Ulam-Hyers stability results for a novel nonlinear Nabla Caputo fractional variable-order difference system

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Abstract: This paper is concerned with a kind of nonlinear Nabla Caputo fractional difference system with variable-order and fixed initial valuable. By applying Krasnoselskii's fixed point theorem, we give some sufficient conditions to guarantee the existence results for the considered fractional discrete equations. In addition, we further consider the Ulam-Hyers stability by means of generalized Gronwall inequality. At last, two typical examples are delineated to demonstrate the effectiveness of our theoretical results.

Key words: Nabla Caputo fractional difference equation, variable-order, existence, Ulam-Hyers stability

1. Introduction

Fractional difference equations play an important role in promoting the development of modern mathematics and have been widely applied, especially in physics, dynamic mechanics, medicines, and communications. There is a growing tendency nowadays that many experts show their great enthusiasms for fractional difference equations, and in the past few years, a lot of achievements have been done. For an extensive collection of such results, we recommend the readers the monographs [4, 6, 14].

As for fractional discrete equations, we usually investigate the existence and stability properties. Such as in [16], Henderson got the existence conditions of solutions by applying Leray–Schauder nonlinear alternative method. In [18, 23, 35], the authors studied fractional difference equations, and the existence of solutions were established by employing Schauder's fixed point theorem. In [10, 19], Luo and Chen investigated the uniqueness results for a class of nonlinear fractional difference system with time delay and gave the proof by contradiction and generalized Gronwall inequality. He et al. gave existence results for fractional discrete equations by means of topological degree methods in [15], and many other conclusions can be seein in [1–3].

On the other hand, stability analysis is also one of the most popular themes for fractional difference system; for example in [13, 19, 34], the authors researched finite-time stability in fractional difference system. In [5, 7, 31], some results about asymptotic stability of fractional order difference equations were given. We also saw that some achievements about Mittag–Leffler stability in fractional difference equations had been mentioned in [33]. To best of our knowledge, there is little published paper which have considered the Ulam–Hyers and

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Ulam–Hyers–Rassias stability in fractional difference equations.

The famous Ulam–Hyers problem goes back to the years 1940–1941 when Ulam [28] and Hyers [17] firstly proposed this issue, and many mathematicians had considered the wide scope of this same problem for fractional equations of different types, such as [20–22, 24–27, 29, 30]. The common characteristic of those systems referred is that the order is fixed. However, we know many physical processes exhibited memory effects that may vary with time or space, and the traditional identical fractional order can not describe the changing rules of things very well. Therefore we wish to research the existence and a novel stability in the variable-order fractional difference equations.

We now introduce the notations $\mathbb{N}_a = \{a, a+1, a+2, ...\}$, $\mathbb{N}_a^T = \{a, a+1, a+2, ...T\}$, $T, a \in \mathbb{N}^+$, and there exists a bounded integer sequence $\{t_{Nl}\}_{N=1}^m$ such that $[t_0, T] = [t_0, t_l] \cup [t_l, t_{2l}] \cup \cdots \cup [t_{(N-1)l}, t_{Nl}] \cup \cdots \cup [t_{(m-1)l}, t_{ml}]$, $t_0 = a$, $t_{ml} = T$, and $t_{Nl} - t_{(N-1)l} > 1$. Inspired by the mentioned papers, we will discuss the existence of solutions and Ulam-Hyers stability of the following variable-order fractional discrete equations

$$\begin{cases} {}_{t_*}^C \nabla_t^{\nu_k} x(t) = Ax(t) + f(t, x(t)), \ t \in \mathbb{N}_a^T, \\ x(t_0) = y(t_0), \end{cases} \tag{1.1}$$

where $0 < \nu_k \leq 1$, $t_* = t_{kl}$, $k = 0, 1, 2, \cdots, m-1$, ${}_{t_*}^C \nabla_t^{\nu_k}$ denotes the Nabla Caputo fractional difference operators, $A : \mathcal{C}\left(\mathbb{N}_a^T; \mathbb{R}^n\right) \to \mathbb{R}^n$ is a bounded linear operator, and $f : \mathbb{N}_a^T \times \mathbb{R}^n \to \mathbb{R}^n$ specified later.

Compared with some literatures just mentioned, the highlights and major contributions of this paper are reflected in the subsequent key aspects:

(1) The fractional difference system we studied is a variable-order system, which is quite different from other systems in the literatures.

(2) An innovative method based on the generalized Gronwall inequality is exploited to discuss the Ulam-Hyers stability of the fractional order difference equations with variable-order and fixed initial valuable. The results established are essentially new.

The article is organized as follows: In Sect. 2, we will recall some known results including useful lemmas and definitions for our considerations. Sect. 3 is devoted to researching the existence of solutions for variable-order Nabla Caputo fractional difference system. Our methods rely on application of the Krasnoselskii's fixed point theorem. Subsequently, we investigate the Ulam–Hyers-Rassias and Ulam–Hyers stabilities of the addressed fractional difference system, and then we will come up with the main theorems in Sect. 4. To explain the results clearly, we finally provide some examples in Sect. 5.

2. Preliminaries
In this section, we plan to introduce some basic definitions, lemmas, and fundamental properties of discrete Nabla fractional calculus which are used throughout this paper.

Definition 2.1 *([14]) Let $f : \mathbb{N}_a \to \mathbb{R}$. We define the Nabla operator (backwards difference operator) ∇ by*

$$\nabla f(t) = f(t) - f(t-1), \quad t \in \mathbb{N}_{a+1}.$$

We introduce the fractional ν-th order Nabla Taylor monomial:

Definition 2.2 *([14]) For $v \neq -1, -2, -3, \cdots$. Then we define the ν-th order Nabla fractional Taylor monomial $H_\nu(t, a)$ by*

$$H_\nu(t, a) := \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+1)}, \quad t \in \mathbb{N}_a,$$

where $t^{\overline{\nu}} = \frac{\Gamma(t+\nu)}{\Gamma(t)}$.

Corollary 2.3 *When $\nu \to 0$, we can know $H_\nu(t, a) \to 1$.*

Lemma 2.4 *([14]) Some properties for $H_v(\cdot, \cdot)$ are given as follows:*

(1) $H_v(a, a) = 0$;

(2) $\nabla H_{v+1}(t, a) = H_v(t, a)$;

(3) $\int_a^t H_v(s, a) \nabla s = H_{v+1}(t, a)$;

(4) $\int_a^t H_v(t, \rho(s)) \nabla s = H_{v+1}(t, a)$.

Definition 2.5 *([14]) Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$ be given and assume $v > 0$. Then*

$$\nabla_a^{-v} f(t) := \int_a^t H_{v-1}(t, \rho(s)) f(s) \nabla s$$

$$= \sum_{s=a+1}^t H_{v-1}(t, \rho(s)) f(s), \quad t \in \mathbb{N}_{a+1},$$

where $\rho(t) := t - 1$ and $\nabla_a^{-v} f(a) := 0$.

Definition 2.6 *([19]) Assume $f : \mathbb{N}_a \to \mathbb{R}$, and $0 < v \leq 1$. Then the ν-th Caputo Nabla fractional difference of f is defined by*

$$\nabla_a^v f(t) := \nabla_a^{-(1-v)} \nabla^1 f(t), \quad t \in \mathbb{N}_{a+1}.$$

Lemma 2.7 *([14]) We consider the Nabla fractional initial value problem (IVP)*

$$\begin{cases} \nabla_a^\nu x(t) = h(t), \ t \in \mathbb{N}_{a+1}, \\ \nabla^k x(a) = c_k, \ 0 \leq k \leq N-1, \end{cases} \tag{2.1}$$

where we always assume that $a, \nu \in \mathbb{R}$, $\nu > 0$, $N := \lceil \nu \rceil$, $c_k \in \mathbb{R}$ for $0 \leq k \leq N-1$, and $h : \mathbb{N}_{a+1} \to \mathbb{R}$. Then unique solution to the IVP (2.1) is given by

$$x(t) = \sum_{k=0}^{N-1} H_k(t, a) c_k + \nabla_a^{-\nu} h(t),$$

for $t \in \mathbb{N}_{a-N+1}$, where by convention $\nabla_a^{-\nu} h(t) = 0$ for $a - N + 1 \leq t \leq a$.

Lemma 2.8 *(Generalized Gronwall inequality [11]) Assume $f(t)$ and $g(t)$ are nonnegative, nondecreasing functions on J. Let $x(t)$ be a nonnegative function on J and $g(t) \leq M < 1$, $t \in J$. Suppose for $0 < v \leq 1$, the following nabla fractional inequality holds:*

$$x(t) \leq f(t) + g(t) \left(\nabla_a^{-v} x \right)(t), \quad t \in J.$$

Then

$$x(t) \leq f(t) \sum_{j=0}^{\infty} g^j(t) H_{jv}(t, a), \quad t \in J,$$

where $H_{jv}(t, a)$ is defined as in Definition 2.2.

Lemma 2.9 *A function $x(t)$ is called the solution of (1.1) if $x(t)$ satisfies*

$$x(t) =$$
$$\begin{cases} x(t_0) + \int_{t_0}^t H_{\nu_0 - 1}(t, s - 1) \left[Ax(s) + f(s, x(s)) \right] \nabla s, \ t \in \mathbb{N}_{t_0}^{t_l}, \\ x(t_0) + \sum_{k=2}^N \int_{t_{(k-2)l}}^{t_{(k-1)l}} H_{\nu_{(k-2)} - 1} \left(t_{(k-1)l}, s - 1 \right) \left[Ax(s) + f(s, x(s)) \right] \nabla s \\ \quad + \int_{t_{(N-1)l}}^t H_{\nu_{(N-1)} - 1}(t, s - 1) \left[Ax(s) + f(s, x(s)) \right] \nabla s, \ t \in \mathbb{N}_{t_{(N-1)l}}^{t_{Nl}}, N = 2, 3, \cdots, m. \end{cases} \tag{2.2}$$

Proof As for $t \in \mathbb{N}_{t_0}^{t_l}$, we let $N = 1$ in Lemma 2.7, then we get that the solution of (1.1) can be expressed

$$x(t) = x(t_0) + \nabla_{t_0}^{-\nu_0} \left[Ax(t) + f(t, x(t)) \right],$$

and by Definition 2.5, we can further obtain that

$$x(t) = x(t_0) + \int_{t_0}^t H_{\nu_0 - 1}(t, s - 1) \left[Ax(s) + f(s, x(s)) \right] \nabla s.$$

Using the same approach, we can get the expression of $x(t)$, for any $t \in \mathbb{N}_{t_l}^{t_{2l}}$,

$$x(t) = x(t_0) + \int_{t_0}^{t_l} H_{\nu_{(0)} - 1} \left(t_l, s - 1 \right) \left[Ax(s) + f(s, x(s)) \right] \nabla s$$

$$+ \int_{t_l}^t H_{\nu_1 - 1}(t, s - 1) \left[Ax(s) + f(s, x(s)) \right] \nabla s.$$

For any $t \in \mathbb{N}_{t_{(N-1)l}}^{t_{Nl}}$, $N = 3, 4, \cdots m$, we can similarly derive $x(t)$ as following

$$x(t) = x(t_0) + \sum_{k=2}^N \int_{t_{(k-2)l}}^{t_{(k-1)l}} H_{\nu_{(k-2)} - 1} \left(t_{(k-1)l}, s - 1 \right) \left[Ax(s) + f(s, x(s)) \right] \nabla s$$

$$+ \int_{t_{(N-1)l}}^t H_{\nu_{(N-1)} - 1}(t, s - 1) \left[Ax(s) + f(s, x(s)) \right] \nabla s,$$

and the proof is completed.

We present the concepts of Ulam–Hyers and Ulam–Hyers–Rassias stabilities:

Definition 2.10 *([22]) If for any function $y(t) : \mathbb{N}_a^T \to \mathbb{R}$ satisfying*

$$\left\| {}^C_{t_*} \nabla^{\nu_k}_t y(t) - Ay(t) - f(t, y(t)) \right\| \leq \Phi(t), \tag{2.3}$$

where $\Phi(t) \in \mathbb{R}$ is a positive function for all $t \in \mathbb{N}_a^T$, there exists a solution $x(t) : \mathbb{N}_a^T \to \mathbb{R}$ of system (1.1) and a constant $C > 0$ with

$$\|x(t) - y(t)\| \leq C\Phi(t),$$

where C is independent of $x(t)$ and $y(t)$, then we say that system (1.1) has the Ulam–Hyers–Rassias stability. If $\Phi(t)$ is a constant function in the above inequalities, we say that system (1.1) has the Ulam–Hyers stability.

Denote that $E = \left\{ x : x(t) \in \mathcal{C}\left(\mathbb{N}_a^T ; \mathbb{R}^n \right) \right\}$ and endowed with the norm $\|x\| = \sup\limits_{t \in \mathbb{N}_a^T} \left(\sum\limits_{i=1}^n x_i^2(t) \right)^{\frac{1}{2}}$ for $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in \mathbb{R}^n$. Then, $(E, \|\cdot\|)$ is a Banach space.

Definition 2.11 *([12]) A set Ω of sequences in E is uniformly Cauchy (or equi-Cauchy) if for every $\varepsilon > 0$, there exists an integer N such that $|x(i) - x(j)| < \varepsilon$ whenever $i, j > N$ for any $x = \{x(n)\}$ in Ω.*

Theorem 2.12 *([12]) (Discrete Arzelà-Ascoli's theorem) A bounded, uniformly Cauchy subset Ω of E is relatively compact.*

Theorem 2.13 *([8]) (Krasnoselskii's fixed point theorem) Let S be a nonempty, closed, convex, and bounded subset of a Banach space E, and let $P : E \to E$ and $Q : S \to E$ be two operators such that*

(1) P is a contraction with constant $L < 1$;

(2) Q is continuous, QS resides in a compact subset of E;

(3) $[x = Px + Qy; y \in S] \implies x \in S$.

Then the operator equation $Px + Qx = x$ has a solution in S.

3. Existence results

In this section, we will consider the existence theorems for a nonlinear variable-order Nabla Caputo fractional difference system (1.1). And we define the operator

$$
\begin{aligned}
Tx(t) = {} & x(t_0) + \sum_{k=2}^N \int_{t_{(k-2)l}}^{t_{(k-1)l}} H_{\nu_{(k-2)}-1}\left(t_{(k-1)l}, s-1\right)\left[Ax(s) + f(s, x(s))\right]\nabla s \\
& + \int_{t_{(N-1)l}}^t H_{\nu_{(N-1)}-1}(t, s-1)\left[Ax(s) + f(s, x(s))\right]\nabla s, \ t \in \mathbb{N}_{t_{(N-1)l}}^{t_{Nl}},
\end{aligned} \tag{3.1}
$$

and $N = 2, 3, \cdots, m$.

It is easily concluded that x is a solution of (1.1) iff x is a fixed point of operator T. We plan to adopt the Krasnoselskii's fixed point theorem to establish existence results, and for any $t \in \mathbb{N}_{t_{(N-1)l}}^{t_{Nl}}$, $N = 2, 3, \cdots, m$,

we also define operators

$$Px(t) = \sum_{k=2}^{N} \int_{t_{(k-2)l}}^{t_{(k-1)l}} H_{\nu_{(k-2)}-1}\left(t_{(k-1)l}, s-1\right) \left[Ax(s) + f\left(s, x(s)\right)\right] \nabla s, \tag{3.2}$$

and

$$Qx(t) = x(t_0) + \int_{t_{(N-1)l}}^{t} H_{\nu_{(N-1)}-1}(t, s-1) \left[Ax(s) + f\left(s, x(s)\right)\right] \nabla s. \tag{3.3}$$

Before stating the main results, we introduce the following assumptions:

(H_1) For all $t \in \mathbb{N}_a^T$, $f(t, x)$ represents a continuous function with respect to x, and there exists a constant $L \in \mathbb{R}^+$ such that

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|.$$

(H_2) For all $t \in \mathbb{N}_a^T$, $x \in \mathbb{R}^n$, there exists a constant $M_A \in \mathbb{R}^+$ such that $\|Ax(t)\| \leq M_A\|x(t)\|$.

Theorem 3.1 *Suppose the validity of $(H_1) - (H_2)$, and for any $(t, x) \in \left(\mathbb{N}_a^T, \mathbb{R}^n\right)$, $f(t, x) \leq M_1$ holds, and there exists positive δ such that $\|x(t_0)\| \leq \delta$, then the fractional discrete equations (1.1) has at least one bounded solution in $\Omega_{r_1} = \{x(t) \in E : \|x\| \leq r_1\}$ provided that*

$$r_1 \geq \frac{\delta M^{-1} + M_1}{M^{-1} - M_A}, \tag{3.4}$$

and

$$0 < (M_A + L) \sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) < 1, \tag{3.5}$$

where $M = \sum\limits_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) + \sup\limits_{t \in \mathbb{N}_{t_{(N-1)l}}^T} H_{\nu_{(N-1)}}\left(t, t_{(N-1)l}\right)$, $0 < M < M_A^{-1}$, for any $t \in \mathbb{N}_{t_{(N-1)l}}^T$.

Proof It is obvious that Ω_{r_1} is a nonempty, closed, bounded, and convex subset of E.

Step 1. We show that Q maps Ω_{r_1} into Ω_{r_1}. By (3.3)–(3.4) and Lemma 2.4, for any $x \in \Omega_{r_1}$, we have

$$\begin{aligned}
\|Qx(t)\| &\leq \|x(t_0)\| + \int_{t_{(N-1)l}}^{t} H_{\nu_{(N-1)}-1}(t, s-1) \left[\|A\| \cdot \|x(s)\| + \|f\left(s, x(s)\right)\|\right] \nabla s \\
&\leq \delta + (M_A \cdot r_1 + M_1) \int_{t_{(N-1)l}}^{t} H_{\nu_{(N-1)}-1}(t, s-1) \nabla s \\
&\leq \delta + (M_A \cdot r_1 + M_1) \sup_{t \in \mathbb{N}_{t_{(N-1)l}}^T} H_{\nu_{(N-1)}}\left(t, t_{(N-1)l}\right) \leq r_1,
\end{aligned} \tag{3.6}$$

which implies $Q\Omega_{r_1} \subset \Omega_{r_1}$.

Step 2. We need to prove that Q is continuous. Let $\{x_n\}$ be a sequence of Ω_{r_1} satisfying $x_n \to x$ $(n \to +\infty)$. And by (H_1) and (3.3), for every $t \in \mathbb{N}^{t_{Nl}}_{t_{(N-1)l}}$, $N = 2, 3, \cdots, m$, one can obtain

$$
\|Qx_n(t) - Qx(t)\|
$$
$$
\leq \|x_n(t_0) - x(t_0)\| +
$$
$$
\int_{t_{(N-1)l}}^{t} H_{\nu_{(N-1)}-1}(t, s-1) \left[M_A \cdot \|x_n(s) - x(s)\| + L \cdot \|x_n(s) - x(s)\| \right] \nabla s \tag{3.7}
$$
$$
= \|x_n(t_0) - x(t_0)\| + (M_A + L) \int_{t_{(N-1)l}}^{t} H_{\nu_{(N-1)}-1}(t, s-1) \cdot \|x_n(s) - x(s)\| \nabla s,
$$

and then we can conclude that $\|Qx_n(t) - Qx(t)\| \to 0$ when $n \to +\infty$, which implies the Q is continuous.

Step 3. We show that Q is relatively compact. We choose t_1, $t_2 \in \mathbb{N}^{t_{Nl}}_{t_{(N-1)l}}$, $t_1 < t_2$, $N = 2, 3, \cdots, m$. By (3.3), we have

$$
\|Qx(t_1) - Qx(t_2)\|
$$
$$
\leq \int_{t_{(N-1)l}}^{t_1} \left\| H_{\nu_{(N-1)}-1}(t_1, s-1) - H_{\nu_{(N-1)}-1}(t_2, s-1) \right\| \cdot \|Ax(s) + f(s, x(s))\| \nabla s
$$
$$
+ \int_{t_1}^{t_2} H_{\nu_{(N-1)}-1}(t_2, s-1) \|Ax(s) + f(s, x(s))\| \nabla s
$$
$$
\leq (M_A \cdot r_1 + M_1) \int_{t_{(N-1)l}}^{t_1} \left\| H_{\nu_{(N-1)}-1}(t_1, s-1) - H_{\nu_{(N-1)}-1}(t_2, s-1) \right\| \nabla s \tag{3.8}
$$
$$
+ (M_A \cdot r_1 + M_1) \int_{t_1}^{t_2} H_{\nu_{(N-1)}-1}(t_2, s-1) \nabla s
$$
$$
\to 0, \text{ as } t_1 \to t_2,
$$

which implies that $\{Qx : x \in \Omega_{r_1}\}$ is a bounded and uniformly Cauchy subset E, together with Discrete Arzelà-Ascoli's theorem, we get $Q\Omega_{r_1}$ is relatively compact.

Step 4. We choose a fixed $y \in \Omega_{r_1}$, $x = Px + Qy$, for all $N = 2, 3, \cdots, m$, and we have

$$
\|x\| \leq \|Px\| + \|Qy\|
$$
$$
\leq \sum_{k=2}^{N} \int_{t_{(k-2)l}}^{t_{(k-1)l}} H_{\nu_{(k-2)}-1}\left(t_{(k-1)l}, s-1\right) \left[M_A \cdot \|x\| + M_1 \right] \nabla s
$$
$$
+ \|y(t_0)\| + \int_{t_{(N-1)l}}^{t} H_{\nu_{(N-1)}-1}(t, s-1) \left[M_A \cdot r_1 + M_1 \right] \nabla s \tag{3.9}
$$
$$
\leq \left[M_A \cdot \|x\| + M_1 \right] \sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) + \delta
$$
$$
+ \left[M_A \cdot r_1 + M_1 \right] \sup_{t \in \mathbb{N}^{T}_{t_{(N-1)l}}} H_{\nu_{(N-1)}}\left(t, t_{(N-1)l}\right).
$$

Further, we can obtain the following inequality

$$\|x\| \leq \frac{MM_1 + \delta + M_A \displaystyle\sup_{t \in \mathbb{N}_{t_{(N-1)l}}^T} H_{\nu_{(N-1)}}\left(t, t_{(N-1)l}\right) \cdot r_1}{1 - M_A \displaystyle\sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right)}. \tag{3.10}$$

By (3.4) and (3.10), we have

$$\|x\| \leq r_1. \tag{3.11}$$

Therefore $x \in \Omega_{r_1}$.

Finally, we proof that P is a contraction. For all x_1, $x_2 \in \Omega_{r_1}$, and by (H_1), (3.2) and (3.5), we obtain

$$\begin{aligned}
&\|Px_1 - Px_2\| \\
&\leq \sum_{k=2}^{N} \int_{t_{(k-2)l}}^{t_{(k-1)l}} H_{\nu_{(k-2)}-1}\left(t_{(k-1)l}, s-1\right) \left[M_A \|x_1 - x_2\| + L \|x_1 - x_2\|\right] \nabla s \\
&= (M_A + L)\|x_1 - x_2\| \sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) \\
&< \|x_1 - x_2\|.
\end{aligned} \tag{3.12}$$

From the Theorem 2.13, $T = P + Q$ has a fixed point in Ω_{r_1} which is a solution of (1.1). The proof is completed.

Theorem 3.2 *Suppose the validity of* $(H_1)-(H_2)$, *and there exists a positive constant* δ *such that* $\|x(t_0)\| \leq \delta$. *If for any* $(t, x) \in \left(\mathbb{N}_a^T, \mathbb{R}^n\right)$, *there exists two nondecreasing functions* $g(\cdot)$ *and* $\psi(\cdot)$ *such that* $f(t, x) \leq g(t)\psi(x)$, *then the fractional discrete equations* (1.1) *has at least one bounded solution in* $\Omega_{r_2} = \{x(t) \in E : \|x\| \leq r_2\}$ *provided that*

$$g(T)\psi(r_2) + \frac{\delta}{M} \leq r_2\left(\frac{1}{M} - M_A\right), \tag{3.13}$$

where $M = \displaystyle\sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) + H_{\nu_{(N-1)}}\left(t, t_{(N-1)l}\right)$, *and for all* $t \in \mathbb{N}_{t_{(N-1)l}}^T$, *the inequality* $0 < (M_A + L) \displaystyle\sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) < 1$ *holds.*

Proof It is easy to identify that Ω_{r_2} is a nonempty, closed, bounded, and convex subset of E.

In the first step, we should prove that the operator Q maps Ω_{r_2} into Ω_{r_2}. By (3.3), (3.13) and Lemma 2.4, for any $x \in \Omega_{r_2}$, we have

$$\begin{aligned}
\|Qx(t)\| &\leq \|x(t_0)\| + \int_{t_{(N-1)l}}^{t} H_{\nu_{(N-1)}-1}(t, s-1)\left[M_A \cdot \|x(s)\| + g(s)\psi(x)\right] \nabla s \\
&\leq \delta + (M_A \cdot r_2 + g(T)\psi(r_2)) \int_{t_{(N-1)l}}^{t} H_{\nu_{(N-1)}-1}(t, s-1)\nabla s \\
&\leq \delta + (M_A \cdot r_2 + g(T)\psi(r_2)) \sup_{t \in \mathbb{N}_{t_{(N-1)l}}^T} H_{\nu_{(N-1)}}\left(t, t_{(N-1)l}\right) \leq r_2,
\end{aligned} \tag{3.14}$$

which yields $Q\Omega_{r_2} \subset \Omega_{r_2}$.

In the next two steps, we need to show that Q is continuous and relatively compact, and this process is similar to (3.7) and (3.8).

Finally, we choose a fixed $y \in \Omega_{r_2}$, $x = Px + Qy$, for all $N = 2, 3, \cdots, m$, $t \in \mathbb{N}_a^T$, and we have

$$
\begin{aligned}
\|x\| \leq & \|Px\| + \|Qy\| \\
\leq & \sum_{k=2}^{N} \int_{t_{(k-2)l}}^{t_{(k-1)l}} H_{\nu_{(k-2)}-1}\left(t_{(k-1)l}, s-1\right)\left[M_A \cdot \|x\| + g(s)\psi\left(x(s)\right)\right] \nabla s + \|y(t_0)\| \\
& + \int_{t_{(N-1)l}}^{t} H_{\nu_{(N-1)}-1}(t, s-1)\left[M_A \cdot \|y\| + g(s)\psi\left(y(s)\right)\right] \nabla s \\
\leq & \left[M_A \cdot \|x\| + g(T)\psi(\|x\|)\right] \sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) + \delta \\
& + \left(M_A \cdot r_2 + g(T)\psi(r_2)\right) \sup_{t \in \mathbb{N}_{t_{(N-1)l}}^{T}} H_{\nu_{(N-1)}}\left(t, t_{(N-1)l}\right).
\end{aligned}
\tag{3.15}
$$

Therefore, by (3.13) we have the following

$$
\|x\| \leq \frac{M \cdot g(T)\psi(r_2) + \delta + M_A \cdot \displaystyle\sup_{t \in \mathbb{N}_{t_{(N-1)l}}^{T}} H_{\nu_{(N-1)}}\left(t, t_{(N-1)l}\right) \cdot r_2}{1 - M_A \displaystyle\sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right)}
\tag{3.16}
$$

$$
\leq r_2,
$$

which implies $x \in \Omega_{r_2}$. Operator P is clearly a contraction, and by Theorem 2.13, T has a fixed point in Ω_{r_2} which is a solution of (1.1). Hence the proof is completed.

4. Stability results

In this section, we present two theorems showing the nonlinear variable-order Nabla Caputo fractional difference system admits Ulam–Hyers-Rassias and Ulam–Hyers stabilities.

Lemma 4.1 *If $y(t)$ solves (2.3), then exists a function $\varphi(t)$ such that ${}_{t_*}^{C}\nabla_t^{\nu_k} y(t) - Ay(t) - f\left(t, y(t)\right) = \varphi(t)$ and $\|\varphi(t)\| \leq \Phi(t)$. And we can get the following equations in a similar way with Lemma 2.9:*

$$
y(t) = \begin{cases} y(t_0) + \int_{t_0}^{t} H_{\nu_0-1}(t, s-1)\left[Ay(s) + f\left(s, y(s)\right) + \varphi(s)\right] \nabla s, \ t \in \mathbb{N}_{t_0}^{t_l}, \\ y(t_0) + \displaystyle\sum_{k=2}^{N} \int_{t_{(k-2)l}}^{t_{(k-1)l}} H_{\nu_{(k-2)}-1}\left(t_{(k-1)l}, s-1\right)\left[Ay(s) + f\left(s, y(s)\right) + \varphi(s)\right] \nabla s \\ \quad + \int_{t_{(N-1)l}}^{t} H_{\nu_{(N-1)}-1}(t, s-1)\left[Ay(s) + f\left(s, y(s)\right) + \varphi(s)\right] \nabla s, \ t \in \mathbb{N}_{t_{(N-1)l}}^{t_{Nl}}, \end{cases}
\tag{4.1}
$$

and $N = 2, 3, \cdots, m$.

Theorem 4.2 *Suppose that the conditions* $(H_1) - (H_2)$ *hold, for all* $t \in \mathbb{N}_{t_{(N-1)l}}^{t_{Nl}}$, $N = 2, 3, \cdots, m$, $\Phi(t)$ *is a nondecreasing function. Then* (1.1) *has Ulam-Hyers-Rassias stability provided that*

$$0 < \sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) + \sup_{t \in \mathbb{N}_{t_{(N-1)l}}^{T}} H_{\nu_{(N-1)}}\left(t, t_{(N-1)l}\right) < \frac{1}{M_A + L}, \tag{4.2}$$

for all $N = 2, 3, \cdots, m$, $t \in \mathbb{N}_{t_{(N-1)l}}^{T}$.

Proof

By (2.2) and (4.1), for all $t \in \mathbb{N}_{t_{(N-1)l}}^{t_{Nl}}$, $N = 2, 3, \cdots, m$, we have

$$
\begin{aligned}
\|y - x\| &\leq \sum_{k=2}^{N} \int_{t_{(k-2)l}}^{t_{(k-1)l}} H_{\nu_{(k-2)}-1}\left(t_{(k-1)l}, s-1\right)\left[(M_A + L)\|y - x\| + \Phi(s)\right] \nabla s \\
&\quad + \int_{t_{(N-1)l}}^{t} H_{\nu_{(N-1)}-1}(t, s-1)\left[(M_A + L)\|y - x\| + \Phi(s)\right] \nabla s \\
&\leq (\Phi(t) + (M_A + L) \cdot \|y - x\|) \times \\
&\quad \left(\sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) + \sup_{t \in \mathbb{N}_{t_{(N-1)l}}^{T}} H_{\nu_{(N-1)}}\left(t, t_{(N-1)l}\right)\right),
\end{aligned}
\tag{4.3}
$$

which yields

$$\|y - x\| \leq \frac{\Phi(t)}{\left[\sum\limits_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) + \sup\limits_{t \in \mathbb{N}_{t_{(N-1)l}}^{T}} H_{\nu_{(N-1)}}\left(t, t_{(N-1)l}\right)\right]^{-1} - M_A - L}. \tag{4.4}$$

We can deduce that the system (1.1) has Ulam-Hyers-Rassias stability, and

$$C = \frac{1}{\left[\sum\limits_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) + \sup\limits_{t \in \mathbb{N}_{t_{(N-1)l}}^{T}} H_{\nu_{(N-1)}}\left(t, t_{(N-1)l}\right)\right]^{-1} - M_A - L},$$

defined in Definition 2.10.

Theorem 4.3 *Suppose that the conditions* $(H_1) - -(H_2)$ *hold, and* $\Phi(t)$ *defined in Definition 2.10 is a fixed constant* ε. *Then* (1.1) *has Ulam-Hyers stability if*

$$
\begin{aligned}
0 <& (M_A + L) \sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) \\
&< \frac{1}{\sum\limits_{j=0}^{\infty} (M_A + L)^{j} \sup\limits_{t \in \mathbb{N}_{t_{(N-1)l}}^{T}} H_{j\nu_{(N-1)}}\left(t, t_{(N-1)l}\right)}
\end{aligned}
\tag{4.5}
$$

holds for all $N = 2, 3, \cdots, m$, $t \in \mathbb{N}_{t_{(N-1)l}}^{T}$.

Proof For any $t \in \mathbb{N}_{t_{(N-1)l}}^{t_{Nl}}$, $N = 2, 3, \cdots, m$, we can get the following inequality by (2.2) and (4.1):

$$
\begin{aligned}
|y(t) - x(t)| \leq & \sum_{k=2}^{N} \int_{t_{(k-2)l}}^{t_{(k-1)l}} H_{\nu_{(k-2)}-1}\left(t_{(k-1)l}, s-1\right)\left[(M_A + L)|y(s) - x(s)| + \Phi(s)\right] \nabla s \\
& + \int_{t_{(N-1)l}}^{t} H_{\nu_{(N-1)}-1}(t, s-1)\left[(M_A + L)|y(s) - x(s)| + \Phi(s)\right] \nabla s \\
= & (M_A + L) \sum_{k=2}^{N} \int_{t_{(k-2)l}}^{t_{(k-1)l}} H_{\nu_{(k-2)}-1}\left(t_{(k-1)l}, s-1\right)|y(s) - x(s)|\nabla s \\
& + \left(\sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) + H_{\nu_{(N-1)}}\left(t, t_{(N-1)l}\right)\right) \cdot \varepsilon \\
& + (M_A + L) \int_{t_{(N-1)l}}^{t} H_{\nu_{(N-1)}-1}(t, s-1)|y(s) - x(s)|\nabla s.
\end{aligned}
\tag{4.6}
$$

We let

$$
\begin{aligned}
p(t) = & (M_A + L) \sum_{k=2}^{N} \int_{t_{(k-2)l}}^{t_{(k-1)l}} H_{\nu_{(k-2)}-1}\left(t_{(k-1)l}, s-1\right)|y(s) - x(s)|\nabla s \\
& + \left(\sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) + H_{\nu_{(N-1)}}\left(t, t_{(N-1)l}\right)\right) \cdot \varepsilon,
\end{aligned}
$$

which is nonnegative, nondecreasing function for all $t \in \mathbb{N}_a^T$. Therefore, from the Lemma 2.8 one can obtain that

$$
|y(t) - x(t)| \leq p(t) \sum_{j=0}^{\infty} (M_A + L)^j H_{j\nu_{(N-1)}}\left(t, t_{(N-1)l}\right),
\tag{4.7}
$$

and then we can conclude that

$$
\begin{aligned}
\|y - x\| \leq & \left[(M_A + L)\|y - x\| \sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) \right. \\
& \left. + \left(\sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) + \sup_{t \in \mathbb{N}_{t_{(N-1)l}}^T} H_{\nu_{(N-1)}}\left(t, t_{(N-1)l}\right)\right) \cdot \varepsilon\right] \\
& \times \sum_{j=0}^{\infty} (M_A + L)^j \sup_{t \in \mathbb{N}_{t_{(N-1)l}}^T} H_{j\nu_{(N-1)}}\left(t, t_{(N-1)l}\right),
\end{aligned}
\tag{4.8}
$$

which implies

$$\|y - x\| \leq$$

$$\frac{\left(\sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) + \sup_{t \in \mathbb{N}_{t_{(N-1)l}}^{T}} H_{\nu_{(N-1)}}\left(t, t_{(N-1)l}\right)\right) \cdot \varepsilon}{\left[\sum_{j=0}^{\infty} (M_A + L)^j \sup_{t \in \mathbb{N}_{t_{(N-1)l}}^{T}} H_{j\nu_{(N-1)}}\left(t, t_{(N-1)l}\right)\right]^{-1} - (M_A + L)\sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right)}. \tag{4.9}$$

Hence, the system (1.1) has Ulam-Hyers stability. The proof of this theorem is completed.

5. Example

In this section, we present the following examples to illustrate our main results.

Example 5.1 *Assume that* $f(t, x) = \frac{\sin x(t)}{10}$, $Ax(t) = \frac{9}{100}e^{-t}x(t)$, $t \in \mathbb{N}_0^{12}$, $[t_0, T] = [0, 3] \cup [3, 6] \cup [6, 9] \cup [9, 12]$, $x(0) = 1$, $\nu_k = \left(\frac{1}{2}\right)^{k+1}$, $t_{kl} = 3 * k$, $k = 0, 1, 2, 3$, *then* $N = 4$, $L = M_1 = 0.100$, $M_A = 0.090$. *And by Mathematica software, we know*

$$\sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) = 4.476,$$

and

$$\sup_{t \in \mathbb{N}_a^{T}} H_{\nu_{(N-1)}}\left(t, t_{(N-1)l}\right) = 1.096, \ t \in \mathbb{N}_9^{12},$$

then $M = 4.476 + 1.096 = 5.572$ *satisfying* $0 < M < M_A^{-1}$, *and we can have*

$$(M_A + L)\sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) = 0.850$$

admitting the inequality (3.5). *Meanwhile,* $r_1 \geq \frac{\delta M^{-1} + M_1}{M^{-1} - M_A} = 3.135$. *From the Theorem* 3.1, *fractional discrete equations* (1.1) *has at least one bounded solution in* $\Omega_{r_1} = \{x(t) \in E : \|x\| \leq r_1\}$, *and* $r_1 \geq 3.135$.

Example 5.2 *Assume that* $f(t, x) = t * \frac{7}{1000}\sin\frac{\pi x}{24}$, $t \in \mathbb{N}_0^{12}$, *and all other data are the same as in the above example, then* $g(t) = t$, $\psi(x) = \frac{7}{1000}\sin\frac{\pi x}{24}$, *and* $L = 0.011$. *And we can have*

$$(M_A + L)\sum_{k=2}^{N} H_{\nu_{(k-2)}}\left(t_{(k-1)l}, t_{(k-2)l}\right) = 0.452 < 1,$$

and (3.13) *can be translate into*

$$g(T)\psi(r_2) + \frac{\delta}{M} \leq g(12) * \frac{7}{1000}\frac{\pi r_2}{24} + \frac{\delta}{M} \leq r_2\left(\frac{1}{M} - M_A\right),$$

which implies $r_2 \geq 2.295$. By Theorem 3.2, we know that the fractional discrete equations (1.1) *has at least one bounded solution in* $\Omega_{r_2} = \{x(t) \in E : \|x\| \leq r_2\}$, *and* $r_2 \geq 2.295$.

In addition, by Mathematica (Walfram Research, Champaign, Illinois, IL, USA) software, we have

$$\left[\frac{1}{\sum\limits_{j=0}^{\infty} (M_A + L)^j \sup\limits_{t \in \mathbb{N}_{t_{(N-1)l}}^T} H_{j\nu_{(N-1)}}\left(t, t_{(N-1)l}\right)} \right]_{min} = 0.889,$$

and it is easy to verify (4.2) *and* (4.5). *Hence system* (1.1) *has Ulam–Hyers-Rassias and Ulam–Hyers stabilities.*

Remark 5.3 *Since there are few papers researching the existence of solutions for the nonlinear Nabla Caputo variable-order fractional difference equations, one can see that all the results in references can not directly be applicable to the two examples just listed. This implies that the results in this paper are essentially new.*

6. Conclusion

In this paper, we are concerned with a class of nonlinear variable-order Nabla Caputo fractional difference system, which is quite different from the related references discussed in the literature [5, 9, 10, 16, 18, 19, 23]. The problem studied in the present paper is more generalized and more practical. By applying the generalized Gronwall inequality and the definition of the Ulam–Hyers stability, we obtain the expected results. Finally, some examples have been provided at the end of this paper to illustrate the effectiveness.

An interesting extension of our study would be to discuss the stability in the variable-order Nabla Caputo fractional difference system with impulses [32, 33]. Impulses may destabilize the stability of fractional difference system, and we are interested in returning some conditions we add to stable state. This topic will be the subject of a forthcoming paper.

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