

On the number of non- G -equivalent minimal abelian codes

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Abstract: Let G be a finite abelian group. Ferraz, Guerreiro, and Polcino Milies (2014) proved that the number of G -equivalence classes of minimal abelian codes is equal to the number of G -isomorphism classes of subgroups for which corresponding quotients are cyclic. In this article, we prove that the notion of G -isomorphism is equivalent to the notion of isomorphism on the set of all subgroups H of G with the property that G/H is cyclic. As an application, we calculate the number of non- G -equivalent minimal abelian codes for some specific family of abelian groups. We also prove that the number of non- G -equivalent minimal abelian codes is equal to the number of divisors of the exponent of G if and only if for each prime p dividing the order of G , the Sylow p -subgroups of G are homocyclic.

Key words: Minimal abelian code, homocyclic group, G -equivalence, G -isomorphism

1. Introduction

According to Berman [2] and MacWilliams [11], an abelian code over a field is defined to be an ideal in a finite group algebra of an abelian group. In a more general context, a group code is an ideal in a finite group algebra. Many important linear codes are group codes [10]. For example, cyclic codes are ideals in a finite group algebra of a cyclic group. In [3], it is shown that Reed–Muller codes are ideals in modular group algebra of an elementary abelian p -group. Group codes have been of interest for many researchers. For an extensive literature review on group codes, see the work in [17]. For some recent studies in the field of group codes, see the work in [1, 4–6, 8, 9, 15, 16]. From these articles, one can deduce that the group codes generate an important family of codes in the framework of coding theory. For instance, in [1] it is shown that group codes coming from metacyclic groups are good or in [5] it is proved that group codes over fields of any characteristic are asymptotically good. A characterization for determining whether a linear code is a group code or not is given in [4]. In [16], cyclic and noncyclic abelian group codes of the same length are considered and efficiency of these group codes are compared.

An abelian code is said to be minimal if the corresponding ideal is minimal in the set of all ideals of the group algebra. Let G be a finite abelian group and \mathbb{F} a finite field of characteristic coprime to the order of G . Under these conditions, Maschke's theorem says that every abelian code is a direct sum of minimal abelian codes. Moreover, as defined in [12], two abelian codes I and J are called **G -equivalent** if there is a group automorphism $\varphi : G \rightarrow G$ whose linear extension to the group algebra maps I onto J . It is easy to see that

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G -equivalent codes have the same weight distribution. However, the converse is not true (see Proposition IV.2 in [7]). Therefore, knowing the number of G -equivalence classes of minimal abelian codes tells us a lot about the nature of codes that can be defined using the group algebra $\mathbb{F}G$.

A one-to-one correspondence between G -equivalence classes of minimal abelian codes and G -isomorphism classes of cocyclic subgroups of G was established by Ferraz et al. (for the details see Proposition III.2, Proposition III.7 and Proposition III.8 in [7]). According to [7], two subgroups H and K of G are called G -isomorphic if there is an automorphism φ of G which maps H onto K . A subgroup $L \leq G$ is called a cocyclic subgroup of G if G/L is cyclic. Note that this definition is not the same definition as in [7]. We take into account G itself also as a cocyclic subgroup to count the minimal abelian code which corresponds to the subgroup G of G . From the definition, it is clear that if two subgroups of G are G -isomorphic, then they are isomorphic. However, the converse of this statement is not true for arbitrary subgroups of G . We observe that the notion of G -isomorphism is equivalent to the notion of isomorphism on the set of cocyclic subgroups of G as follows.

Proposition 1.1 *Let G be a finite abelian group and let H, K be cocyclic subgroups of G . Then H and K are G -isomorphic if and only if they are isomorphic.*

This proposition, together with Proposition III.2, Proposition III.7, and Proposition III.8 in [7] leads us to write the following theorem.

Theorem 1.2 *Let G be a finite abelian group. The number of non- G -equivalent minimal abelian codes over \mathbb{F} is equal to the number of isomorphism classes of cocyclic subgroups of G .*

Let $\eta(\mathbb{F}G)$ denote the number of non- G -equivalent minimal abelian codes over \mathbb{F} . As an application of Theorem 1.2, we prove the following results. Among these, the first result is the following. By using Theorem V.6 in [7], we have that $\eta(\mathbb{F}(C)^m) = \eta(\mathbb{F}C)$ where C is a cyclic group. In this sense, Theorem 1.3 below is a kind of generalization of Theorem V.6 in [7].

Theorem 1.3 *Let H be a finite abelian group and let G be a direct product of finite number of copies of H . Then we have that $\eta(\mathbb{F}G) = \eta(\mathbb{F}H)$.*

There is no formula for $\eta(\mathbb{F}(H \times K))$ in terms of $\eta(\mathbb{F}H)$ and $\eta(\mathbb{F}K)$ in general. For example in [7], Proposition IV.3 gives that $\eta(\mathbb{F}_2(C_{p^n} \times C_p)) = 2n$. On the other hand, $\eta(\mathbb{F}_2C_{p^n}) = n + 1$ and $\eta(\mathbb{F}_2C_p) = 2$. We observe that under an assumption on the exponent of the direct factors, multiplying a finite abelian group by a homocyclic group does not change the number $\eta(\mathbb{F}G)$. Here a homocyclic group is a direct product of pairwise isomorphic cyclic groups.

Theorem 1.4 *Let K be a finite homocyclic group and H a finite abelian group such that $\exp(K) = \exp(H)$. If $G = K \times H$, then we have that $\eta(\mathbb{F}G) = \eta(\mathbb{F}H)$.*

In fact, this result is another generalization of Theorem V.6 in [7].

As emphasized in [12], the codes arising from the group algebra $\mathbb{F}_2(C_m \times C_n)$, where m and n are positive odd integers, are referred to as two-dimensional linear recurring arrays, linear recurring planes, or two-dimensional cyclic codes in [13] and [14]. These codes are related to the problem of constructing perfect maps

and have applications to x -ray photography. In [12, Theorem 3.6], it is stated that the number of nonequivalent minimal codes of $\mathbb{F}_2(C_m \times C_n)$ is equal to the number of divisors of the exponent of the corresponding group. Ferraz et al. point out that this result is not true by calculating the number of nonequivalent minimal codes of $\mathbb{F}_2(C_{p^n} \times C_p)$ as $2n$ where p is an odd prime. (see [7, Propostion IV.3]). The following theorem generalizes this result.

Theorem 1.5 *If $G = C_{p^n} \times C_{p^m}$ and $n > m$, then $\eta(\mathbb{F}G) = (n - m + 1)(m + 1)$.*

As a corollary we obtain the following result.

Corollary 1.6 *Let n be a positive integer such that $n = p_1^{k_1} p_2^{k_2} \dots p_t^{k_t}$ where p_i 's are distinct prime numbers and k_i 's are positive integers. Then for $G = C_{n^l} \times C_{n^s}$ where n, l, s are positive integers and $l > s$ we have that $\eta(\mathbb{F}G) = \prod_{i=1}^t (k_i l - k_i s + 1)(k_i s + 1)$.*

In [12], for an abelian group G of odd order, it is proved that the number of non- G -equivalent minimal abelian codes over \mathbb{F}_2 is equal to the number of divisors of the exponent of G . In [7], it is shown that this statement is not true. Moreover, it is shown that if G is homocyclic, the number of non- G -equivalent minimal abelian codes over \mathbb{F} is equal to the number of divisors of exponent of G (see Theorem V.6 in [7]). In the following theorem, we extend this result and give a characterization of an abelian group whose number of nonequivalent minimal codes is equal to the number of divisors of its exponent.

Theorem 1.7 *Let G be a finite abelian group and \mathbb{F} a finite field of characteristic coprime to order of G . The number of non- G -equivalent minimal abelian codes over \mathbb{F} is equal to the number of divisors of exponent of G if and only if for each prime p dividing the order of G , the Sylow p -subgroups of G are homocyclic.*

Note that Theorem V.6 in [7] now follows from Theorem 1.7 as a corollary.

The structure of the paper is as follows. In Section 2, we establish some results on isomorphic cocyclic subgroups of a finite abelian group and give the proofs of Proposition 1.1 and Theorem 1.2, Theorem 1.3 and Theorem 1.4. We also present some important examples related to Theorem 1.4. In Section 3, we prove Theorem 1.5. In Section 4, we present the proof of Theorem 1.7.

2. Proof of Proposition 1.1 and its consequences

It is not very easy to determine whether two subgroups of a given group G are G -isomorphic or not. We begin by showing that isomorphisms between cocyclic subgroups of G can be extended to an automorphism of G . Then, we continue to prove some other facts to give a proof for Proposition 1.1.

Lemma 2.1 *Let G be an abelian p -group of exponent p^n and H a cocyclic subgroup of G . If the exponent of H is strictly less than p^n , then for any $z \in G$ of order p^n , zH generates the cyclic group G/H .*

Proof Let z be any element in G of order p^n . As $\exp(H) < p^n$, we have that $z \notin H$, so zH is a nontrivial element in G/H . Since H is a cocyclic subgroup of G , $G/H \cong C_{p^m}$ for some $1 \leq m \leq n$. Then there exists some $b \in G$ such that $G/H = \langle bH \rangle$. Since zH is a nontrivial element in G/H , we have that $zH = (bH)^{p^i}$ for some i where $0 \leq i < m$. This implies that $(b^{p^i})^{-1}z \in H$. As $\exp(H) < p^n$, this is possible only if $z = b^{p^i}$. Since z has order p^n , i should be equal to zero so that $z = b$; hence, $G/H = \langle zH \rangle$. \square

Lemma 2.2 *Let G be an abelian p -group of exponent p^n and $H < G$ a cocyclic subgroup of G . Then there exists an element $x \in G$ of order p^n so that $G/H = \langle xH \rangle$.*

Proof There are two cases to consider.

Case 1: $\exp(H) < p^n$: Follows from Lemma 2.1.

Case 2: $\exp(H) = p^n$: As H is cocyclic we have that $G/H \cong C_{p^m}$ for some $1 \leq m \leq n$. Then there exists $a \in G - H$ so that $G/H = \langle aH \rangle$ and then $a^{p^m} \in H$ and m is the smallest integer satisfying this property. If the order of a^{p^m} is equal to p^{n-m} , then one can take $x = a$. If the order of a^{p^m} is less than p^{n-m} , as $\exp(H) = p^n$, there exists some $y \in H$ of order p^n so one can take $x = ya$. It is clear that xH generates G/H . □

Lemma 2.3 *Let G be an abelian p -group of exponent p^n . Assume that H and K are subgroups of G such that for some $x, y \in G$ of order p^n , we have that $G/H = \langle xH \rangle \cong C_{p^m} \cong G/K = \langle yK \rangle$ where $0 \leq m \leq n$. Then, we have that $H/\langle x^{p^m} \rangle \cong K/\langle y^{p^m} \rangle$.*

Proof As $\exp(G) = p^n$ and $|x| = |y| = p^n$, we can write

$$G = \langle x \rangle \times A = \langle y \rangle \times B$$

where $A \cong B$, so that $G/\langle x \rangle \cong G/\langle y \rangle$. On the other hand, the second isomorphism theorem gives us that

$$G/\langle x \rangle = \langle x \rangle H / \langle x \rangle \cong H / H \cap \langle x \rangle = H / \langle x^{p^m} \rangle$$

and similarly $G/\langle y \rangle = \langle y \rangle K / \langle y \rangle \cong K / K \cap \langle y \rangle = K / \langle y^{p^m} \rangle$. Therefore, we deduce that

$$H / \langle x^{p^m} \rangle \cong K / \langle y^{p^m} \rangle.$$

□

Lemma 2.4 *Let H be a finite abelian p -group and let x be a nongenerator of H . Then there exists a generator a of H such that $\langle x \rangle \leq \langle a \rangle$.*

Proof Recall that the Frattini subgroup of H , denoted by $\Phi(H)$ is the set of all nongenerators of H . Then, if $H = \langle a_1 \rangle \times \dots \times \langle a_r \rangle$, it is easy to see that $\Phi(H) = \langle a_1^p \rangle \times \dots \times \langle a_r^p \rangle$. As x is a nongenerator, $x \in \Phi(H)$ and $x = (a_1^p)^{i_1} \dots (a_r^p)^{i_r}$ where $0 \leq i_j \leq |a_j|$, that is, $x = (a_1^{i_1} \dots a_r^{i_r})^p$. If at least one of i_k 's is not a multiple of p , set $a = a_1^{i_1} \dots a_r^{i_r}$, then $a \notin \Phi(H)$, which means that a is a generator of H and $\langle a^p \rangle = \langle x \rangle \leq \langle a \rangle$. If all i_k 's are multiple of p , then for $a = a_1^{\frac{i_1}{p}} \dots a_r^{\frac{i_r}{p}}$ we have that $x = a^{p^2}$. Thus, $\langle x \rangle \leq \langle a \rangle$. If one of $\frac{i_k}{p}$'s is not a multiple of p , then $a \notin \Phi(H)$ and a is a generator. If all $\frac{i_k}{p}$'s are a multiple of p then one can continue the process until getting an $\frac{i_k}{p}$ which is not a multiple of p for some $1 \leq k \leq r$. □

Lemma 2.5 *Let G be an abelian p -group of exponent p^n . Assume that H and K are isomorphic subgroups of G such that for some $x, y \in G$ of order p^n , we have that*

$$G/H = \langle xH \rangle \cong C_{p^m} \cong \langle yK \rangle = G/K$$

where $0 \leq m \leq n$. Then x^{p^m} is a generator of H if and only if y^{p^m} is a generator of K .

Proof Suppose that x^{p^m} is a generator of H , then $H = \langle x^{p^m} \rangle \times H_1$ for some $H_1 \leq H$. Suppose for a contradiction that y^{p^m} is not a generator of K . Then by Lemma 2.4, there exists a generator $a \in K$ such that $y^{p^m} = a^{p^t}$ for some t with $t > 0$. Note that $K = \langle a \rangle \times K_1$ for some $K_1 \leq K$. On the other hand, since the orders of x and y are equal, we have that $\langle x^{p^m} \rangle \not\cong \langle a \rangle$. Hence, since $H \cong K$, there exists $b \in H_1$ with $\langle b \rangle \cong \langle a \rangle$ and there exists $z \in K_1$ with $\langle z \rangle \cong \langle x^{p^m} \rangle$, such that $K = \langle a \rangle \times \langle z \rangle \times K_2$ and $H = \langle b \rangle \times \langle x^{p^m} \rangle \times H_2$ where $K_2 \cong H_2$. It follows that

$$\langle a \rangle / \langle a^{p^t} \rangle \times \langle z \rangle \times K_2 \cong K / \langle y^{p^m} \rangle \not\cong H / \langle x^{p^m} \rangle \cong \langle b \rangle \times H_2.$$

However, this contradicts with Lemma 2.3. The converse implication is similar. \square

Proposition 2.6 *Let G be an abelian p -group of exponent p^n . Assume that H and K are isomorphic subgroups of G such that for some $x, y \in G$ of order p^n , $G/H = \langle xH \rangle \cong C_{p^m} \cong \langle yK \rangle = G/K$, where $0 \leq m \leq n$. Then, there exists an isomorphism $\theta : H \rightarrow K$ such that $\theta(x^{p^m}) = y^{p^m}$.*

Proof If $m = n$ we have that $x^{p^m} = y^{p^m} = 1$, then for any isomorphism $\theta : H \rightarrow K$, we definitely have $\theta(x^{p^m}) = y^{p^m}$. Thus, we can assume that $m < n$. Then we have that $x^{p^m} \in H$ and $y^{p^m} \in K$. There are two cases.

Case 1: x^{p^m} is a generator of H .

By Lemma 2.5, y^{p^m} is also a generator of K . Then $H = \langle x^{p^m} \rangle \times H_1$ and $K = \langle y^{p^m} \rangle \times K_1$ where $H_1 \cong K_1$. Thus, one can choose $\theta_1 : H_1 \rightarrow K_1$ as an isomorphism, and define $\theta : H \rightarrow K$ as

$$\theta((x^{p^m})^l h) = (y^{p^m})^l \theta_1(h)$$

for $0 \leq l \leq p^{n-m} - 1$ and $h \in H_1$. It is clear that θ is an isomorphism from H onto K which satisfies $\theta(x^{p^m}) = y^{p^m}$.

Case 2: x^{p^m} is a nongenerator of H .

By Lemma 2.5, y^{p^m} is also a nongenerator of K . In this case, by Lemma 2.4, there exists a generator $a \in H$ such that $\langle x^{p^m} \rangle \leq \langle a \rangle$. Similarly, there exists a generator $b \in K$ such that $\langle y^{p^m} \rangle \leq \langle b \rangle$. We claim that $\langle a \rangle \cong \langle b \rangle$. Assume that $\langle a \rangle \not\cong \langle b \rangle$. Then as $H \cong K$, we have that $H = \langle a \rangle \times \langle b_1 \rangle \times H_1$ and $K = \langle a_1 \rangle \times \langle b \rangle \times K_1$ where $\langle a \rangle \cong \langle a_1 \rangle$, $\langle b_1 \rangle \cong \langle b \rangle$, $H_1 \cong K_1$. Note that $\langle a_1 \rangle \not\cong \langle b_1 \rangle$ since $\langle a \rangle \not\cong \langle b \rangle$. We have that

$$(\langle a \rangle / \langle x^{p^m} \rangle) \times \langle b_1 \rangle \times H_1 \cong H / \langle x^{p^m} \rangle \not\cong K / \langle y^{p^m} \rangle \cong \langle a_1 \rangle \times (\langle b \rangle / \langle y^{p^m} \rangle) \times K_1.$$

This contradicts with Lemma 2.3. Thus, $\langle a \rangle \cong \langle b \rangle$. Hence, $H = \langle a \rangle \times H_1$, $K = \langle b \rangle \times K_1$ where $H_1 \cong K_1$. Choose $\theta_1 : H_1 \rightarrow K_1$ as one of those isomorphisms. Then one can define $\theta : H \rightarrow K$ as

$$\theta(a^i \cdot h) = b^i \cdot \theta_1(h)$$

for $0 \leq i \leq |a| - 1$ and $h \in H_1$. Since the orders of a and b are equal, now it is easy to see that θ is an isomorphism between H and K which takes x^{p^m} to y^{p^m} . \square

Lemma 2.7 *Assume that $G = H \times K$, where $(|H|, |K|) = 1$. Then, we have that G_1 is a cocyclic subgroup of G if and only if $G_1 = H_1 \times K_1$ where H_1 is a cocyclic subgroup of H and K_1 is a cocyclic subgroup of K .*

Proof If H_1 is a cocyclic subgroup of H and K_1 is a cocyclic subgroup of K , it is easy to see that $H_1 \times K_1$ is a cocyclic subgroup of $H \times K$ since the orders of H and K are coprime. Conversely, if G_1 is a cocyclic subgroup of $H \times K$. Then, since H and K are groups of coprime order, we have that $G_1 = H_1 \times K_1$, where $H_1 \leq H$ and $K_1 \leq K$. However, since G_1 is a cocyclic subgroup of $H \times K$, we have that H_1 is a cocyclic subgroup of H and K_1 is a cocyclic subgroup of K . \square

Now we are ready to prove Proposition 1.1.

Proof [Proof of Proposition 1.1] Since any pair of G -isomorphic subgroups of G are isomorphic by definition, it is enough to prove that isomorphic cocyclic subgroups of G are G -isomorphic. Let us prove this statement first for p -groups. Assume that G is an abelian p -group of exponent p^n and let H and K be two cocyclic isomorphic subgroups of G . If $H = K = G$ there is nothing to do, so assume that $H, K < G$. By Lemma 2.2, there exist $x, y \in G$ of order p^n such that $G/H = \langle xH \rangle$ and $G/K = \langle yK \rangle$. Since H and K are isomorphic, their cyclic quotient groups G/H and G/K are isomorphic. Let $G/H = \langle xH \rangle \cong C_{p^m} \cong \langle yK \rangle = G/K$ for some m with $0 < m \leq n$. By using Proposition 2.6, we can choose an isomorphism $\theta : H \rightarrow K$ so that $\theta(x^{p^m}) = y^{p^m}$. For each $g \in G$ there exist a unique i where $0 \leq i \leq p^m - 1$ and a unique $h \in H$ such that $g = x^i h$. Now, define $\varphi : G \rightarrow G$ as $\varphi(g) = y^i \theta(h)$. We claim that φ is an automorphism of G . It is easy to see that φ is a bijection. To show that it is a homomorphism let us choose $g_1 = x^{i_1} h_1$ and $g_2 = x^{i_2} h_2$ for $0 \leq i_1, i_2 < p^m$ and $h_1, h_2 \in H$. Note that $0 \leq i_1 + i_2 < 2p^m$, we will show that φ is a homomorphism by considering two separate cases depending on the value of this sum.

Case 1: $0 \leq i_1 + i_2 < p^m$

In this case $\varphi(g_1 g_2) = \varphi(x^{i_1+i_2} h_1 h_2) = y^{i_1+i_2} \theta(h_1 h_2) = (y^{i_1} \theta(h_1))(y^{i_2} \theta(h_2)) = \varphi(g_1) \varphi(g_2)$.

Case 2: $p^m \leq i_1 + i_2 < 2p^m$

In this case, we have that

$$\varphi(g_1 g_2) = \varphi((x^{i_1+i_2-p^m})(x^{p^m} h_1 h_2)) = y^{i_1+i_2-p^m} \theta(x^{p^m} h_1 h_2) = y^{i_1+i_2-p^m} \theta(x^{p^m}) \theta(h_1) \theta(h_2)$$

from the definition of φ and from the fact that $x^{p^m} \in H$. However, this last expression is equal to $y^{i_1+i_2} \theta(h_1 h_2)$ since $y^{p^m} = \theta(x^{p^m})$. Moreover, $\varphi(g_1) \varphi(g_2) = (y^{i_1} \theta(h_1))(y^{i_2} \theta(h_2)) = y^{i_1+i_2} \theta(h_1 h_2)$. Hence, $\varphi(g_1 g_2) = \varphi(g_1) \varphi(g_2)$ in this case, too.

In both of the cases, we have shown that φ is an automorphism of G and it is easy to observe that φ takes H onto K . Hence, H and K are G -isomorphic.

Now, let G be a finite abelian group whose order is composite. For $1 \leq i \leq r$, let G_{p_i} denote a Sylow p_i -subgroup of G . Then $G = G_{p_1} \times G_{p_2} \times \dots \times G_{p_r}$. If H and K are two cocyclic subgroups of G which are isomorphic, $H = H_{p_1} \times H_{p_2} \times \dots \times H_{p_r}$ and $K = K_{p_1} \times K_{p_2} \times \dots \times K_{p_r}$ where for each $1 \leq i \leq r$, one has $H_{p_i} \cong K_{p_i}$. Moreover, by Lemma 2.7, for each i , the groups H_{p_i} and K_{p_i} are cocyclic subgroups of G_{p_i} . Hence, by the first part of the proof, H_{p_i} and K_{p_i} are G_{p_i} -isomorphic subgroups or equivalently there exists an automorphism φ_i of G_{p_i} which takes H_{p_i} onto K_{p_i} . Now let us set $\varphi := (\varphi_1, \varphi_2, \dots, \varphi_r)$, then it is easy to see that φ is an automorphism of G which takes H onto K . \square

Proof [Proof of Theorem 1.2] Follows from Proposition III.2, Proposition III.7, and Proposition III.8 in [7], together with Proposition 1.1. \square

For the proofs of Theorem 1.3, Theorem 1.4, and Theorem 1.7 we need to consider direct products of groups whose orders are relatively prime.

When G is equal to the direct product of subgroups of coprime order, the number of isomorphism classes of cocyclic subgroups, hence the number of non- G -equivalent minimal abelian codes is calculated easily as follows.

Theorem 2.8 *Let $G = H \times K$ where $(|H|, |K|) = 1$. Then we have that $\eta(\mathbb{F}G) = \eta(\mathbb{F}K) \eta(\mathbb{F}H)$.*

Proof By Lemma 2.7, any cocyclic subgroup of G is of the form $H_1 \times K_1$ where H_1 is a cocyclic of H and K_1 is a cocyclic of K . Thus, the number of isomorphism classes of cocyclic subgroups of G is the product of number of isomorphism classes of cocyclic subgroups of H and the number of isomorphism classes of cocyclic subgroups of K . Now the result follows from Theorem 1.2. \square

Now, thanks to Theorem 1.2, to count the number of non- G -equivalent minimal abelian codes over \mathbb{F} , we just need to count the number of isomorphism classes of cocyclic subgroups of G .

Proof [Proof of Theorem 1.3] By using classification of finitely generated abelian groups and Theorem 2.8, it is enough to prove the result when H is a finite p -group. Let $H = C_{p^{a_1}} \times \dots \times C_{p^{a_n}}$ where $a_i \geq 1$ are integers and $G = H^k$ for $k \geq 1$, then $G = G_1 \times \dots \times G_n$ where $G_i = (C_{p^{a_i}})^k$ for $i = 1, \dots, n$. Let L be a cocyclic subgroup of G . For each i , we have that

$$G_i/G_i \cap L \cong G_i L/L \leq G/L$$

, which implies that $G_i/G_i \cap L$ is cyclic. Thus, $G_i \cap L$ should contain a subgroup L_i which is isomorphic to $(C_{p^{a_i}})^{k-1}$ (for example by [7, Theorem V.2]). Moreover, it is easy to see that for each i , there exists an element $x_i \in G_i$ of order p^{a_i} such that $G_i = L_i \times \langle x_i \rangle$. Hence,

$$G = \left(\prod_{i=1}^n L_i \right) \times \left(\prod_{i=1}^n \langle x_i \rangle \right),$$

where the first term of the product is isomorphic to $(H)^{k-1}$ and the second is isomorphic to H . Now, by the use of the correspondence theorem, there is a bijection between the subgroups of G containing $\prod_{i=1}^n L_i$ and the subgroups of $\prod_{i=1}^n \langle x_i \rangle$. Under this bijection, L corresponds to a cocyclic subgroup C_L of $\prod_{i=1}^n \langle x_i \rangle$, where $L = \left(\prod_{i=1}^n L_i \right) \times C_L$. By Theorem 1.2, the result follows since $\prod_{i=1}^n \langle x_i \rangle$ is isomorphic to H . \square

Proof [Proof of Theorem 1.4] It is enough to prove the result when H and K are finite p -groups by the classification of finitely generated abelian groups and Lemma 2.8. Let p^n be the exponent of H and K . Then there exists $x \in H$ of order p^n such that $H = \langle x \rangle \times \hat{H}$ where \hat{H} is a finite p -group of exponent less or equal than p^n and $K \cong (C_{p^n})^r$ for some positive integer r . Let $G_1 \leq G = H \times K$ with $G_1 \cong (C_{p^n})^{r+1}$ so that $G = G_1 \times \hat{H}$ and let L be a cocyclic subgroup of G . Then by a similar reasoning to that in the proof of Theorem 1.3, we deduce that $G_1/G_1 \cap L$ is cyclic, so $G_1 \cap L$ contains a subgroup isomorphic to $(C_{p^n})^r$, call this subgroup K_L . Then there exists an element $x_1 \in G_1$ of order p^n such that $G_1 = K_L \times \langle x_1 \rangle$. Thus, $G = K_L \times \langle x_1 \rangle \times \hat{H}$ and letting $H_L = \langle x_1 \rangle \times \hat{H}$, G is equal to $K_L \times H_L$, where K_L and H_L are isomorphic to K and H , respectively. Since L is a cocyclic subgroup of $K_L \times H_L$ containing K_L , by the correspondence theorem, L corresponds to a cocyclic subgroup C_L of H_L . Therefore, $L = K_L \times C_L$ where K_L is isomorphic to K and C_L is cocyclic subgroup of H_L . Now, the result follows from Theorem 1.2. \square

In Theorem 1.4, the assumption on the exponents of the groups is important. We end this section by presenting the significance of this assumption with the following examples.

Example 2.1 For an odd prime p , if we take $H = C_p \times C_p$ and $K = C_{p^2} \times C_{p^2}$, then $\eta(\mathbb{F}H) = 2$, $\eta(\mathbb{F}K) = 3$ and $\eta(\mathbb{F}(H \times K)) = 4$ where the characteristic of \mathbb{F} is coprime to p .

Example 2.2 Let K be a finite homocyclic group and H be a finite abelian group which is not homocyclic and $\exp(K) > \exp(H)$. Let \mathbb{F} be a finite field of characteristic coprime to 3. If $G = K \times H$, $\eta(\mathbb{F}G) = \eta(\mathbb{F}H)$ is no longer true. Indeed, if $K = C_{27} \times C_{27}$ and $H = C_9 \times C_3$, then $\eta(\mathbb{F}H) = 4$, $\eta(\mathbb{F}K) = 4$ and $\eta(\mathbb{F}G) = 8$. However, for $G = C_{27} \times C_{27} \times C_9 \times C_3$ if we write $G = H \times K$ where $K = C_{27}$ and $H = C_{27} \times C_9 \times C_3$ then $\eta(\mathbb{F}G) = \eta(\mathbb{F}H)$.

Example 2.3 Consider $G = C_{27} \times C_9 \times C_3 \times C_3$ and take $H = C_{27} \times C_9$ and $K = C_3 \times C_3$. $\eta(\mathbb{F}H) = 6$, $\eta(\mathbb{F}K) = 2$ and $\eta(\mathbb{F}G) = 8$, where the characteristic of \mathbb{F} is coprime to 3.

3. Calculation for $C_{p^n} \times C_{p^m}$

For the proof of Theorem 1.5, we need the following lemma.

Lemma 3.1 Let $G = \langle a \rangle \times \langle b \rangle$ where $\langle a \rangle \cong C_{p^n}$ and $\langle b \rangle \cong C_{p^m}$ with $n > m \geq 1$. Assume that L is a cocyclic subgroup of G which is not cyclic. Then $L \cong C_{p^i} \times C_{p^j}$ where $m \leq i \leq n$ and $1 \leq j \leq m$.

Proof Let L be such a cocyclic subgroup of G . If the exponent of L is p^n , then $L \cong C_{p^n} \times C_{p^j}$ where $1 \leq j \leq m$. If the exponent of L is strictly less than p^n , by Lemma 2.1, $G/L = \langle aL \rangle$, so that $G = L\langle a \rangle$. Thus, we have that

$$C_{p^m} \cong G/\langle a \rangle = L\langle a \rangle/\langle a \rangle \cong L/L \cap \langle a \rangle$$

that is, L has a quotient isomorphic to C_{p^m} ; hence, L has a subgroup isomorphic to C_{p^m} . Thus, the exponent of L is at least p^m , in this case. Hence, $L \cong C_{p^i} \times C_{p^j}$ for $i \geq m$ and $j \geq 1$. As $|L| \geq p^{m+1}$, $|G/L| \leq p^{(m+n)-(m+1)} = p^{n-1}$. Therefore, the index of L in G is at most p^{n-1} .

We prove the required result by induction on the index of the cocyclic subgroup L in G . Clearly, the statement holds when $|G/L| = 1$. If $|G/L| = p$, then either $L \cong C_{p^{n-1}} \times C_{p^m}$ or $L \cong C_{p^n} \times C_{p^{m-1}}$. Assume the statement holds for any noncyclic cocyclic subgroup of G with index strictly less than p^s where $1 \leq s \leq n-1$. Now let L be a cocyclic subgroup of G such that $|G/L| = p^s$. Then there exists a cocyclic subgroup L_1 of G such that $L < L_1 \leq G$ where $|G/L_1| = p^{s-1}$. By our induction hypothesis, $L_1 \cong C_{p^i} \times C_{p^j}$ where $m \leq i \leq n$ and $1 \leq j \leq m$. Moreover, i and j satisfy $m+n-(i+j) = s-1 \leq n-2$. We also have that $L_1/L \cong C_p$. Thus, if $i \neq m$ and $j \neq 1$, then we deduce that $L \cong C_{p^{i-1}} \times C_{p^j}$ or $L \cong C_{p^i} \times C_{p^{j-1}}$. If $i = m$, then $j \geq 2$ by the inequality $m+n-(i+j) = s-1 \leq n-2$. Since the exponent of L is at least p^m , we shall have that $L \cong C_{p^m} \times C_{p^{j-1}}$. If $j = 1$, then by the same inequality we have that $i \geq m+1$. As L is not cyclic, we shall have that $L \cong C_{p^{i-1}} \times C_p$. Therefore, we deduce that $L \cong C_{p^i} \times C_{p^j}$ where $m \leq i \leq n$ and $1 \leq j \leq m$. □

Proposition 3.2 *Let $G = \langle a \rangle \times \langle b \rangle$ where $\langle a \rangle \cong C_{p^n}$ and $\langle b \rangle \cong C_{p^m}$ with $n > m \geq 1$. Then any cocyclic subgroup of G is isomorphic to one of the following subgroups in the following set*

$$\{H_k \times K_j \mid H_k = \langle a^{p^k} b \rangle, K_j = \langle b^{p^j} \rangle, 0 \leq k \leq n - m, 0 \leq j \leq m\}.$$

Proof There are two cases to consider.

Case 1 (Cocyclic subgroups of G which are cyclic): For each $k \in \{0, \dots, n - m\}$, H_k is a cocyclic subgroup of G , because $G/H_k = \langle aH_k \rangle \cong C_{p^{m+k}}$. Notice that there are exactly $n - m + 1$ such subgroups of G . Up to isomorphism, there is no other cyclic cocyclic subgroup of G . Indeed, if there is one such subgroup H which is not isomorphic to any H_k for $0 \leq k \leq n - m$, then $|H| = p^s$ where $s \in \{0, \dots, m - 1\}$. In this case, $G/H \cong C_{p^{n+m-s}}$ where $n + m - s \geq n + 1$, but this is impossible since the exponent of G is equal to p^n . Note that $H_k \cong H_k \times K_m$ because $K_m = 1$.

Case 2 (Cocyclic subgroups of G which are not cyclic): From Case 1, we know that H_k is a cocyclic subgroup of G , for each k , so that G/H_k is cyclic. Since for each j , the quotient $G/(H_k \times K_j)$ is isomorphic to a subgroup of G/H_k , we have that $H_k \times K_j$ is a cocyclic subgroup of G for any $j \in \{0, \dots, m\}$. Conversely, using Lemma 3.1, we deduce that any cocyclic subgroup is isomorphic to one of $H_k \times K_j$ since $H_k \cong C_{p^{n-k}}$ where $k \in \{0, \dots, n - m, \}$ and $K_j \cong C_{p^{m-j}}$ for $j \in \{0, \dots, m - 1\}$. □

Proof [Proof of Theorem 1.5] By Proposition 3.2, the number of isomorphism classes of cocyclic subgroups of G is $(n - m + 1)(m + 1)$. By Theorem 1.2, $\eta(\mathbb{F}G) = (n - m + 1)(m + 1)$. □

An immediate consequence of Theorem 1.5 and Theorem 1.4 is the following result.

Corollary 3.3 *Let n, m and s be positive integers such that $n > m$. If $G = (C_{p^n} \times C_{p^m})^s$ for $s \in \mathbb{N}$, then $\eta(\mathbb{F}G) = (n - m + 1)(m + 1)$. Moreover, if $G = C_{p^n} \times C_{p^m} \times (C_{p^n})^s$, then $\eta(\mathbb{F}G) = (n - m + 1)(m + 1)$.*

4. Proof of Theorem 1.7

Let $\tau(G)$ denote the number of divisors of the exponent of G . It is not difficult to see that the number of non- G -equivalent minimal abelian codes is greater than or equal to $\tau(G)$ when G is a finite abelian group. Therefore, if the exponent of G is given, Theorem 1.7 gives a complete characterization of the groups having $\tau(G)$ non- G -equivalent minimal abelian codes, that is, having the least possible number of non- G -equivalent abelian codes. For the proof of Theorem 1.7, first of all we find the number of non- G -equivalent minimal abelian group codes for homocyclic p -groups and prove the following.

Theorem 4.1 *Let G be a finite abelian p -group. The number of non- G -equivalent minimal abelian codes is equal to $\tau(G)$ if and only if G is homocyclic.*

Proof Assume that G is homocyclic, that is, $G \cong (C_{p^n})^s$. Then by Theorem 1.3, $\eta(\mathbb{F}G) = \eta(\mathbb{F}C_{p^n})$. Now it is clear that the number of isomorphism classes of subgroups of C_{p^n} is equal to the number of divisors of p^n . For the converse implication, assume that G is not homocyclic. If the exponent of G is p^r for some $r \geq 1$, then $G \cong C_{p^r} \times H$ where $H \cong K \times C_{p^i}$ for some $1 \leq i \leq r - 1$ for some subgroup K of H . Then $H, H \times C_p, H \times C_{p^2}, \dots, H \times C_{p^{r-1}}$ is a family of nonisomorphic cocyclic subgroups of G . Obviously, $K \times C_{p^r}$ is another cocyclic subgroup which is not isomorphic to none of the elements of this family. Thus, we have at

least $r + 2$ nonisomorphic cocyclic subgroups. Hence, $\eta(\mathbb{F}G)$ is at least $r + 2$. This leads to a contradiction because $\tau(G) = r + 1$. □

Proof [Proof of Theorem 1.7] Let $G = G_{p_1} \times G_{p_2} \times \dots \times G_{p_k}$ where each G_{p_i} is a homocyclic Sylow p_i -subgroup of G . If the exponent of each G_{p_i} is p^{e_i} , then by Theorem 4.1, $\eta(\mathbb{F}G_{p_i})$ is equal to $\tau(G_{p_i}) = e_i + 1$. By Lemma 2.8, $\eta(\mathbb{F}G)$ is equal to $\prod_{i=1}^k (e_i + 1)$ which is equal to $\tau(G)$.

For the converse implication, assume for some i , a Sylow p_i -subgroup G_{p_i} is not homocyclic. Then by Theorem 4.1, $\eta(\mathbb{F}G_{p_i}) > \tau(G_{p_i}) = e_i + 1$, which gives a contradiction. □

5. Concluding remarks

Let $I \subseteq \mathbb{F}G$ be a code. Any element $\alpha \in I$ is written as $\alpha = \sum_{g \in G} \alpha_g g$ where $\alpha_g \in \mathbb{F}$ for each $g \in G$. The weight of an element $\alpha \in I$ is defined to be $w(\alpha) = |\{g \in G \mid \alpha_g \neq 0\}|$. The weight of I is defined to be $w(I) = \min\{w(\alpha) \mid \alpha \text{ is a nonzero element in } I\}$.

If two codes I and J in $\mathbb{F}G$ are G -equivalent, then they have equal dimensions, equal lengths, and equal weights. The converse of this statement is not true in general. For example, Proposition IV.2 and Table I in [7] show that there are non- G -equivalent codes which have equal weights and equal dimensions. By using Table VI in [7], for the group $G = (C_p^r)^m$ we have that two minimal abelian codes $I, J \subseteq \mathbb{F}G$ are G -equivalent if and only if they have equal weights. Naturally, one can ask the following question:

Question 5.1 *For which abelian groups G can we say that any two minimal codes are G -equivalent if and only if their weights (or dimensions) are equal?*

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