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On f-Kenmotsu 3-manifolds with respect to the Schouten–van Kampen connection

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Abstract: In this paper we study some semisymmetry conditions and some soliton types on f-Kenmotsu 3-manifolds with respect to the Schouten–van Kampen connection.

Key words: Schouten–van Kampen connection, f-Kenmotsu manifolds, projective semisymmetric, conharmonical semisymmetric, Einstein manifold, η-Einstein manifold, solitons

1. Introduction

Connected almost contact metric manifolds whose automorphism groups have the maximum dimension were classified in [29] by considering the constant sectional curvature c of plane section containing the characteristic vector field. In case of c being a negative constant, it is well known that the manifold is a warped product space $\mathbb{R}\times_f \mathbb{C}^n$. Kenmotsu studied such type of manifolds and introduced an important class of almost contact Riemannian manifolds, namely Kenmotsu manifolds [15].

A $(2n+1)$-dimensional differentiable almost contact Riemannian manifold (ψ,ξ,η,g) is called [3, 10, 15, 16, 23]:

i) normal, if the almost complex structure defined on the product manifold $M\times\mathbb{R}$ is integrable (equivalently, $[\psi,\psi]+2d\eta\otimes\xi=0$),

ii) almost cosymplectic, if $d\eta=0$ and $d\Phi=0$,

iii) cosymplectic, if it is normal and almost cosymplectic (equivalently, $\nabla\psi=0$, ∇ being covariant differentiation with respect to the Levi-Civita connection),

iv) almost Kenmotsu, if η is closed and $d\Phi=2\eta\wedge\Phi$,

iv) Kenmotsu, if it is normal and almost Kenmotsu,

where Φ denotes the fundamental 2-form of the manifold defined by $\Phi(X,Y)=g(X,\psi Y)$, for all $X,Y\in\chi(M)$. Here $\chi(M)$ is the Lie algebra of differentiable vector fields on M.

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Normal locally conformal almost cosymplectic manifolds were studied by Olszak and Rosca [20] and they gave an differential geometric interpretation of such manifolds which are called f-Kenmotsu manifolds by investigating some curvature properties. Among others they proved that a Ricci symmetric f-Kenmotsu manifold is an Einstein manifold.

The Schouten–van Kampen connection, which is one of the most natural connections adapted to a pair of complementary distributions on a differentiable manifold endowed with an affine connection [2, 12, 24], was used by Solov'ev to investigate hyperdistributions in Riemannian manifolds [25–28]. Then, Olszak studied an almost contact metric structure with respect to the Schouten–van Kampen connection and characterized some classes of almost contact metric manifolds admitting such connection by obtaining certain curvature properties of the Schouten–van Kampen connection on these manifolds [19]. Also, Yıldız studied projectively flat and conharmonically flat f-Kenmotsu 3-manifolds with respect to the Schouten–van Kampen connection [30].

As well known, an almost Ricci soliton is a generalization of an Einstein metric. In a Riemannian manifold (M, g), g is called an almost Ricci soliton if [11]

$$L_V g + 2Ric + 2\mu g = 0, \tag{1.1}$$

where L is the Lie derivative, Ric is the Ricci tensor, V is a complete vector field on M, and μ is a smooth function. Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t} g = -2Ric$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. An almost Ricci soliton is said to be shrinking, steady, and expanding if μ is negative, zero, and positive, respectively. Also, Cho and Kimura introduced the notion of η-Ricci soliton [6]. A Riemannian manifold (M, g) is called an almost η-Ricci soliton if there exists a smooth vector field V such that the Ricci tensor satisfies the following equation

$$L_V g + 2Ric + 2\mu g + 2\delta\eta \otimes \eta = 0, \tag{1.2}$$

where μ and δ are some smooth functions. If the vector field V is the gradient of a potential function $-k$, then g is called an almost gradient Ricci soliton and equation (1.1) assumes the form

$$\nabla\nabla k = Ric + \mu g. \tag{1.3}$$

In dimension 2 and in dimension 3, a Ricci soliton on a compact manifold has constant curvature [11, 13]. For details we refer to Chow and Knopf [7] and Derdzinski [8].

We also recall that a Ricci soliton on a compact manifold is a gradient Ricci soliton [21]. Then Sharma studied Ricci solitons in K-contact manifolds [22]. In a K-contact manifold, the structure vector field ξ is Killing, that is, $L_\xi g = 0$, which is not, in general, true for f-Kenmotsu manifolds.

The concept of Yamabe flow was defined by Hamilton in [11] to solve the Yamabe problem. Yamabe solitons are self-similar solutions for Yamabe flows and they seem to be as singularity models. More clearly, the Yamabe soliton comes from the blow-up procedure along the Yamabe flow, so such solitons have been studied intensively. For further reading we refer the reader to [1, 4, 5, 9, 17].

Almost Yamabe solitons can be viewed as a generalization of Yamabe solitons. If there exists a vector field V on a Riemannian manifold (M, g) satisfying [1]

$$\frac{1}{2}(L_V g) = (scal - \gamma)g, \tag{1.4}$$

where $scal$ is the scalar curvature of M, γ is a smooth function, V is a soliton field, and L is the Lie-derivative, then M is said to be an almost Yamabe soliton. We denote an almost Yamabe soliton by (M, V, γ). Also, an almost Yamabe soliton is said to be steady, expanding, or shrinking, if $\gamma = 0$, $\gamma < 0$, or $\gamma > 0$, respectively. If γ is a constant, then an almost Yamabe soliton becomes a Yamabe soliton. Moreover, it is easy to see that Einstein manifolds are always almost Yamabe solitons. Note that if (M, g) is of constant scalar curvature $scal$, then the Riemannian metric g is called a Yamabe metric.

In the present paper we study some semisymmetry properties and some soliton types of f-Kenmotsu 3-manifolds with the Schouten–van Kampen connection. The paper is organized as follows. After introduction, in the preliminaries, we give f-Kenmotsu manifolds and the Schouten–van Kampen connection. Then we adapt the Schouten–van Kampen connection on f-Kenmotsu 3-manifolds. Section 4 is devoted to projectively semisymmetric f-Kenmotsu 3-manifolds with respect to the Schouten–van Kampen connection. In Section 5, we study conharmonical semisymmetric f-Kenmotsu 3-manifolds with respect to the Schouten–van Kampen connection. In the last section, we focus on some soliton types, namely, almost Ricci solitons, almost η-Ricci solitons, and almost Yamabe solitons on f-Kenmotsu 3-manifolds with respect to the Schouten–van Kampen connection and we give an example.

2. Preliminaries

Let M be a real $(2n + 1)$-dimensional differentiable manifold endowed with an almost contact structure (ψ, ξ, η, g) satisfying

$$\begin{aligned} \psi^2 &= -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \\ \psi\xi &= 0, \quad \eta \circ \psi = 0, \quad \eta(U) = g(U, \xi), \\ g(\psi U, \psi W) &= g(U, W) - \eta(U)\eta(W), \end{aligned} \tag{2.1}$$

for any vector fields $U, W \in \chi(M)$, where I is the identity of the tangent bundle TM, ψ is a tensor field of $(1, 1)$-type, η is a 1-form, ξ is a vector field, and g is a metric tensor field. We say that (M, ψ, ξ, η, g) is an f-Kenmotsu manifold if the Levi-Civita connection of g satisfies [18]:

$$(\nabla_U \psi)(W) = f\{g(\psi U, W)\xi - \eta(W)\psi U\}, \tag{2.2}$$

where $f \in C^\infty(M)$ such that $df \wedge \eta = 0$. If $f = \alpha =$ constant $\neq 0$, then the manifold is an α-Kenmotsu manifold [14]. 1-Kenmotsu manifold is a Kenmotsu manifold [15]. If $f = 0$, then the manifold is cosymplectic [14]. An f-Kenmotsu manifold is said to be *regular* if $f^2 + f' \neq 0$, where $f' = \xi(f)$. For an f-Kenmotsu manifold from (2.1) and (2.2) it follows that

$$\nabla_U \xi = f\{U - \eta(U)\xi\}. \tag{2.3}$$

Then using (2.3), we have

$$(\nabla_U \eta)(W) = f\{g(U, W) - \eta(U)\eta(W)\}. \tag{2.4}$$

The condition $df \wedge \eta = 0$ holds if $\dim M \geq 5$. This does not hold in general if $\dim M = 3$ [20].

On the other hand we have two naturally defined distributions in the tangent bundle TM of M as follows:

$$H = \ker \eta, \qquad V = \operatorname{span}\{\xi\}.$$

Then we have $TM = H \oplus V$, $H \cap V = \{0\}$, and $H \perp V$. This decomposition allows one to define the Schouten–van Kampen connection $\overset{\star}{\nabla}$ over an almost contact metric structure. The Schouten–van Kampen connection $\overset{\star}{\nabla}$ on an almost contact metric manifold with respect to Levi-Civita connection ∇ is defined by [25]

$$\overset{\star}{\nabla}_U W = \nabla_U W - \eta(W)\nabla_U \xi + (\nabla_U \eta)(W)\xi. \tag{2.5}$$

Thus, with the help of the Schouten–van Kampen connection (2.5), many properties of some geometric objects connected with the distributions H, V can be characterized [25–27]. For example g, ξ, and η are parallel with respect to $\overset{\star}{\nabla}$, that is, $\overset{\star}{\nabla}\xi = 0$, $\overset{\star}{\nabla}g = 0$, $\overset{\star}{\nabla}\eta = 0$. Also, the torsion $\overset{\star}{T}$ of $\overset{\star}{\nabla}$ is defined by

$$\overset{\star}{T}(U,W) = \eta(U)\nabla_W \xi - \eta(W)\nabla_U \xi + 2d\eta(U,W)\xi.$$

As it is well known on a 3-dimensional Riemannian manifold, we have

$$\begin{aligned} R(U,W)Z &= g(W,Z)QU - g(U,Z)QW + Ric(W,Z)U - Ric(U,Z)W \\ &\quad -\frac{scal}{2}\{g(W,Z)U - g(U,Z)W\}. \end{aligned} \tag{2.6}$$

Thus, for an f-Kenmotsu 3-manifold M, we write [20]

$$\begin{aligned} R(U,W)Z &= (\frac{scal}{2} + 2f^2 + 2f')\{g(W,Z)U - g(U,Z)W\} \\ &\quad -(\frac{scal}{2} + 3f^2 + 3f')\{g(W,Z)\eta(U)\xi - g(U,Z)\eta(W)\xi \\ &\quad +\eta(W)\eta(Z)U - \eta(U)\eta(Z)W\}, \end{aligned} \tag{2.7}$$

$$Ric(U,W) = (\frac{scal}{2} + f^2 + f')g(U,W) - (\frac{scal}{2} + 3f^2 + 3f')\eta(U)\eta(W), \tag{2.8}$$

$$QU = (\frac{scal}{2} + f^2 + f')U - (\frac{scal}{2} + 3f^2 + 3f')\eta(U)\xi, \tag{2.9}$$

where R denotes the curvature tensor, Ric is the Ricci tensor, Q is the Ricci operator, and $scal$ is the scalar curvature of M. From (2.7) and (2.8), we have

$$R(U,W)\xi = -(f^2 + f')\{\eta(W)U - \eta(U)W\}, \tag{2.10}$$

$$R(\xi,U)W = -(f^2 + f')\{g(U,W)\xi - \eta(W)W\}, \tag{2.11}$$

and

$$Ric(U,\xi) = -2(f^2 + f')\eta(U). \tag{2.12}$$

3. f-Kenmotsu 3-manifolds with respect to the Schouten–van Kampen connection

Let M be an f-Kenmotsu 3-manifold with respect to the Schouten-van Kampen connection. Then using (2.3) and (2.4) in (2.5), we get

$$\overset{\star}{\nabla}_U W = \nabla_U W + f(g(U,W)\xi - \eta(W)U). \tag{3.1}$$

Let R and $\overset{\star}{R}$ be the curvature tensors of the Levi-Civita connection ∇ and the Schouten–van Kampen connection $\overset{\star}{\nabla}$, respectively. Then since we have

$$R(U,W) = [\nabla_U, \nabla_W] - \nabla_{[U,W]}, \qquad \overset{\star}{R}(U,W) = [\overset{\star}{\nabla}_U, \overset{\star}{\nabla}_W] - \overset{\star}{\nabla}_{[U,W]},$$

then by using (3.1), the following formula connecting $\overset{\star}{R}$ and R on an f-Kenmotsu 3-manifold M [30]:

$$\begin{aligned} \overset{\star}{R}(U,W)Z &= R(U,W)Z \\ &+f^2\{g(W,Z)U - g(U,Z)W\} \\ &+f'\{g(W,Z)\eta(U)\xi - g(U,Z)\eta(W)\xi + \eta(W)\eta\star(Z)U - \eta(U)\eta(Z)W\}. \end{aligned} \tag{3.2}$$

If taking the inner product with T in (2.2), then the relations between the Riemann-Christoffel curvature $(0,4)$-tensors $\overset{\star}{R}$ and R; the Ricci tensors $\overset{\star}{Ric}$ and Ric; the Ricci operators $\overset{\star}{Q}$ and Q; the scalar curvatures $\overset{\star}{scal}$ and $scal$ of the connections $\overset{\star}{\nabla}$ and ∇ are given by [30]

$$\begin{aligned} \overset{\star}{R}(U,W,Z,T) &= R(U,W,Z,T) \\ &+f^2\{g(W,Z)g(U,T) - g(U,Z)g(W,T)\} \\ &+f'\{g(W,Z)\eta(U)\eta(T) - g(U,Z)\eta(W)\eta(T) \\ &+g(U,T)\eta(W)\eta(Z) - g(W,T)\eta(U)\eta(Z)\}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \overset{\star}{Ric}(W,Z) &= Ric(W,Z) \\ &+(2f^2 + f')g(W,Z) + f'\eta(W)\eta(Z), \end{aligned} \tag{3.4}$$

$$\overset{\star}{Q}U = QU + (2f^2 + f')U + f'\eta(U)\xi, \tag{3.5}$$

$$\overset{\star}{scal} = scal + 6f^2 + 4f', \tag{3.6}$$

respectively, where $\overset{\star}{R}(U,W,Z,T) = g(\overset{\star}{R}(U,W)Z,T)$ and $R(U,W,Z,T) = g(R(U,W)Z,T)$.

An f-Kenmotsu 3-manifold with respect to the Schouten–van Kampen connection is called η-Einstein if

$$\overset{\star}{Ric}(W,Z) = ag(W,Z) + b\eta(W)\eta(Z),$$

for some real numbers a and b.

4. Projectively semisymmetric f-Kenmotsu 3-manifolds with the Schouten–van Kampen connection

In this section, we study projectively semisymmetric f-Kenmotsu 3-manifolds with respect to the Schouten–van Kampen connection. In an f-Kenmotsu 3-manifold, the projective curvature tensor with respect to the Schouten–van Kampen connection is given by

$$\begin{aligned}\overset{\star}{P}(U,W)Z &= P(U,W)Z - \frac{1}{2}f'\{g(W,Z)U - g(U,Z)W\} \\ &\quad + \frac{1}{2}f'\{\eta(W)\eta(Z)U - \eta(U)\eta(Z)W\} \\ &\quad + f'\{g(W,Z)\eta(U)\xi - g(U,Z)\eta(W)\xi\},\end{aligned} \tag{4.1}$$

where $P(U,W)Z$ is the projective curvature tensor with respect to the Levi-Civita connection and defined by

$$P(U,W)Z = R(U,W)Z - \frac{1}{2}\{Ric(W,Z)U - Ric(U,Z)W\}. \tag{4.2}$$

It is well known that if an f-Kenmotsu manifold with respect to the Schouten–van Kampen connection satisfies the condition

$$\overset{\star}{R}(U,W)\cdot\overset{\star}{P} = L\hat{Q}(g,\overset{\star}{P}),$$

then the manifold is called projectively pseudosymmetric f-Kenmotsu manifold with respect to the Schouten–van Kampen connection, where L is a function and

$$\begin{aligned}\hat{Q}(g,\overset{\star}{P})(Z,E,T;U,W) &= ((U\Lambda W)\overset{\star}{P})(Z,E)T \\ &= -\overset{\star}{P}((U\Lambda W)Z,E)T - \overset{\star}{P}(Z,(U\Lambda W)E)T \\ &\quad -\overset{\star}{P}(Z,E)(U\Lambda W)T,\end{aligned}$$

and

$$(U\Lambda W)Z = g(W,Z)U - g(U,Z)W,$$

respectively. If $L=0$, then the manifold M is called projectively semisymmetric manifold with respect to the Schouten–van Kampen connection.

Let M be a projectively semisymmetric f-Kenmotsu 3-manifold with respect to the Schouten–van Kampen connection. Then, we have

$$(\overset{\star}{R}(U,W)\cdot\overset{\star}{P})(Z,E)T = 0, \tag{4.3}$$

which satisfies

$$\begin{aligned}&\overset{\star}{R}(U,W)\overset{\star}{P}(Z,E)T - \overset{\star}{P}(\overset{\star}{R}(U,W)Z,E)T \\ &-\overset{\star}{P}(Z,\overset{\star}{R}(U,W)E)T - \overset{\star}{P}(Z,E)\overset{\star}{R}(U,W)T = 0.\end{aligned} \tag{4.4}$$

Using (4.1) in (4.4), we have

$$\begin{aligned}&\overset{\star}{R}(U,W)P(Z,E)T - P(\overset{\star}{R}(U,W)Z,E)T \\ &-P(Z,\overset{\star}{R}(U,W)E)T - P(Z,E)\overset{\star}{R}(U,W)T = 0.\end{aligned} \tag{4.5}$$

Now using (3.2) in (4.5), we get

$$
\begin{aligned}
& R(U,W)P(Z,E)T - P(R(U,W)Z,E)T - P(Z,R(U,W)E)T \\
& -P(Z,E)R(U,W)T + f^2\{g(W,P(Z,E)T)U - g(U,P(Z,E)T)W \\
& -g(W,Z)P(U,E)T + g(U,Z)P(W,E)T - g(W,E)P(Z,U)T \\
& +g(U,E)P(Z,W)T - g(W,T)P(Z,E)U + g(U,T)P(Z,E)W\} \\
& +f'\{g(P(Z,E)T,W)\eta(U)\xi - g(P(Z,E)T,U)\eta(W)\xi \\
& +\eta(P(Z,E)T)\eta(W)U - \eta(P(Z,E)T)\eta(U)W \\
& -g(W,Z)\eta(P(U,E)T)\xi + g(U,Z)\eta(P(W,E)T)\xi \\
& -\eta(W)\eta(Z)P(U,E)T + \eta(U)\eta(Z)P(W,E)T \\
& -g(W,E)\eta(P(Z,U)T)\xi + g(U,E)\eta(P(Z,W)T)\xi \\
& -\eta(W)\eta(E)P(Z,U)T + \eta(U)\eta(E)P(Z,W)T \\
& -g(W,T)\eta(P(Z,E)U)\xi + g(U,T)\eta(P(Z,E)W)\xi \\
& -\eta(W)\eta(T)P(Z,E)U + \eta(U)\eta(T)P(Z,E)W\} = 0.
\end{aligned} \tag{4.6}
$$

Now from (4.6), we can say: If $0 \neq f = \text{constant}$ (say $f = \alpha$) then $f' = 0$. Hence, we get $R \cdot P = -\alpha^2 Q(g,P)$. Therefore, the manifold M is a projectively pseudosymmetric α-Kenmotsu manifold. Using (2.7) and (4.2) in (4.6), we obtain

$$
\begin{aligned}
& R(U,W)R(Z,E)T - R(R(U,W)Z,E)T - R(Z,R(U,W)E)T \\
& -R(Z,E)R(U,W)T \\
= & \frac{1}{2}\{Ric(R(U,W)Z,T)E + Ric(Z,R(U,W)T)E - Ric(R(U,W)E,T)Z \\
& -Ric(R(U,W)T,E)Z\} - f^2\{g(W,R(Z,E)T)U - g(U,R(Z,E)T)W \\
& -g(W,Z)R(U,E)T + g(U,Z)R(W,E)T \\
& -g(W,E)R(Z,U)T + g(U,E)R(Z,W)T - g(W,T)R(Z,E)U \\
& +g(U,T)R(Z,E)W + \frac{f^2}{2}\{Ric(W,T)g(U,E)Z - Ric(U,E)g(W,T)Z \\
& +Ric(W,E)g(U,T)Z - Ric(W,Z)g(U,T)Z \\
& -Ric(W,T)g(U,Z)E + Ric(U,Z)g(W,T)E\}.
\end{aligned} \tag{4.7}
$$

Again using (2.8) in (4.7), we get

$$\begin{aligned}
& R(U,W)R(Z,E)T - R(R(U,W)Z,E)T - R(Z,R(U,W)E)T \\
& -R(Z,E)R(U,W)T \\
= \; & \frac{B}{2}\{\eta(R(U,W)E)\eta(T)Z - \eta(R(U,W)Z)\eta(T)E + \eta(R(U,W)T)\eta(E)Z \\
& -\eta(R(U,W)T)\eta(Z)E + \frac{Af^2}{2}\{g(W,E)g(U,T)Z - g(W,Z)g(U,T)E\} \\
& -f^2\{g(W,R(Z,E)T)U - g(U,R(Z,E)T)W - g(W,Z)R(U,E)T \\
& +g(U,Z)R(W,E)T - g(W,E)R(Z,U)T + g(U,E)R(Z,W)T \\
& -g(W,T)R(Z,E)U + g(U,T)R(Z,E)W\} \\
& +\frac{Bf^2}{2}\{g(U,Z)\eta(W)\eta(T)E - g(U,E)\eta(W)\eta(T)Z \\
& +g(W,T)\eta(U)\eta(E)Z - g(W,T)\eta(U)\eta(Z)E \\
& -g(U,T)\eta(W)\eta(E)Z + g(U,T)\eta(W)\eta(Z)E\},
\end{aligned} \tag{4.8}$$

where $A = \frac{scal}{2} + 2f^2$ and $B = \frac{scal}{2} + 3f^2$.

Also putting $U = \xi$ in (4.8), we have

$$\begin{aligned}
& R(\xi,W)R(Z,E)T - R(R(\xi,W)Z,E)T - R(Z,R(\xi,W)E)T \\
& -R(Z,E)R(\xi,W)T \\
= \; & \frac{B}{2}\{\eta(R(\xi,W)E)\eta(T)Z - \eta(R(\xi,W)Z)\eta(T)E + \eta(R(\xi,W)T)\eta(E)Z \\
& -\eta(R(\xi,W)T)\eta(Z)E\} - f^2\{g(W,R(Z,E)T)\xi - \eta(R(Z,E)T)W \\
& -g(W,Z)R(\xi,E)T + \eta(Z)R(W,E)T - g(W,E)R(Z,\xi)T + \eta(E)R(Z,W)T \\
& -g(W,T)R(Z,E)\xi + \eta(T)R(Z,E)W\} \\
& +\frac{Af^2}{2}\{g(W,E)\eta(T)Z - g(W,Z)\eta(T)E\} \\
& +\frac{Bf^2}{2}\{\eta(Z)\eta(W)\eta(T)E - \eta(E)\eta(W)\eta(T)Z + g(W,T)\eta(E)Z \\
& -g(W,T)\eta(Z)E - \eta(T)\eta(W)\eta(E)Z + \eta(T)\eta(W)\eta(Z)E\}.
\end{aligned} \tag{4.9}$$

Taking the inner product with ξ in (4.9), we get

$$\begin{aligned}
&\eta(R(\xi,W)R(Z,E)T) - \eta(R(R(\xi,W)Z,E)T) - \eta(R(Z,R(\xi,W)E)T)\\
&-\eta(R(Z,E)R(\xi,W)T)\\
= &\frac{B}{2}\{\eta(R(\xi,W)E)\eta(T)\eta(Z) - \eta(R(\xi,W)Z)\eta(T)\eta(E)\}\\
&-f^2\{g(W,R(Z,E)T) - \eta(R(Z,E)T)\eta(W) - g(W,Z)\eta(R(\xi,E)T)\\
&+\eta(Z)\eta(R(W,E)T) - g(W,E)\eta(R(Z,\xi)T) + \eta(E)\eta(R(Z,W)T)\\
&+\eta(T)\eta(R(Z,E)W)\} + \frac{Af^2}{2}\{g(W,E)\eta(T)\eta(Z) - g(W,Z)\eta(T)\eta(E)\}.
\end{aligned} \tag{4.10}$$

Let $\{e_i\}$ $(1 \leq i \leq 3)$ be an orthonormal basis of the tangent space at any point of M. Then the sum for $1 \leq i \leq 3$ of the relation (4.10) for $W = Z = e_i$ gives

$$\begin{aligned}
&\eta(R(\xi,e_i)R(e_i,E)T) - \eta(R(R(\xi,e_i)e_i,E)T) - \eta(R(e_i,R(\xi,e_i)E)T)\\
&-\eta(R(e_i,E)R(\xi,e_i)T)\\
= &\frac{B}{2}\{\eta(R(\xi,e_i)E)\eta(T)\eta(e_i) - \eta(R(\xi,e_i)e_i)\eta(T)\eta(E)\}\\
&-f^2\{g(e_i,R(e_i,E)T) - \eta(R(e_i,E)T)\eta(e_i) - g(e_i,e_i)\eta(R(\xi,E)T)\\
&+\eta(e_i)\eta(R(e_i,E)T) - g(e_i,E)\eta(R(e_i,\xi)T) + \eta(E)\eta(R(e_i,e_i)T)\\
&+\eta(T)\eta(R(e_i,E)e_i)\} + \frac{Af^2}{2}\{g(e_i,E)\eta(T)\eta(e_i) - g(e_i,e_i)\eta(T)\eta(E)\},
\end{aligned} \tag{4.11}$$

which is equal to

$$\begin{aligned}
&\eta(R(\xi,e_i)R(e_i,E)T) - \eta(R(R(\xi,e_i)e_i,E)T) - \eta(R(e_i,R(\xi,e_i)E)T)\\
&-\eta(R(e_i,E)R(\xi,e_i)T)\\
= &\frac{-B}{2}\eta(R(\xi,e_i)e_i)\eta(T)\eta(E) - f^2\{g(e_i,R(e_i,E)T)\\
&-3\eta(R(\xi,E)T) - g(e_i,E)\eta(R(e_i,\xi)T)\\
&+\eta(T)\eta(R(e_i,E)e_i)\} - Af^2\eta(T)\eta(E).
\end{aligned} \tag{4.12}$$

Using (2.10) and (2.11) in (4.12), we obtain

$$2f^2 Ric(E,T) = (-2f^4 - 3f^2)g(E,T) + (-2f^4 + 3f^2)\eta(E)\eta(T), \tag{4.13}$$

i.e.

$$Ric(E,T) = (-f^2 - \frac{3}{2})g(E,T) - (f^2 - \frac{3}{2})\eta(E)\eta(T). \tag{4.14}$$

Thus, the manifold M is an η-Einstein manifold with respect to the Levi-Civita connection. Also, using (4.14) in (3.4), we have

$$\overset{\star}{Ric}(E,T) = (f^2 - \frac{3}{2})g(E,T) - (f^2 - \frac{3}{2})\eta(E)\eta(T). \tag{4.15}$$

Hence, the manifold M is an η-Einstein manifold with respect to the Schouten–van Kampen connection.

If f is not a constant, then using $U = \xi$ in (4.6), we get

$$\begin{aligned}
&R(\xi,W)P(Z,E)T - P(R(\xi,W)Z,E)T - P(Z,R(\xi,W)E)T \\
&-P(Z,E)R(\xi,W)T + f^{2}\{g(W,P(Z,E)T)\xi - \eta(P(Z,E)T)W \\
&-g(W,Z)P(\xi,E)T + \eta(Z)P(W,E)T - g(W,E)P(Z,\xi)T \\
&+\eta(E)P(Z,W)T - g(W,T)P(Z,E)\xi + \eta(T)P(Z,E)W\} \\
&+f'\{g(P(Z,E)T,W)\xi - \eta(P(Z,E)T)\eta(W)\xi \\
&+\eta(P(Z,E)T)\eta(W)\xi - \eta(P(Z,E)T)W \\
&-g(W,Z)\eta(P(\xi,E)T)\xi + \eta(Z)\eta(P(W,E)T)\xi \\
&-\eta(W)\eta(Z)P(\xi,E)T + \eta(Z)P(W,E)T \\
&-g(W,E)\eta(P(Z,\xi)T)\xi + \eta(E)\eta(P(Z,W)T)\xi \\
&-\eta(W)\eta(E)P(Z,\xi)T + \eta(E)P(Z,W)T \\
&-g(W,T)\eta(P(Z,E)\xi)\xi + \eta(T)\eta(P(Z,E)W)\xi \\
&-\eta(W)\eta(T)P(Z,E)\xi + \eta(T)P(Z,E)W\} = 0,
\end{aligned}$$

which gives

$$\begin{aligned}
&R(\xi,W)P(Z,E)T - P(R(\xi,W)Z,E)T - P(Z,R(\xi,W)E)T \\
&-P(Z,E)R(\xi,W)T \\
&+(f^{2}+f')\{g(W,P(Z,E)T)\xi - g(\xi,P(Z,E)T)W \\
&-g(W,Z)P(\xi,E)T + g(\xi,Z)P(W,E)T - g(W,E)P(Z,\xi)T \\
&+g(\xi,E)P(Z,W)T - g(W,T)P(Z,E)\xi + g(\xi,T)P(Z,E)W\} = 0.
\end{aligned}$$

Combining the above results, we have the following:

Theorem 4.1 *Let M be a projectively semisymmetric f-Kenmotsu 3-manifold with respect to the Schouten–van Kampen connection. Then we get the following: i) If $0 \neq f =$ constant (say $f = \alpha$), then M is a projectively pseudosymmetric α-Kenmotsu manifold. In this case, M is an η-Einstein manifold both with respect to the Levi-Civita connection and the Schouten–van Kampen connection. ii) If f is not a constant, then M is projectively semisymmetric with respect to the Levi-Civita connection.*

5. Conharmonically semisymmetric f-Kenmotsu 3-manifolds with the Schouten–van Kampen connection

In this section, we study conharmonically semisymmetric f-Kenmotsu 3-manifolds with respect to the Schouten–van Kampen connection. In an f-Kenmotsu 3-manifold, the conharmonic curvature tensor with respect to the Schouten–van Kampen connection is given by

$$\begin{aligned}
\overset{\star}{K}(U,W)Z &= \overset{\star}{R}(U,W)Z - \{\overset{\star}{Ric}(W,Z)U - \overset{\star}{Ric}(U,Z)W \\
&\quad + g(W,Z)\overset{\star}{Q}U - g(U,Z)\overset{\star}{Q}W\}.
\end{aligned} \tag{5.1}$$

Using (3.2), (3.4), and (3.5) in (5.1), we have

$$\overset{\star}{K}(U,W)Z = K(U,W)Z - (3f^{2}+2f')\{g(W,Z)U - g(U,Z)W\}. \tag{5.2}$$

Also, it is well known that if an f-Kenmotsu manifold with respect to the Schouten–van Kampen connection satisfies the condition

$$\overset{\star}{R}(U,W)\cdot\overset{\star}{K} = L\hat{Q}(g,\overset{\star}{K}),$$

then the manifold is called conharmonically pseudosymmetric f-Kenmotsu manifold with respect to the Schouten–van Kampen connection, where L is a function and

$$\begin{aligned}\hat{Q}(g,\overset{\star}{K})(Z,E,T;U,W) &= ((U\Lambda W)\overset{\star}{K})(Z,E)T\\ &= -\overset{\star}{K}((U\Lambda W)Z,E)T - \overset{\star}{K}(Z,(U\Lambda W)E)T\\ &\quad -\overset{\star}{K}(Z,E)(U\Lambda W)T.\end{aligned}$$

If $L=0$, then the manifold M is called conharmonically semisymmetric manifold with respect to the Schouten–van Kampen connection.

Let M be a conharmonically semisymmetric f-Kenmotsu 3-manifold with respect to the Schouten–van Kampen connection. Then, we have

$$(\overset{\star}{R}(U,W)\cdot\overset{\star}{K})(Z,E)T = 0, \tag{5.3}$$

which satisfies

$$\begin{aligned}&\overset{\star}{R}(U,W)\overset{\star}{K}(Z,E)T - \overset{\star}{K}(\overset{\star}{R}(U,W)Z,E)T\\ &-\overset{\star}{K}(Z,\overset{\star}{R}(U,W)E)T - \overset{\star}{K}(Z,E)\overset{\star}{R}(U,W)T = 0.\end{aligned} \tag{5.4}$$

Using (5.2) in (5.4), we get

$$\begin{aligned}&\overset{\star}{R}(U,W)K(Z,E)T - K(\overset{\star}{R}(U,W)Z,E)T\\ &-K(Z,\overset{\star}{R}(U,W)E)T - K(Z,E)\overset{\star}{R}(U,W)T = 0,\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}K(U,W)Z &= R(U,W)Z\\ &\quad -\{Ric(W,Z)U - Ric(U,Z)W\\ &\quad +g(W,Z)QU - g(U,Z)QW\}.\end{aligned} \tag{5.6}$$

Now using (3.2) in (5.5), we have

$$\begin{aligned}
&R(U,W)K(Z,E)T - K(R(U,W)Z,E)T - K(Z,R(U,W)E)T\\
&-K(Z,E)R(U,W)T + f^{2}\{g(W,K(Z,E)T)U - g(U,K(Z,E)T)W\\
&-g(W,Z)K(U,E)T + g(U,Z)K(W,E)T - g(W,E)K(Z,U)T\\
&+g(U,E)K(Z,W)T - g(W,T)K(Z,E)U + g(U,T)K(Z,E)W\}\\
&+f'\{g(K(Z,E)T,W)\eta(U)\xi - g(K(Z,E)T,U)\eta(W)\xi\\
&+\eta(K(Z,E)T)\eta(W)U - \eta(K(Z,E)T)\eta(U)W\\
&-g(W,Z)\eta(K(U,E)T)\xi + g(U,Z)\eta(K(W,E)T)\xi\\
&-\eta(W)\eta(Z)K(U,E)T + \eta(U)\eta(Z)K(W,E)T\\
&-g(W,E)\eta(K(Z,U)T)\xi + g(U,E)\eta(K(Z,W)T)\xi\\
&-\eta(W)\eta(E)K(Z,U)T + \eta(U)\eta(E)K(Z,W)T\\
&-g(W,T)\eta(K(Z,E)U)\xi + g(U,T)\eta(K(Z,E)W)\xi\\
&-\eta(W)\eta(T)K(Z,E)U + \eta(U)\eta(T)K(Z,E)W\} = 0.
\end{aligned} \tag{5.7}$$

Now from (5.7), we can say: If $0 \neq f =$ constant (say $f = \alpha$) then $f' = 0$. Hence, we get $R\cdot K = -\alpha^2 Q(g,K)$. Therefore, the manifold M is a conharmonically pseudosymmetric α-Kenmotsu manifold. Thus, using (5.6) in (5.7), we obtain

$$\begin{aligned}
&R(U,W)R(Z,E)T - R(R(U,W)Z,E)T - R(Z,R(U,W)E)T\\
&-R(Z,E)R(U,W)T\\
&-A\{2g(E,T)R(U,W)Z - 2g(Z,T)R(U,W)E\\
&-g(R(U,W)Z,T)E + g(Z,R(U,W)T)E + g(R(U,W)E,T)Z\\
&-g(R(U,W)T,E)Z\}\\
&+B\{g(E,T)\eta(Z)R(U,W)\xi - g(Z,T)\eta(E)R(U,W)\xi\\
&+g(E,T)\eta(R(U,W)Z)\xi - 2g(R(U,W)Z,T)\eta(E)\xi\\
&+2g(R(U,W)E,T)\eta(Z)\xi - g(Z,T)\eta(R(U,W)E)\xi\}\\
&+Ric(R(U,W)E,T)Z + Ric(E,R(U,W)T)Z\\
&-Ric(R(U,W)Z,T)E - Ric(Z,R(U,W)T)E = 0,
\end{aligned} \tag{5.8}$$

which satisfies

$$\begin{aligned}
&R(U,W)R(Z,E)T - R(R(U,W)Z,E)T - R(Z,R(U,W)E)T \\
&-R(Z,E)R(U,W)T \\
&-2A\{g(E,T)R(U,W)Z - g(Z,T)R(U,W)E \\
&-g(R(U,W)Z,T)E + g(R(U,W)E,T)Z\} \\
&+B\{g(E,T)\eta(Z)R(U,W)\xi - g(Z,T)\eta(E)R(U,W)\xi \\
&+g(E,T)\eta(R(U,W)Z)\xi - g(Z,T)\eta(R(U,W)E)\xi \\
&+2g(R(U,W)E,T)\eta(Z)\xi - 2g(R(U,W)Z,T)\eta(E)\xi\} \\
&+\eta(T)\eta(R(U,W)Z)E + \eta(Z)\eta(R(U,W)T)E \\
&-\eta(T)\eta(R(U,W)E)Z - \eta(E)\eta(R(U,W)T)Z = 0.
\end{aligned} \tag{5.9}$$

Now putting $U = \xi$ and using (2.11) in (5.9), we obtain

$$\begin{aligned}
&-f^2\{g(W,R(Z,E)T)\xi - \eta(R(Z,E)T)W - g(W,Z)R(\xi,E)T \\
&+\eta(Z)R(W,E)T - g(W,E)R(Z,\xi)T + \eta(E)R(Z,W)T \\
&-g(W,T)R(Z,E)\xi + \eta(T)R(Z,E)W\} \\
&+2Af^2\{g(E,T)g(W,Z)\xi - g(E,T)\eta(Z)W - g(Z,T)g(W,E)\xi \\
&+g(Z,T)\eta(E)W - g(W,Z)\eta(T)E + g(W,T)\eta(Z)E \\
&+g(W,E)\eta(T)Z - g(W,T)\eta(E)Z\} \\
&-Bf^2\{g(E,T)\eta(Z)\eta(W)\xi - g(E,T)\eta(Z)W - g(Z,T)\eta(E)\eta(W)\xi \\
&+g(Z,T)\eta(E)W + g(E,T)g(W,Z)\xi - g(E,T)\eta(W)\eta(Z)\xi \\
&-g(Z,T)g(W,E)\xi + g(Z,T)\eta(W)\eta(E)\xi - g(W,E)\eta(T)Z \\
&+\eta(W)\eta(E)\eta(T)Z - g(W,T)\eta(E)Z + \eta(W)\eta(E)\eta(T)Z \\
&+g(W,Z)\eta(T)E - \eta(T)\eta(W)\eta(Z)E + g(W,T)\eta(Z)E \\
&-\eta(Z)\eta(W)\eta(T)E\} = 0.
\end{aligned} \tag{5.10}$$

Taking the inner product with ξ in (5.10), we get

$$\begin{aligned}
&-f^2[\{g(W,R(Z,E)T) - \eta(R(Z,E)T)\eta(W) - g(W,Z)\eta(R(\xi,E)T) \\
&+\eta(Z)\eta(R(W,E)T) - g(W,E)\eta(R(Z,\xi)T) + \eta(E)\eta(R(Z,W)T) \\
&-g(W,T)\eta(R(Z,E)\xi) + \eta(T)\eta(R(Z,E)W)\} \\
&-2A\{g(E,T)g(W,Z) - g(E,T)\eta(Z)\eta(W) - g(Z,T)g(W,E) \\
&+g(Z,T)\eta(E)\eta(W) - g(W,Z)\eta(T)\eta(E) + g(W,E)\eta(T)\eta(Z)\} \\
&+B\{g(E,T)g(W,Z) - g(E,T)\eta(W)\eta(Z) \\
&-g(Z,T)g(W,E) + g(Z,T)\eta(W)\eta(E) - g(W,E)\eta(T)\eta(Z) \\
&+g(W,Z)\eta(T)\eta(E)\}] = 0.
\end{aligned} \tag{5.11}$$

Since $f^2 \neq 0$, from (5.11) we have

$$\begin{aligned}
&g(W,R(Z,E)T)-\eta(R(Z,E)T)\eta(W)-g(W,Z)\eta(R(\xi,E)T)\\
&+\eta(Z)\eta(R(W,E)T)-g(W,E)\eta(R(Z,\xi)T)+\eta(E)\eta(R(Z,W)T)\\
&-g(W,T)\eta(R(Z,E)\xi)+\eta(T)\eta(R(Z,E)W)\\
&-2A\{g(E,T)g(W,Z)-g(E,T)\eta(Z)\eta(W)-g(Z,T)g(W,E)\\
&+g(Z,T)\eta(E)\eta(W)-g(W,Z)\eta(T)\eta(E)+g(W,E)\eta(T)\eta(Z)\}\\
&+B\{g(E,T)g(W,Z)-g(E,T)\eta(W)\eta(Z)\\
&-g(Z,T)g(W,E)+g(Z,T)\eta(W)\eta(E)-g(W,E)\eta(T)\eta(Z)\\
&+g(W,Z)\eta(T)\eta(E)\}=0.
\end{aligned} \tag{5.12}$$

Let $\{e_i\}$, $(1 \leq i \leq 3)$ be an orthonormal basis of the tangent space at any point of M. Then the sum for $1 \leq i \leq 3$ of the relation (5.12) for $W = Z = e_i$ gives

$$Ric(E,T)=(\frac{scal}{2}+3f^{2})g(E,T)-(\frac{scal}{2}+5f^{2})\eta(E)\eta(T).$$

Thus, the manifold M is an η-Einstein manifold with respect to the Levi-Civita connection. Also, using (3.4), we have

$$\overset{\star}{Ric}(E,T)=(\frac{scal}{2}+5f^{2})g(E,T)-(\frac{scal}{2}+5f^{2})\eta(E)\eta(T), \tag{5.13}$$

which implies that M is an η-Einstein manifold with respect to the Schouten–van Kampen connection.

If f is not a constant, then using $U = \xi$ in (5.7), we get

$$\begin{aligned}
&R(\xi,W)K(Z,E)T-K(R(\xi,W)Z,E)T-K(Z,R(\xi,W)E)T\\
&-K(Z,E)R(\xi,W)T+f^{2}\{g(W,K(Z,E)T)\xi-g(\xi,K(Z,E)T)W\\
&-g(W,Z)K(\xi,E)T+g(\xi,Z)K(W,E)T-g(W,E)K(Z,\xi)T\\
&+g(\xi,E)K(Z,W)T-g(W,T)K(Z,E)\xi+g(\xi,T)K(Z,E)W\}\\
&+f'\{g(K(Z,E)T,W)\xi-g(K(Z,E)T,\xi)\eta(W)\xi\\
&+\eta(K(Z,E)T)\eta(W)\xi-\eta(K(Z,E)T)W\\
&-g(W,Z)\eta(K(\xi,E)T)\xi+g(\xi,Z)\eta(K(W,E)T)\xi\\
&-\eta(W)\eta(Z)K(\xi,E)T+\eta(Z)K(W,E)T\\
&-g(W,E)\eta(K(Z,\xi)T)\xi+g(\xi,E)\eta(K(Z,W)T)\xi\\
&-\eta(W)\eta(E)K(Z,\xi)T+\eta(E)K(Z,W)T\\
&+g(U,T)\eta(K(Z,E)W)\xi\\
&-\eta(W)\eta(T)K(Z,E)\xi+\eta(T)K(Z,E)W\}=0.
\end{aligned}$$

Combining the above results, we have the following:

Theorem 5.1 *Let M be a conharmonically semisymmetric f-Kenmotsu 3-manifold with respect to the Schouten–van Kampen connection. Then we get the following: i) If $0 \neq f =$ constant (say $f = \alpha$), then*

M is a conharmonically pseudosymmetric α-Kenmotsu manifold. In this case, M is an η-Einstein manifold both with respect to the Levi-Civita connection and the Schouten–van Kampen connection. ii) If f is not a constant, then M is conharmonically semisymmetric with respect to the Levi-Civita connection.

6. Some soliton types on f-Kenmotsu 3-manifolds with respect to the Schouten–van Kampen connection

In this section we study some soliton types on f-Kenmotsu 3-manifolds with respect to the Schouten–van Kampen connection.

In an f-Kenmotsu 3-manifold with the Schouten–van Kampen connection, since $\overset{\star}{\nabla} g = 0$ and $\overset{\star}{T} \neq 0$, by using (3.1), we get

$$(\overset{\star}{L}_V g)(U, W) = g(\nabla_U V, W) + g(U, \nabla_W V) = (L_V g)(U, W), \tag{6.1}$$

where $\overset{\star}{L}$ denotes the Lie derivative on the manifold with respect to the Schouten–van Kampen connection.

Now we consider an almost Ricci soliton on an f-Kenmotsu 3-manifold with respect to the Schouten–van Kampen connection. From (1.1), we can write

$$(\overset{\star}{L}_V g + 2\overset{\star}{Ric} + 2\mu g)(U, W) = 0, \tag{6.2}$$

that is

$$g(\nabla_U V, W) + g(U, \nabla_W V) + 2\overset{\star}{Ric}(U, W) + 2\mu g(U, W) = 0, \tag{6.3}$$

via (6.1). Putting $V = \xi$ in (6.3) and using (2.3), we obtain

$$\overset{\star}{Ric}(U, W) = -(\mu + f)g(U, W) + f\eta(U)\eta(W). \tag{6.4}$$

Also, using (3.4) in (6.4), we have

$$Ric(U, W) = -(f' + 2f^2 + f + \mu)g(U, W) + (-f' + f)\eta(U)\eta(W). \tag{6.5}$$

Hence, we have the following:

Theorem 6.1 *Let M be an f-Kenmotsu 3-manifold bearing an almost Ricci soliton (ξ, μ, g) with respect to the Schouten–van Kampen connection. Then M is an η-Einstein manifold both with respect to the Schouten–van Kampen connection and the Levi-Civita connection.*

Putting $U = \xi$ and using (3.4), we give the following:

Corollary 6.2 *An almost Ricci soliton (ξ, μ, g) on an f-Kenmotsu 3-manifold with respect to the Schouten–van Kampen connection is always steady.*

On the other hand, from (2.8) and (3.4), it is easy to see that an f-Kenmotsu 3-manifold is always η-Einstein with respect to the Schouten–van Kampen connection of the form $\overset{\star}{Ric} = ag + b\eta \otimes \eta$, where $a = -b = \frac{scal}{2} + 3f^2 + 2f'$. Then, we write

$$(\overset{\star}{L}_\xi g + 2\overset{\star}{Ric} + 2\mu g)(U, W) = \left((2f + 2a + 2\mu)\, g + (-2f + 2b)\, \eta \otimes \eta\right)(U, W),$$

for all $U, W \in \chi(M)$, which implies that the manifold M admits an almost Ricci soliton (ξ, μ, g) if $f+a+\mu=0$ and $-f+b=0$. Thus, we give the following:

Theorem 6.3 *An f-Kenmotsu 3-manifold admits a steady almost Ricci soliton (ξ, μ, g) with respect to the Schouten–van Kampen connection provided $f = -\frac{\overset{\star}{scal}}{2}$.*

By using (2.1) in (6.4), we can also state the following:

Corollary 6.4 *The scalar curvature of an f-Kenmotsu 3-manifold bearing an almost Ricci soliton (ξ, μ, g) with respect to the Schouten–van Kampen connection is $\overset{\star}{scal} = -3\mu - 2f$.*

From Theorem 4.1 and (4.15), we can say the following:

Corollary 6.5 *A projectively semisymmetric 3-dimensional α-Kenmotsu manifold with respect to the Schouten–van Kampen connection admits a steady almost Ricci soliton (ξ, μ, g) provided $f = -1 \pm \sqrt{7}$.*

Also Theorem 5.1 and (5.13) give the following:

Corollary 6.6 *A conharmonically semisymmetric 3-dimensional α-Kenmotsu manifold with respect to the Schouten–van Kampen connection cannot admit a steady almost Ricci soliton (ξ, μ, g).*

Again let us consider equations (6.2) and (6.3). Using (3.4), we obtain

$$g(\nabla_U V, W) + g(U, \nabla_W V) + 2Ric(U, W) + 2(2f^2 + f' + \mu)g(U, W) + f'\eta(U)\eta(W) = 0.$$

Thus, we write

$$(L_V g)(U, W) + 2Ric(U, W) + 2(2f^2 + f' + \mu)g(U, W) + f'\eta(U)\eta(W) = 0.$$

This last equation shows that if (V, μ, g) is an almost Ricci soliton and f is not a constant on an f-Kenmotsu 3-manifold with respect to the Schouten–van Kampen connection, then the manifold admits an almost η-Ricci soliton $(V, 2f^2 + f' + \mu, f', g)$ with respect to the Levi-Civita connection. If $0 \neq f =$ constant, then $f' = 0$. Thus, we have

$$(L_V g)(U, W) + 2Ric(U, W) + 2(2f^2 + \mu)g(U, W) = 0.$$

Thus, we have the following:

Theorem 6.7 *Let M be an f-Kenmotsu 3-manifold bearing an almost Ricci soliton (V, μ, g) with respect to the Schouten–van Kampen connection . Then we have*

(i) *If $0 \neq f =$ constant then M admits an almost Ricci soliton $(V, 2f^2 + \mu, g)$ with respect to the Levi-Civita connection.*

(ii) *If $f \neq$ constant then M admits an almost η-Ricci soliton $(V, f' + 2f^2 + \mu, f', g)$ with respect to the Levi-Civita connection.*

Now, let us consider an almost η-Ricci soliton (V,μ,δ,g) on an f-Kenmotsu 3-manifold with respect to the Levi-Civita connection. Using (6.1) and (3.4) in (1.2), we can write

$$(\overset{\star}{L}_V g)(U,W) + 2\overset{\star}{Ric}(U,W) + 2(-2f^{\,2} - f^{\,\prime} + \mu)g(U,W) + 2(\delta - f^{\,\prime})\eta(U)\eta(W) = 0.$$

Hence, we give the following:

Theorem 6.8 *An f-Kenmotsu 3-manifold bearing an almost η-Ricci soliton (V,μ,δ,g) with respect to the Levi-Civita connection admits an almost η-Ricci soliton $(V,-f^{\,\prime}-2f^{\,2}+\mu,-f^{\,\prime}+\delta,g)$ with respect to the Schouten–van Kampen connection.*

Conversely, let us consider an almost η-Ricci soliton (V,μ,δ,g) on an f-Kenmotsu 3-manifold with respect to the Schouten–van Kampen connection. Then, in view of (1.2) we get

$$(L_V g)(U,W) + 2Ric(U,W) + 2(2f^{\,2} + f^{\,\prime} + \mu)g(U,W) + 2(\delta + f^{\,\prime})\eta(U)\eta(W) = 0.$$

Thus, we have the following:

Theorem 6.9 *An f-Kenmotsu 3-manifold bearing an almost η-Ricci soliton (V,μ,δ,g) with respect to the Schouten–van Kampen connection admits an almost η-Ricci soliton $(V,2f^{\,2}+f^{\,\prime}+\mu,f^{\,\prime}+\delta,g)$ with respect to the Levi-Civita connection.*

Assume that M is a Ricci flat f-Kenmotsu 3-manifold with respect to the Schouten–van Kampen connection. In this case, by using (2.4) and (6.1), we get

$$(\overset{\star}{L}_\xi g + 2\overset{\star}{Ric} + 2\mu g + 2\delta\eta\otimes\eta)(U,W) = 2\,(f+\mu)\,g(U,W) + 2\,(-f+\delta)\,\eta(U)\eta(W),$$

for all $U,W\in\chi(M)$. Hence, we give the following:

Theorem 6.10 *A Ricci flat f-Kenmotsu 3-manifold with respect to the Schouten–van Kampen connection admits an almost η-Ricci soliton (ξ,μ,δ,g) provided $-\mu=\delta=f$. Moreover, in this case the almost η-Ricci soliton (ξ,μ,δ,g) is shrinking (resp. expanding) if $f>0$ (resp. $f<0$).*

Example 6.11 *Let $M=\{(u,v,t)\in R^3 : t\neq 0\}$ be a 3-dimensional manifold endowed with the standard coordinate system in R^3. We define linearly independent vector fields at each point of M by*

$$X_1 = t^2\frac{\partial}{\partial u},\quad X_2 = t^2\frac{\partial}{\partial v},\quad X_3 = \frac{\partial}{\partial t},$$

and a Riemannian metric by

$$\begin{aligned} g(X_1,X_1) &= g(X_2,X_2) = g(X_3,X_3) = 1,\\ g(X_1,X_2) &= g(X_1,X_3) = g(X_2,X_3) = 0. \end{aligned}$$

Assume that η is a 1-form given by $\eta(T)=g(T,X_3)$, for any $T\in\chi(M)$, and ψ is a $(1,1)$ tensor field defined by

$$\psi(X_1) = -X_2,\quad \psi(X_2) = X_1,\quad \psi(X_3) = 0.$$

Then one can easily show that the quadruple (ψ, ξ, η, g) *is an almost contact metric structure on* M *by choosing* $\xi = X_3$. *By direct calculations, we see that the nonzero components of the Levi-Civita connection* ∇ *on* M *are*

$$\left\{ \begin{array}{ll} \nabla_{X_1} X_1 = \frac{2}{t} X_3, & \nabla_{X_1} X_3 = -\frac{2}{t} X_1, \\ \nabla_{X_2} X_2 = \frac{2}{t} X_3, & \nabla_{X_2} X_3 = -\frac{2}{t} X_2, \end{array} \right. \tag{6.6}$$

which implies that (2.3) is satisfied for the function $f = -\frac{2}{t}$. *Hence,* M *is a* 3*-dimensional regular* f*-Kenmotsu manifold [31]. Now we consider the Schouten–van Kampen connection* $\overset{\star}{\nabla}$ *on the* f*-Kenmotsu* 3*-manifold* M *defined above. By using (2.5) and (6.6), we see that* $\overset{\star}{\nabla}_{X_i} X_j = 0$, *for* $1 \leq i, j \leq 3$. *Thus, the manifold* M *reduces to a Ricci-flat manifold with respect to the Schouten–van Kampen connection. For any* $Z, T \in \chi(M)$, *we write*

$$\begin{aligned} U &= a_1 X_1 + a_2 X_2 + a_3 X_3, \\ W &= b_1 X_1 + b_2 X_2 + b_3 X_3, \end{aligned}$$

where a_i *and* b_j, $(1 \leq i, j \leq 3)$, *are real functions. Then, from (6.1) and (6.6) we obtain*

$$\begin{aligned} (\overset{\star}{L}_\xi g)(U, W) &= g(\nabla_U \xi, W) + g(U, \nabla_W \xi) \\ &= -\frac{4}{t}(a_1 b_1 + a_2 b_2), \end{aligned}$$

which implies that

$$\left\{ \begin{array}{l} (\overset{\star}{L}_\xi g)(U, W) + 2\overset{\star}{Ric}(U, W) \\ +2\mu g(U, W) + 2\delta \eta(U)\eta(W) \end{array} \right. = \begin{array}{c} -\frac{4}{t}(a_1 b_1 + a_2 b_2) \\ +2\mu(a_1 b_1 + a_2 b_2) + 2\delta a_3 b_3. \end{array}$$

If $\mu = \frac{2}{t}$ *and* $\delta = -\frac{2}{t}$, *then* M *admits an almost* η*-Ricci soliton* (ξ, μ, δ, g) *with respect to the Schouten–van Kampen connection. Moreover, such an almost* η*-Ricci soliton is shrinking (resp. expanding) if* $t < 0$ *(resp.* $t > 0$*).*

Furthermore, by using Theorems 4.1 and 5.1, we state the following:

Corollary 6.12 *A Ricci flat projectively semisymmetric (resp. conharmonically semisymmetric)* 3*-dimensional* α*-Kenmotsu manifold with respect to the Schouten–van Kampen connection admits an* η*-Ricci soliton* $(\xi, -\alpha, \alpha, g)$.

If the vector field V is the gradient of a potential function $-k$, then g is called an almost gradient Ricci soliton that is $V = -grad\, k$. In this case equation (1.1) becomes

$$\nabla grad\, k = Ric + \mu g,$$

where ∇ is the Levi-Civita connection. Now assume that M is an f-Kenmotsu 3-manifold with respect to the Schouten–van Kampen connection. If we take $V = -grad\, k$ in (6.1), we write

$$(\overset{\star}{L}_{grad\, k} g)(U, W) = (L_{grad\, k} g)(U, W) = g(\nabla_U grad\, k, W) + g(U, \nabla_W grad\, k).$$

We can easily see that

$$g(\nabla_U grad\, k, W) = g(U, \nabla_W grad\, k),$$

which implies that

$$\overset{\star}{L}_{grad\,k}g - 2\overset{\star}{Ric} - 2\mu g = 0,$$

is equal to

$$g(\nabla_U grad\,k, W) = \overset{\star}{Ric}(U, W) + \mu g(U, W).$$

This reduces to

$$\nabla_U grad\,k = \overset{\star}{Q}U + \mu U.$$

Now we want to compute $R(U, W)grad\,k = (\nabla_U Q)W$ -$(\nabla_W Q)U$. For this, we can write

$$\begin{aligned}
\nabla_U QW &= \nabla_U\{(\frac{scal}{2} + f^2 + f')W - (\frac{scal}{2} + 3f^2 + 3f')\eta(W)\xi\} \\
&= (\frac{1}{2}U(scal) + 2fU(f) + U(f'))W + (\frac{scal}{2} + f^2 + f')\nabla_U W \\
&\quad -(\frac{1}{2}U(scal) + 6fU(f) + 3U(f'))\eta(W)\xi \\
&\quad -(\frac{scal}{2} + 3f^2 + 3f')(\eta(\nabla_U W) + g(W, \nabla_U\xi))\xi \\
&\quad -(\frac{scal}{2} + 3f^2 + 3f')\eta(W)\nabla_U\xi,
\end{aligned}$$

and

$$Q\nabla_U W = (\frac{scal}{2} + f^2 + f')\nabla_U W - (\frac{scal}{2} + 3f^2 + 3f')\eta(\nabla_U W)\xi.$$

Then we have

$$\begin{aligned}
(\nabla_U Q)W &= \nabla_U QW - Q\nabla_U W \\
&= (\frac{1}{2}U(scal) + 2fU(f) + U(f'))W - (\frac{1}{2}U(scal) + 6fU(f) + 3U(f'))\eta(W)\xi \\
&\quad -(\frac{scal}{2} + 3f^2 + 3f')g(W, \nabla_U\xi)\xi - (\frac{scal}{2} + 3f^2 + 3f')\eta(W)\nabla_U\xi,
\end{aligned}$$

which is equal to

$$\begin{aligned}
(\nabla_U Q)W &= (\frac{1}{2}U(scal) + 2fU(f) + U(f'))W - (\frac{1}{2}U(scal) + 6fU(f) + 3U(f'))\eta(W)\xi \\
&\quad -(\frac{scal}{2} + 3f^2 + 3f')(fg(U, W) - f\eta(U)\eta(W))\xi \\
&\quad -(\frac{scal}{2} + 3f^2 + 3f')(f\eta(W)U - f\eta(U)\eta(W)\xi).
\end{aligned} \tag{6.7}$$

Putting $U = \xi$ in (6.7), we have

$$(\nabla_\xi Q)W = (\frac{1}{2}\xi(scal) + 2ff' + f'')W - (\frac{1}{2}\xi(scal) + 6ff' + 3f'')\eta(W)\xi. \tag{6.8}$$

Similarly, if we take $W=\xi$ in (6.7), we also have

$$(\nabla_U Q)\xi = (-4fU(f) - 2U(f'))\xi. \tag{6.9}$$

Using (6.8) and (6.9), we obtain

$$g((\nabla_U Q)\xi - (\nabla_\xi Q)U, \xi) = (4ff' + 2f'')\eta(U) - 4fU(f) - 2U(f').$$

Thus, we can write

$$g(R(\xi, U)grad\, k, \xi) = (4ff' + 2f'')\eta(U) - 4fU(f) - 2U(f'). \tag{6.10}$$

On the other hand, we have

$$\begin{aligned} R(\xi, U)grad\, k &= (\frac{scal}{2} + 2f^2 + 2f')\{g(U, grad\, k)\xi - g(\xi, grad\, k)U\} \\ &\quad -(\frac{scal}{2} + 3f^2 + 3f')\{g(U, grad\, k)\xi - g(\xi, grad\, k)\eta(U)\xi \\ &\quad +\eta(U)\eta(grad\, k)\xi - \eta(grad\, k)U\}, \end{aligned}$$

which implies that

$$g(R(\xi, U)grad\, k, \xi) = -(f^2 + f')\{g(U, grad\, k) - g(\xi, grad\, k)\eta(U)\}. \tag{6.11}$$

From (6.10) and (6.11), we obtain

$$(4ff' + 2f'')\xi - 4f grad\, f - 2grad\, f' = -(f^2 + f')grad\, k + (f^2 + f')\xi(k)\xi. \tag{6.12}$$

If f is a constant, then we have $grad\, k = \xi(k)\xi$. Using $g(\nabla_W grad\, k, U) = \overset{\star}{Ric}(U, W) + \mu g(U, W)$, we get

$$\begin{aligned} \overset{\star}{Ric}(U, W) + \mu g(U, W) &= g(\nabla_W \xi(k)\xi, U) \\ &= W(\xi k)\eta(U) + (\xi k)g(U, W) - (\xi k)\eta(U)\eta(W). \end{aligned} \tag{6.13}$$

Putting $U=\xi$ in (6.13) and using (2.12), we obtain

$$W(\xi k) = \mu\eta(W). \tag{6.14}$$

Now using (6.14) in (6.13), we get

$$Ric(U, W) = \left(-2f^2 - \mu + (\xi k)\right) g(U, W) + (\mu - (\xi k))\,\eta(U)\eta(W).$$

Hence, we give the following:

Theorem 6.13 *An f-Kenmotsu 3-manifold M bearing a gradient Ricci soliton with respect to the Schouten–van Kampen connection is an η-Einstein manifold provided f is a constant. In particular, if $\xi k = \mu$ then the manifold is an Einstein manifold with respect to the Levi-Civita connection.*

Finally, we study almost Yamabe solitons on an f-Kenmotsu 3-manifold with respect to the Schouten–van Kampen connection. Assume that (M, V, γ, g) is an almost Yamabe soliton on an f-Kenmotsu 3-manifold with respect to the Schouten–van Kampen connection. Then, from (1.4), we write

$$\frac{1}{2}(\overset{\star}{L}_V g)(U, W) = (\overset{\star}{scal} - \gamma)g(U, W). \tag{6.15}$$

From (6.1) and (3.6), we write

$$\frac{1}{2}(L_V g)(U, W) = \left(scal + 6f^{\,2} + 4f^{\,\prime} - \gamma\right) g(U, W). \tag{6.16}$$

By virtue of (1.4) and (6.16), we state the following:

Theorem 6.14 *An almost Yamabe soliton (M, V, γ, g) on an f-Kenmotsu 3-manifold is invariant under the Schouten–van Kampen connection if and only if $3f^{\,2} + 2f^{\,\prime} = 0$.*

Since on an α-Kenmotsu manifold, $f = \alpha =$ constant and an α-Kenmotsu manifold is cosymplectic if α vanishes, then from the last theorem above we have the following:

Corollary 6.15 *An almost Yamabe soliton (M, V, γ, g) on a 3-dimensional α-Kenmotsu manifold is invariant under the Schouten–van Kampen connection if and only if M is a cosymplectic manifold.*

Corollary 6.16 *An almost Yamabe soliton (M, V, γ, g) on a 3-dimensional α-Kenmotsu $(\alpha \neq 0)$ manifold cannot be invariant under the Schouten–van Kampen connection.*

Let us consider that an f-Kenmotsu 3-manifold admits an almost Yamabe soliton (M, ξ, γ, g). In this case, from (6.16) and (2.3), we have

$$f\left(g(U, W) - \eta(U)\eta(W)\right) = (\overset{\star}{scal} - \gamma)g(U, W),$$

which implies the following:

Theorem 6.17 *The scalar curvature $\overset{\star}{scal}$ of an f-Kenmotsu 3-manifold bearing an almost Yamabe soliton (M, ξ, γ, g) with respect to the Schouten–van Kampen connection is equal to γ.*

Thus, we give the following:

Corollary 6.18 *An f-Kenmotsu 3-manifold bearing a Yamabe soliton (M, ξ, γ, g) with respect to the Schouten–van Kampen connection is of constant scalar curvature with respect to the Schouten–van Kampen connection.*

Corollary 6.19 *If an f-Kenmotsu 3-manifold bearing a Yamabe soliton (M, ξ, γ, g) with respect to the Schouten–van Kampen connection, then the Riemannian metric g is a Yamabe metric.*

Corollary 6.20 *There does not exist a steady almost Yamabe soliton with respect to the Schouten–van Kampen connection on a Ricci flat f-Kenmotsu 3-manifold with respect to the Schouten–van Kampen connection.*

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