^n\times\mathbf{R}^m}\left(\sum_{k,j\in\mathbf{Z}}\left|g_{k,j}(x,y)\right|^{\lambda}\right)dxdy,
\end{aligned}$$

which implies (3.23) for the case $p=\lambda$.

Case 2. $1/\left(\min\{1,\frac{1}{\lambda}+\frac{1}{q'}\}\right)<p<\lambda$. By duality, there exist functions $f=f_{k,j,t,s}(x,y)$ defined on $\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R}_+\times\mathbf{R}_+$ with $\left\|\left\|\|f_{k,j,t,s}\|_{L^{\lambda'}([\theta^k,\theta^{k+1}]\times[\theta^j,\theta^{j+1}],\frac{dtds}{ts})}\right\|_{l^{\lambda'}}\right\|_{L^{p'}}\leq 1$ such that

$$\begin{aligned}
&\left\|\left(\sum_{k,j\in\mathbf{Z}}\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}}\left|\Phi_{t,s}*g_{k,j}\right|^{\lambda}\frac{dtds}{ts}\right)^{1/\lambda}\right\|_{p} \\
=\ & \int_{\mathbf{R}^n\times\mathbf{R}^m}\sum_{k,j\in\mathbf{Z}}\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}}\left(\Phi_{t,s}*g_{k,j}(x,y)\right)f_{k,j,t,s}(x,y)\frac{dtds}{ts}dxdy \\
\leq\ & C_p(q-1)^{-\frac{2}{\lambda}}\left\|\left(\sum_{k,j\in\mathbf{Z}}\left|g_{k,j}\right|^{\lambda}\right)^{1/\lambda}\right\|_{p}\left\|(H(f))^{1/\lambda'}\right\|_{p'}, \qquad (3.27)
\end{aligned}$$

where

$$Hf(x,y)=\sum_{k,j\in\mathbf{Z}}\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}}\left|\Phi_{t,s}*f_{k,j,t,s}(x,y)\right|^{\lambda'}\frac{dtds}{ts}.$$

Since $p'>\lambda'$, there is a nonnegative function $F\in L^{(p'/\lambda')'}(\mathbf{R}^n\times\mathbf{R}^m)$ such that $\|F\|_{(p'/\lambda')'}\leq 1$ and

$$\begin{aligned}
&\|H(f)\|_{p'/\lambda'} \\
=\ & \sum_{k,j\in\mathbf{Z}}\int_{\mathbf{R}^n\times\mathbf{R}^m}\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}}\left|\Phi_{t,s}*f_{k,j,t,s}(x,y)\right|^{\lambda'}\frac{dtds}{ts}F(x,y)dxdy. \qquad (3.28)
\end{aligned}$$

Again, we shall divide the discussion into two subcases.

Case 2 (i): $q<\lambda$. By Hölder's inequality we get

$$\begin{aligned}\left|\Phi_{t,s}*f_{k,j,t,s}(x,y)\right|^{\lambda'} &\leq C\left(\int_{\mathbf{R}^n\times\mathbf{R}^m}\left|\Phi_{t,s}(x-u,y-v)\right|^{q}dudv\right)^{(\lambda'/\lambda)}\times\\ &\left(\int_{\mathbf{R}^n\times\mathbf{R}^m}\left|\Phi_{t,s}(x-u,y-v)\right|^{(\frac{\lambda-q}{\lambda-1})}\left|f_{k,j,t,s}(u,v)\right|^{\lambda'}dudv\right)\end{aligned}$$

and hence

$$\begin{aligned}\left|\Phi_{t,s}*f_{k,j,t,s}(x,y)\right|^{\lambda'} &\leq C\left(t^{n(1-q)}s^{m(1-q)}\left\|\Phi\right\|^{q}_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\right)^{(\lambda'/\lambda)}\times\\ &\left(\int_{\mathbf{R}^n\times\mathbf{R}^m}\left|\Phi_{t,s}(x-u,y-v)\right|^{(\frac{\lambda-q}{\lambda-1})}\left|f_{k,j,t,s}(u,v)\right|^{\lambda'}dudv\right). \quad (3.29)\end{aligned}$$

By (3.28)–(3.29) we easily get

$$\begin{aligned}&\left\|H(f)\right\|_{p'/\lambda'}\\ &\leq C\left(t^{n(1-q)}s^{m(1-q)}\left\|\Phi\right\|^{q}_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\right)^{(\lambda'/\lambda)}\int_{\mathbf{R}^n\times\mathbf{R}^m}\sum_{k,j\in\mathbf{Z}}\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}}\left|f_{k,j,t,s}(u,v)\right|^{\lambda'}\\ &\times\left(\int_{\mathbf{R}^n\times\mathbf{R}^m}\left|\Phi_{t,s}(x-u,y-v)\right|^{(\frac{\lambda-q}{\lambda-1})}F(x,y)dxdy\right)\frac{dtds}{ts}dudv. \quad (3.30)\end{aligned}$$

By Hölder's inequality and the support of Φ, we obtain

$$\begin{aligned}&\int_{\mathbf{R}^n\times\mathbf{R}^m}\left|\Phi_{t,s}(x-u,y-v)\right|^{(\frac{\lambda-q}{\lambda-1})}F(x,y)dxdy\\ &\leq\left(\int_{\mathbf{R}^n\times\mathbf{R}^m}\left|\Phi_{t,s}(u,v)\right|^{q}dudv\right)^{\frac{\lambda-q}{q(\lambda-1)}}\left(\int_{|y-v|\leq s}\int_{|x-u|\leq t}\left|F(x,y)\right|^{\frac{q'}{\lambda'}}dxdy\right)^{\frac{\lambda'}{q'}}\\ &\leq s^{\frac{-m(\lambda-q)}{q'(\lambda-1)}}t^{\frac{-n(\lambda-q)}{q'(\lambda-1)}}\left(\left\|\Phi\right\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\right)^{(\frac{\lambda-q}{\lambda-1})}\left(\int_{|y-v|\leq s}\int_{|x-u|\leq t}\left|F(x,y)\right|^{\frac{q'}{\lambda'}}dxdy\right)^{\frac{\lambda'}{q'}}\\ &\leq s^{\frac{-m(\lambda-q)}{q'(\lambda-1)}}t^{\frac{-n(\lambda-q)}{q'(\lambda-1)}}\left(\left\|\Phi\right\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\right)^{(\frac{\lambda-q}{\lambda-1})}\left(M_{\mathbf{R}^n\times\mathbf{R}^m}\left(\left|F(u,v)\right|^{\frac{q'}{\lambda'}}\right)\right)^{\frac{\lambda'}{q'}}, \quad (3.31)\end{aligned}$$

where $M_{\mathbf{R}^n\times\mathbf{R}^m}$ denotes the strong Hardy–Littlewood maximal function on $\mathbf{R}^n\times\mathbf{R}^m$. Thus by (3.30)–(3.31), Hölder's inequality, the L^p $(1<p\leq\infty)$ boundedness of $M_{\mathbf{R}^n\times\mathbf{R}^m}$ and the choice of F, we obtain

$$\left\|H(f)\right\|_{p'/\lambda'}\leq C\left\|\Phi\right\|^{\lambda'}_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\left\|\sum_{k,j\in\mathbf{Z}}\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}}\left|f_{k,j,t,s}(\cdot,\cdot)\right|^{\lambda'}\frac{dtds}{ts}\right\|_{p'/\lambda'}. \quad (3.32)$$

Thus by (3.27) and (3.32), we get (3.23) in the case $1/\left(\frac{1}{\lambda}+\frac{1}{q'}\right)<p<\lambda$ and $q<\lambda$.

Case 2 (ii): $q \geq \lambda$. We shall again follow the same argument as in the proof of (3.32). By Hölder's inequality and the support of Φ, we obtain

$$|\Phi_{t,s} * f_{k,j,t,s}(x,y)|^{\lambda'} \leq$$

$$\begin{aligned}
&\left(\int_{|y-v|\leq s}\int_{|x-u|\leq t}|\Phi_{t,s}(x-u,y-v)|^{\lambda}\,dudv\right)^{(\lambda'/\lambda)} \\
&\times\left(\int_{|y-v|\leq s}\int_{|x-u|\leq t}|f_{k,j,t,s}(u,v)|^{\lambda'}\,dudv\right) \\
\leq\;& s^{-\frac{m\lambda'}{q'}}t^{-\frac{n\lambda'}{q'}}\left(\|\Phi\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\right)^{\lambda'}\left(\int_{|y-v|\leq s}\int_{|x-u|\leq t}dudv\right)^{\lambda'(\frac{1}{\lambda}-\frac{1}{q})} \\
&\times\left(\int_{|y-v|\leq s}\int_{|x-u|\leq t}|f_{k,j,t,s}(u,v)|^{\lambda'}\,dudv\right) \\
\leq\;& s^{-m}t^{-n}\left(\|\Phi\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\right)^{\lambda'}\left(\int_{|y-v|\leq s}\int_{|x-u|\leq t}|f_{k,j,t,s}(u,v)|^{\lambda'}\,dudv\right), \qquad (3.33)
\end{aligned}$$

which when combined with (3.28) implies that

$$\begin{aligned}
&\|H(f)\|_{p'/\lambda'} \\
\leq\;& Ct^{-n}s^{-m}\left(\|\Phi\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\right)^{\lambda'}\int_{\mathbf{R}^n\times\mathbf{R}^m}\sum_{k,j\in\mathbf{Z}}\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}}|f_{k,j,t,s}(u,v)|^{\lambda'}\times \\
&\left(\int_{|y-v|\leq s}\int_{|x-u|\leq t}F(x,y)dxdy\right)\frac{dtds}{ts}dudv \\
\leq\;& \left(\|\Phi\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\right)^{\lambda'}\int_{\mathbf{R}^n\times\mathbf{R}^m}\sum_{k,j\in\mathbf{Z}}\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}}|f_{k,j,t,s}(u,v)|^{\lambda'}\times \\
&M_{\mathbf{R}^n\times\mathbf{R}^m}F(u,v)\frac{dtds}{ts}dudv. \qquad (3.34)
\end{aligned}$$

By Hölder's inequality, the L^p $(1 < p \leq \infty)$ boundedness of $M_{\mathbf{R}^n\times\mathbf{R}^m}$, and the choices of f and F, we get

$$\|H(f)\|_{p'/\lambda'} \leq C\left(\|\Phi\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\right)^{\lambda'}\|F\|_{(p'/\lambda')'}, \qquad (3.35)$$

which in turn ends the proof of (3.23) in the case $p < \lambda$ and $q \geq \lambda$.

Lemma 3.9. *Suppose $n > 1$. Let K_1 and K_2 be two compact sets in $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Let Φ be a function supported in $K_1 \times K_2$ and $|\Phi(x,y)| \leq \tilde{\Phi}(x,y)$ for some function $\tilde{\Phi} \in L^q(\mathbf{R}^n \times \mathbf{R}^m)$ for some $q > 1$ with $\tilde{\Phi}(x,y) = \varphi(|x|,|y|)$ for some function φ* defined on $(0,\infty)\times(0,\infty)$. *Let λ be a real number with $\lambda > q$. Then for $(\lambda\omega q)/(\lambda\omega q - \lambda + q) < p < \infty$ there exists a positive constant C_p such that the following* *inequality*

$$\begin{aligned} & \left\| \left(\sum_{k,j\in\mathbf{Z}} \int_{2^j}^{2^{(j+1)}} \int_{2^k}^{2^{(k+1)}} \left| \Phi_{t,s} * g_{k,j} \right|^{\lambda} dt/t \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \\ \leq \quad & C_p \left\| \Phi \right\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)} \left\| \left(\sum_{k,j\in\mathbf{Z}} \left| g_{k,j} \right|^{\lambda} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \end{aligned} \tag{3.36}$$

holds for arbitrary functions $\{g_{k,j}(\cdot,\cdot)\}_{k,j\in\mathbf{Z}}$ *on* $\mathbf{R}^n\times\mathbf{R}^m$, *where* $\omega=\min\{n,m\}$.

Proof. The proof of (3.36) for the case $p\geq\lambda$ follows from Lemma 3.8. So we just need to prove (3.36) for the case $(\lambda\omega q)/(\lambda\omega q-\lambda+q)<p<\lambda$. We follow the steps of the proof of (3.23) for the case $p<\lambda$ until we reach (3.29). By Hölder's inequality and the support of Φ, we obtain

$$\begin{aligned} & \int_{\mathbf{R}^n\times\mathbf{R}^m} \left| \Phi_{t,s}(x-u,y-v) \right|^{\left(\frac{\lambda-q}{\lambda-1}\right)} F(x,y)dxdy \\ \leq \quad & \int_0^\infty \int_0^\infty \left| \varphi_{t,s}(r,w) \right|^{\left(\frac{\lambda-q}{\lambda-1}\right)} \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} F(x-ru,y-wv)d\sigma(u)d\sigma(v)r^{n-1}w^{m-1}drdw \\ \leq \quad & \left(\int_0^\infty \int_0^\infty \left| \varphi_{t,s}(r,w) \right|^{q} r^{n-1}w^{m-1}drdw \right)^{\frac{\lambda-q}{q(\lambda-1)}} \times \\ & \left(\int_0^s \int_0^t \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} F(x-ru,y-wv)d\sigma(u)d\sigma(v) \right)^{\frac{q'}{\lambda'}} r^{n-1}w^{m-1}drdw \right)^{\frac{\lambda'}{q'}} \\ \leq \quad & s^{\frac{m(q-1)}{(\lambda-1)}} t^{\frac{n(q-1)}{(\lambda-1)}} \left(\left\| \tilde{\Phi} \right\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)} \right)^{\left(\frac{\lambda-q}{\lambda-1}\right)} N_P^{\left(\frac{q'}{\lambda'}\right)} F(x,y). \end{aligned} \tag{3.37}$$

Thus by (3.30) and (3.37), Hölder's inequality, Lemma 3.6 and the choice of F, we obtain

$$\left\| H(f) \right\|_{p'/\lambda'} \leq C \left\| \Phi \right\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}^{\lambda'} \left\| \sum_{k,j\in\mathbf{Z}} \int_{2^j}^{2^{(j+1)}} \int_{2^k}^{2^{(k+1)}} \left| f_{k,t}(\cdot,\cdot) \right|^{\lambda'} \frac{dtds}{ts} \right\|_{p'/\lambda'}. \tag{3.38}$$

Therefore, by (3.27) and (3.38) we get (3.36) for $(\lambda\omega q)/(\lambda\omega q-\lambda+q)<p<\lambda$. This completes the proof of Lemma 3.9.

Lemma 3.10. *Let* $n>1$ *and let* λ *be a real number with* $\lambda>1$. *Let* Φ *be an* $L^1(\mathbf{R}^n\times\mathbf{R}^m)$ *function, which is supported in* $K_1\times K_2$ *and* $\Phi(x,y)=\varphi(|x|,|y|)$ *for some function* φ *defined on* $(0,\infty)\times(0,\infty)$. *Then* *for* $(\lambda\omega)' < p < \infty$ *there exists a positive constant* C_p *such that the following inequality*

$$\left\| \left(\sum_{k,j\in\mathbf{Z}} \int_{2^j}^{2^{(j+1)}} \int_{2^k}^{2^{(k+1)}} \left|\Phi_{t,s} * g_{k,j}\right|^{\lambda} \frac{dtds}{ts} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}$$

$$\leq \quad C_p \left\|\Phi\right\|_{L^1(\mathbf{R}^n\times\mathbf{R}^m)} \left\| \left(\sum_{k,j\in\mathbf{Z}} \left|g_{k,j}\right|^{\lambda} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \tag{3.39}$$

holds for arbitrary functions $\{g_{k,j}(\cdot,\cdot)\}_{k,j\in\mathbf{Z}}$ *on* $\mathbf{R}^n\times\mathbf{R}^m$, *where* $\omega = \min\{n,m\}$.

Proof. We first prove (3.39) in the case $p \geq \lambda$. We follow the steps of the proof of (3.23) in the case $p > \lambda$ until we reach (3.25). By a change of variable we have

$$\left\| \left(\sum_{k,j\in\mathbf{Z}} \int_{2^j}^{2^{(j+1)}} \int_{2^k}^{2^{(k+1)}} \left|\Phi_{t,s} * g_{k,j}\right|^{\lambda} \frac{dtds}{ts} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}^{\lambda}$$

$$\leq C \left(\left\|\Phi\right\|_{L^1(\mathbf{R}^n\times\mathbf{R}^m)} \right)^{(\lambda/\lambda')} \int_{\mathbf{R}^n\times\mathbf{R}^m} \int_{2^j}^{2^{(j+1)}} \int_{2^k}^{2^{(k+1)}} \int_{\mathbf{R}^n\times\mathbf{R}^m} \left|\Phi_{t,s}(u,v)\right| \times$$

$$\left|g_{k,j}(u,v)\right|^{\lambda} b(x+u,y+v)\frac{dtds}{ts}dxdy. \tag{3.40}$$

By the assumption that $\Phi(x,y) = \varphi(|x|,|y|)$, we obtain

$$\int_{2^j}^{2^{(j+1)}} \int_{2^k}^{2^{(k+1)}} \int_{\mathbf{R}^n\times\mathbf{R}^m} \left|\Phi_{t,s}(u,v)\right| b(x+u,y+v)\frac{dtds}{ts}dxdy$$

$$\leq \quad \int_0^{\infty}\int_0^{\infty} \left|\varphi(r,w)\right| \int_{2^j}^{2^{(j+1)}} \int_{2^k}^{2^{(k+1)}} \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}$$

$$b(u+rt\rho, v+ws\eta)d\sigma(\rho)d\sigma(\eta)\frac{dtds}{ts}r^{n-1}s^{m-1}drdw$$

$$\leq \quad C\left\|\Phi\right\|_{L^1(\mathbf{R}^n\times\mathbf{R}^m)} M_{\mathbf{R}^n\times\mathbf{R}^m}\tilde{b}(u,v), \tag{3.41}$$

where $\tilde{b}(x,y) = b(-x,-y)$. By (3.40)–(3.41) we obtain

$$\left\| \left(\sum_{k,j\in\mathbf{Z}} \int_{2^j}^{2^{(j+1)}} \int_{2^k}^{2^{(k+1)}} \left|\Phi_{t,s} * g_{k,j}\right|^{\lambda} \frac{dtds}{ts} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}^{\lambda}$$

$$\leq \quad C \left(\left\|\Phi\right\|_{L^1(\mathbf{R}^n\times\mathbf{R}^m)} \right)^{(\lambda/\lambda'+1)} \int_{\mathbf{R}^n\times\mathbf{R}^m} \sum_{k,j\in\mathbf{Z}} \left|g_{k,j}(u,v)\right|^{\lambda} M_{\mathbf{R}^n\times\mathbf{R}^m}(\tilde{b})(u,v)dudv.$$

By Hölder's inequality, the L^p $(1 < p \le \infty)$ boundedness of $M_{\mathbf{R}^n\times\mathbf{R}^m}$ and the choice of b, we obtain

$$\begin{aligned} &\left\|\left(\sum_{k,j\in\mathbf{Z}}\int_{2^j}^{2^{(j+1)}}\int_{2^k}^{2^{(k+1)}}\left|\Phi_{t,s}*g_{k,j}\right|^{\lambda}\frac{dtds}{ts}\right)^{1/\lambda}\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}^{\lambda} \\ \le\ & C\left(\|\Phi\|_{L^1(\mathbf{R}^n\times\mathbf{R}^m)}\right)^{(\lambda/\lambda'+1)}\left\|\sum_{k,j\in\mathbf{Z}}\left|g_{k,j}\right|^{\lambda}\right\|_{L^{p/\lambda}(\mathbf{R}^n\times\mathbf{R}^m)}, \end{aligned}$$

which in turn ends the proof of (3.39) for the case $p > \lambda$. The proof of (3.39) in the case $p = \lambda$ will be the same as in the proof of (3.23). We omit the details. Now, we need to prove (3.39) for the case $(\lambda\omega)' < p < \lambda$. We follow the same lines of the proof of (3.23) for the case $p < \lambda$ until we reach (3.28). By Hölder's inequality we get

$$\begin{aligned} \left|\Phi_{t,s}*f_{k,j,t,s}(x,y)\right|^{\lambda'} \le\ & C\left(\int_{\mathbf{R}^n\times\mathbf{R}^m}\left|\Phi_{t,s}(x-u,y-v)\right|dudv\right)^{(\lambda'/\lambda)}\times \\ & \int_{\mathbf{R}^n\times\mathbf{R}^m}\left|\Phi_{t,s}(x-u,y-v)\right|\left|f_{k,j,t,s}(u,v)\right|^{\lambda'}dudv, \end{aligned} \tag{3.42}$$

and hence

$$\begin{aligned} \left|\Phi_{t,s}*f_{k,j,t,s}(x,y)\right|^{\lambda'} \le\ & C\left(s^{m(1-q)}t^{n(1-q)}\|\Phi\|_{L^1(\mathbf{R}^n\times\mathbf{R}^m)}\right)^{(\lambda'/\lambda)}\times \\ & \left(\int_{\mathbf{R}^n\times\mathbf{R}^m}\left|\Phi_{t,s}(x-u,y-v)\right|\left|f_{k,j,t,s}(u,v)\right|^{\lambda'}dudv\right). \end{aligned} \tag{3.43}$$

By (3.28) and (3.34) we get

$$\begin{aligned} &\|H(f)\|_{p'/\lambda'} \\ \le\ & C(s^{m(1-q)}t^{n(1-q)}\|\Phi\|_{L^1(\mathbf{R}^n\times\mathbf{R}^m)})^{(\lambda'/\lambda)}\int_{\mathbf{R}^n\times\mathbf{R}^m}\sum_{k,j\in\mathbf{Z}}\int_{2^j}^{2^{(j+1)}}\int_{2^k}^{2^{(k+1)}}\left|f_{k,j,t,s}(u,v)\right|^{\lambda'} \\ & \times\left(\int_{\mathbf{R}^n\times\mathbf{R}^m}\left|\Phi_{t,s}(x-u,y-v)\right|F(x,y)dxdy\right)\frac{dtds}{ts}dudv. \end{aligned} \tag{3.44}$$

Since $\Phi(x,y) = \varphi(|x|,|y|)$, we get

$$\begin{aligned} &\int_{\mathbf{R}^n\times\mathbf{R}^m}\left|\Phi_{t,s}(x-u,y-v)\right|F(x,y)dxdy \\ \le\ & \int_0^{\infty}\int_0^{\infty}|\varphi(r,w)|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}F(u+rt\rho,v+ws\eta) \\ & d\sigma(\rho)d\sigma(\eta)\frac{dtds}{ts}r^{n-1}s^{m-1}drdw \\ \le\ & C\|\Phi\|_{L^1(\mathbf{R}^n\times\mathbf{R}^m)}N_P^{(\infty)}\tilde{F}(u,v), \end{aligned} \tag{3.45}$$

where $\tilde{F}(x,y)=F(-x,-y)$. Thus by (3.44)–(2.45), Hölder's inequality, the L^p $(p>\max\{n',m'\})$ boundedness of $N_P^{(\infty)}$ and the choice of F we obtain

$$\|H(f)\|_{p'/\lambda'}\le C\,\|\Phi\|_{L^1(\mathbf{R}^n\times\mathbf{R}^m)}^{(\lambda'/\lambda+1)}\left\|\sum_{k,j\in\mathbf{Z}}\int_{2^j}^{2^{(j+1)}}\int_{2^k}^{2^{(k+1)}}|f_{k,j,t,s}(\cdot,\cdot)|^{\lambda'}\frac{dtds}{ts}\right\|_{p'/\lambda'}. \tag{3.46}$$

Therefore by (3.28) and (3.44)–(3.46) we get (3.36) in the case $(\lambda\omega)'<p<\lambda$.

4. Proof of main results

Proof of Theorem 2.1. For $q>1$, let $\theta=2^{q'}$. Let $\{\psi_k\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0,\infty)$ adapted to the intervals $I_k=[\theta^{-(k+1)},\ \theta^{-(k-1)}]$. More precisely, we require the following:

$$\begin{aligned}\psi_k&\in C^\infty,\ 0\le\psi_k\le1,\ \sum_k\psi_k(t)=1;\\ \text{supp }\psi_k&\subseteq I_k;\\ \left|\frac{d^s\psi_k(t)}{dt^s}\right|&\le\frac{C}{t^s},\end{aligned}$$

where C can be chosen to be independent of q. For $k\in\mathbf{Z}$ and $\xi\in\mathbf{R}^n$, let $\widehat{\Psi_{n,k}}(\xi)=\psi_k(|\xi|)$. Decompose

$$\begin{aligned}&f*\Phi_{t,s}(x,y)\\ =&\sum_{\mu,\nu\in\mathbf{Z}}\sum_{k,j\in\mathbf{Z}}\left((\Psi_{m,j+\nu}\oplus\Psi_{n,k+\mu})*f*\Phi_{t,s}\right)(x,y)\chi_{[\theta^j,\theta^{(j+1)})}(s)\chi_{[\theta^k,\theta^{(k+1)})}(t)\\ :=&\sum_{\mu,\nu\in\mathbf{Z}}\Upsilon_{\mu,\nu}(x,y,t,s),\end{aligned}$$

and define

$$T_{\Phi,\mu,\nu}^{(\lambda)}(f)(x,y)=\left(\int_0^\infty\int_0^\infty|\Upsilon_{\mu,\nu}(x,y,t,s)|^\lambda\frac{dtds}{ts}\right)^{\frac{1}{\lambda}}.$$

Then

$$T_\Phi^{(\lambda)}(f)\le\sum_{\mu,\nu\in\mathbf{Z}}T_{\Phi,\mu,\nu}^{(\lambda)}(f). \tag{4.1}$$

Therefore, by the last inequality we notice that (2.1) is proved if we show that

$$\begin{aligned}&\left\|T_{\Phi,\mu,\nu}^{(\lambda)}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}\\ \le\ &C2^{-\vartheta|\mu|}2^{-\vartheta|\nu|}(q-1)^{-2/\lambda}\,\|\Phi\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\,\|f\|_{F_p^{0,\lambda}(\mathbf{R}^n\times\mathbf{R}^m)}\end{aligned} \tag{4.2}$$

for $1/\left(\min\{1,\frac{1}{\lambda}+\frac{1}{q'}\}\right)<p<\infty$ for some positive constants C and ϑ. The proof is based on a sharp L^2 estimate and a cruder L^p estimate. We start by proving the L^2 estimate. First we need to get some Fourier transform estimates. By definition,

$$\widehat{(\Phi_{t,s})}(\xi,\eta)=\int_{\mathbf{R}^n\times\mathbf{R}^m} e^{-it\eta\cdot y}e^{-it\xi\cdot x}\Phi(x,y)dxy.$$

Since Φ is supported in the compact set $K_1\times K_2$ we easily get

$$\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}}\left|\widehat{(\Phi_{t,s})}(\xi,\eta)\right|^2\frac{dtds}{ts}\le C(q-1)^{-2}\left\|\Phi\right\|^2_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}. \tag{4.3}$$

Next, since Φ satisfies (1.1) and it is of compact support, we get

$$\left|\widehat{(\Phi_{t,s})}(\xi,\eta)\right|\le C\left|t\xi\right|\left\|\Phi\right\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\qquad \text{for } t,s\in\mathbf{R}_+ \tag{4.4}$$

and

$$\left|\widehat{(\Phi_{t,s})}(\xi,\eta)\right|\le C\left|s\eta\right|\left\|\Phi\right\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\qquad \text{for } t,s\in\mathbf{R}_+. \tag{4.5}$$

Therefore, by (4.4) we have

$$\begin{aligned}&\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}}\left|\widehat{(\Phi_{t,s})}(\xi,\eta)\right|^2\frac{dtds}{ts}\\ \le\quad & C(q-1)^{-2}\left|\theta^{(k+1)}\xi\right|^2\left\|\Phi\right\|^2_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}.\end{aligned} \tag{4.6}$$

By combining the estimates (4.3) and (4.6) we obtain

$$\begin{aligned}&\left(\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}}\left|\widehat{(\Phi_{t,s})}(\xi,\eta)\right|^2\frac{dtds}{ts}\right)^{1/2}\\ \le\quad & C(q-1)^{-1}\left\|\Phi\right\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\left|\theta^k\xi\right|^{\frac{\beta}{2q'}}\ \text{ for some }\beta>0.\end{aligned} \tag{4.7}$$

Similarly, by (4.5) we have

$$\begin{aligned}&\left(\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}}\left|\widehat{(\Phi_{t,s})}(\xi,\eta)\right|^2\frac{dtds}{ts}\right)^{1/2}\\ \le\quad & C(q-1)^{-1}\left\|\Phi\right\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\left|\theta^j\eta\right|^{\frac{\beta}{2q'}}\ \text{ for some }\beta>0.\end{aligned} \tag{4.8}$$

Finally, by the arguments employed in the proof in Lemma 3.7 we have

$$\left(\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}}\left|\widehat{(\Phi_{t,s})}(\xi)\right|^2\frac{dtds}{ts}\right)^{1/2}\le C(q-1)^{-1}\left|\theta^k\xi\right|^{-\frac{\beta}{2q'}}\left\|\Phi\right\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)} \tag{4.9}$$

and

$$\left(\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}}\left|\widehat{(\Phi_{t,s})}(\xi)\right|^2\frac{dtds}{ts}\right)^{1/2}\le C(q-1)^{-1}\left|\theta^j\eta\right|^{-\frac{\beta}{2q'}}\left\|\Phi\right\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}. \tag{4.10}$$

By (4.3), (4.7)–(4.10) we have

$$\left(\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}}\left|\widehat{(\Phi_{t,s})}(\xi)\right|^2\frac{dtds}{ts}\right)^{1/2}\leq C(q-1)^{-1}\left|\theta^k\xi\right|^{\pm\frac{\beta}{q'}}\left|\theta^j\eta\right|^{\pm\frac{\beta}{q'}}\|\Phi\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)} \tag{4.11}$$

for some positive constants C and β.

Now, by Plancherel's theorem we have

$$\begin{aligned}
&\left\|T^{(\lambda)}_{\Phi,\mu,\nu}(f)\right\|^2_{L^2(\mathbf{R}^n\times\mathbf{R}^m)}\\
=\;&\sum_{k,j\in\mathbf{Z}}\int_{\mathbf{R}^n\times\mathbf{R}^m}\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}}\left|(\Psi_{n,k+\mu}\oplus\Psi_{m,j+\nu})*\Phi_{t,s}*f(x,y)\right|^2\frac{dtds}{ts}dxdy\\
\leq\;&\sum_{k,j\in\mathbf{Z}}\int_{I_{j+v}}\int_{I_{k+\mu}}\left(\int_{\theta^j}^{\theta^{(j+1)}}\int_{\theta^k}^{\theta^{(k+1)}}\left|\widehat{(\Phi_{t,s})}(\xi,\eta)\right|^2\frac{dtds}{ts}\right)\left|\hat{f}(\xi,\eta)\right|^2d\xi d\eta\\
\leq\;&C(q-1)^{-2}\|\Phi\|^2_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\\
&\times\sum_{k,j\in\mathbf{Z}}\int_{I_{j+v}}\int_{I_{k+\mu}}\left(\left|\theta^j\eta\right|^{\pm\frac{\beta}{q'}}\left|\theta^k\xi\right|^{\pm\frac{\beta}{q'}}\right)\left|\hat{f}(\xi,\eta)\right|^2d\xi d\eta\\
\leq\;&C(q-1)^{-2}2^{-\vartheta|\mu|}2^{-\vartheta|\nu|}\|\Phi\|^2_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\sum_{k,j\in\mathbf{Z}}\int_{I_{j+v}}\int_{I_{k+\mu}}\left|\hat{f}(\xi,\eta)\right|^2d\xi d\eta\\
\leq\;&C(q-1)^{-2}2^{-\vartheta|\mu|}2^{-\vartheta|\nu|}\|\Phi\|^2_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\|f\|^2_{L^2(\mathbf{R}^n\times\mathbf{R}^m)}.
\end{aligned} \tag{4.12}$$

By (4.12) and $\|f\|_{F^{\vec{0}}_{2,2}(\mathbf{R}^n\times\mathbf{R}^m)}=\|f\|_{L^2(\mathbf{R}^n\times\mathbf{R}^m)}$, we obtain (4.2) in the case $p=\lambda=2$.

Now by Lemma 3.8 we have

$$\left\|T^{(\lambda)}_{\Phi,\mu,\nu}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}\leq C(q-1)^{-\frac{2}{\lambda}}\|\Phi\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\|f\|_{F^{\vec{0}}_{p,\lambda}(\mathbf{R}^n\times\mathbf{R}^m)} \tag{4.13}$$

for $1/\left(\min\{1,\frac{1}{\lambda}+\frac{1}{q'}\}\right)<p<\infty$. By interpolating (4.12) with (4.13), we get (4.2). Hence the proof of Theorem 1.1 is complete.

Proof of Theorem 2.2. We point out that this theorem can be proved by the same method as used in the proof of Theorem 2.1, except that one will be using Lemma 3.9 instead of Lemma 3.8. Details are omitted.

Proof of Theorem 2.3. We shall follow an argument similar to the one used in the proof of Theorem 2.1. Let $\theta=2$. By the support of Φ, we have

$$\left|\hat{\Phi}(\xi,\eta)\right|\leq C\,|\xi|\;\text{ and }\;\left|\hat{\Phi}(\xi,\eta)\right|\leq C\,|\eta|$$

for some positive constant C. Therefore,

$$\left(\int_{2^j}^{2^{(j+1)}}\int_{2^k}^{2^{(k+1)}}\left|\widehat{(\Phi_{t,s})}(\xi,\eta)\right|^2\frac{dtds}{ts}\right)^{1/2}\leq C\min\{1,\left|2^{(k+1)}\xi\right|^2\} \tag{4.14}$$

and

$$\left(\int_{2^j}^{2^{(j+1)}}\int_{2^k}^{2^{(k+1)}}\left|\widehat{(\Phi_{t,s})}(\xi,\eta)\right|^2\frac{dtds}{ts}\right)^{1/2}\leq C\min\{1,\left|2^{(j+1)}\eta\right|^2\}. \tag{4.15}$$

Let

$$\mu_k(u)=\int_{\mathbf{S}^{k-1}}e^{iu\cdot v}d\sigma(v)\text{ for }u\in\mathbf{R}^k.$$

It is well-known that

$$|\mu_k(\xi)|\leq C_k\min\{1,|\xi|^{-\frac{(k-1)}{2}}\}. \tag{4.16}$$

Since Φ is supported in $K_1\times K_2$ with $K_1\subseteq\mathbf{R}^n-\{0\}$ and $K_2\subseteq\mathbf{R}^m-\{0\}$ and since $\Phi(x,y)=\varphi(|x|,|y|)$, we may assume that supp(φ)$\subseteq[a,\infty)\times[b,\infty)$ for some positive numbers a and b.

Now

$$\widehat{(\Phi)}(\xi,\eta)=\int_b^\infty\int_a^\infty\varphi(r,w)r^{n-1}w^{m-1}\mu_n(r\xi)\mu_m(w\eta)drdw.$$

By (4.16) we have

$$\left|\widehat{(\Phi_{t,s})}(\xi,\eta)\right|\leq C\min\{1,a^{-\frac{(n-1)}{2}}|t\xi|^{-\frac{(n-1)}{2}}\}\min\{1,b^{-\frac{(m-1)}{2}}|s\eta|^{-\frac{(m-1)}{2}}\}.$$

Thus we have

$$\left(\int_{2^j}^{2^{(j+1)}}\int_{2^k}^{2^{(k+1)}}\left|\widehat{(\Phi_{t,s})}(\xi,\eta)\right|^2\frac{dtds}{ts}\right)^{1/2}\leq C\left|2^k\xi\right|^{\pm\alpha}(\left|2^j\eta\right|^{\pm\alpha}. \tag{4.17}$$

By using (4.17) and following an argument similar to the one used in the proof of (4.12), we get

$$\left\|S^{(\lambda)}_{\Phi,\mu,\nu}(f)\right\|_{L^2(\mathbf{R}^n\times\mathbf{R}^m)}\leq C2^{-\gamma|\mu|}2^{-\gamma|\nu|}\|f\|_{L^2(\mathbf{R}^n\times\mathbf{R}^m)}. \tag{4.18}$$

By invoking Lemma 3.10, we get

$$\left\|S^{(\lambda)}_{\Phi,j}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}\leq C\|f\|_{\dot{F}^{\vec{0}}_{p,\lambda}(\mathbf{R}^n\times\mathbf{R}^m)} \tag{4.19}$$

for $(\lambda\omega)'<p<\infty$, where $\omega=\min\{n,m\}$. By interpolation between (4.18)–(4.19) we obtain

$$\left\|S^{(\lambda)}_{\Omega,h,j}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}\leq C2^{-\eta|\mu|}2^{-\eta|\nu|}\|f\|_{\dot{F}^{0,\lambda}_p(\mathbf{R}^n\times\mathbf{R}^m)} \tag{4.20}$$

for $(\lambda\omega)'<p<\infty$. The proof of Theorem 2.3 is complete.

Proof of Theorem 2.4. Define the maximal operators ζ^*_Φ on $\mathbf{R}^n\times\mathbf{R}^m$ by

$$\zeta^*_\Phi(f)=\sup_{t,s\in\mathbf{R}_+}||\Phi_{t,s}|*f|.$$

Now we shall prove the L^p boundedness of $\zeta_\Phi^*(f)$. We follow an argument similar to the proof of (3.2) in [11]. To this end, we notice that

$$\begin{aligned}
&|\Phi_{t,s} * f(x,y)| \\
\leq\ & \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega(x,y) \int_0^\infty\int_0^\infty f(x-ru, y-wv) h_1(r) h_2(w) r^{n-1} w^{m-1} dr dw d\sigma(u) d\sigma(v) \\
\leq\ & \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega(x,y) \left(M_v^{(m)} \circ M_u^{(n)}\right) f(x,y) \times \\
& \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2^{n(k+1)}\, 2^{m(j+1)} h_1(2^k) h_2(2^j) dr dw d\sigma(u) d\sigma(v) \\
\leq\ & \|h_1(|\cdot|)\|_{L^1(\mathbf{R}^n)} \|h_2(|\cdot|)\|_{L^1(\mathbf{R}^m)} \times \\
& \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega(x,y)| \left(H_v^{(m)} \circ H_u^{(n)}\right) f(x,y) d\sigma(u) d\sigma(v), \qquad (4.21)
\end{aligned}$$

where

$$H_u^{(n)} g(x) = \sup_{r>0} \left(\frac{1}{r} \int_0^r f(x-ru) dr \right)$$

is the Hardy–Littlewood maximal function in the direction of $u \in \mathbf{S}^{n-1}$. By (4.24) and the boundedness of $H_u^{(n)}$ on $L^p (1 < p < \infty)$ with a bound independent of u, we get

$$\|\zeta_\Phi^*(f)\|_p \leq C_p \|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \|h_1(|\cdot|)\|_{L^1(\mathbf{R}^n)} \|h_2(|\cdot|)\|_{L^1(\mathbf{R}^m)} \|f\|_p \qquad (4.22)$$

for $1 < p < \infty$. It is easy to see that

$$M_\Phi(f)(x,y) \leq C(q-1)^{-2} \zeta_\Phi^* f(x,y),$$

which in turn implies that

$$\begin{aligned}
\|M_\Phi(f)\|_p \ \leq\ & C_p (q-1)^{-2} \|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \\
& \times \|h_1(|\cdot|)\|_{L^1(\mathbf{R}^n)} \|h_2(|\cdot|)\|_{L^1(\mathbf{R}^m)} \|f\|_p \qquad (4.23)
\end{aligned}$$

for $1 < p < \infty$. By the proof of Lemma 3.8 we get

$$\begin{aligned}
&\left\| \left(\sum_{k,j\in\mathbf{Z}} \int_{\theta^j}^{\theta^{(j+1)}} \int_{\theta^k}^{\theta^{(k+1)}} |\Phi_{t,s} * g_{k,j}|^\lambda \frac{dtds}{ts} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \leq C_p (q-1)^{-\frac{2}{\lambda}} \|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \\
&\times \|h_1(|\cdot|)\|_{L^1(\mathbf{R}^n)} \|h_2(|\cdot|)\|_{L^1(\mathbf{R}^m)} \left\| \left(\sum_{k,j\in\mathbf{Z}} |g_{k,j}|^\lambda \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \qquad (4.24)
\end{aligned}$$

holds for $1 < p < \infty$ and for arbitrary functions $\{g_{k,j}(\cdot,\cdot)\}_{k,j\in\mathbf{Z}}$ on $\mathbf{R}^n \times \mathbf{R}^m$. By the assumptions on Φ and the proof employed in Lemma 2 in [29], we get

$$\left(\int_{\theta^j}^{\theta^{(j+1)}} \int_{\theta^k}^{\theta^{(k+1)}} \left|\widehat{(\Phi_{t,s})}(\xi,\eta)\right|^2 \frac{dtds}{ts}\right)^{1/2}$$
$$\leq \quad C(q-1)^{-2} \left|\theta^k \xi\right|^{\pm\frac{\alpha}{q'}} \left|\theta^j \eta\right|^{\pm\frac{\alpha}{q'}} \|h_1(|\cdot|)\|_{L^q(\mathbf{R}^n)} \|h_2(|\cdot|)\|_{L^q(\mathbf{R}^m)} \|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}. \tag{4.25}$$

By (4.24)–(4.25) and following an argument similar to one in the proof of Theorem 2.1, we get (2.2) which ends the proof of Theorem 2.4.

Proof of Theorems 2.5, 2.6 and 2.7. We can prove Theorem 2.5 by an extrapolation method similar to the one employed in [29] and [2] along with the estimate in (2.2). A proof of Theorems 2.6 and 2.7 can be obtained by an extrapolation method similar to the one employed in [2] along with the estimate (2.2). We omit the details.

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