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Star edge coloring of graphs with Mad(G)$<\frac{14}{5}$

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Abstract: A star edge coloring of a graph G is a proper edge coloring such that there is no bicolored path or cycle of length four. The minimum number of colors needed for a graph G to admit a star edge coloring is called the star chromatic index and it is denoted by $\chi_s^{'}(G)$. In this paper, we consider graphs of maximum degree $\Delta \geq 4$ and show that if the maximum average degree of a graph is less than $\frac{14}{5}$ then $\chi_s^{'}(G) \leq 2\Delta + 1$.

Key words: Graph coloring, star edge coloring, star chromatic index, maximum average degree

1. Introduction

For a simple graph G, with vertex set $V(G)$ and edge set $E(G)$, a proper edge coloring is an assignment of a color to each edge of G so that no two edges with a common endpoint receive the same color. A star edge coloring is a proper edge coloring such that there is no bicolored path of length four or cycle of length four. The star chromatic index of G, denoted by $\chi_s^{'}(G)$ is the minimum number of colors required for G to admit a star edge coloring. This coloring was introduced by Liu and Deng [11] in 2008.

In 2013, Dvořák et al. showed that even determining the star chromatic index of complete graphs is a hard problem and gave the following bound for the star chromatic index of complete graphs [2].

$$(2 + o(1))n \leq \chi_s^{'}(K_n) \leq n\frac{2^{2\sqrt{2}(1+o(1))\sqrt{log(n)}}}{(log(n))^{\frac{1}{4}}}.$$

They also showed that for a subcubic graph G (graph with maximum degree at most 3), $\chi_s^{'}(G) \leq 7$. There are cubic graphs like K_4 with one subdivided edge, $K_{3,3}$ and Heawood graph with the star chromatic index equal to 6 but there is no known example of a subcubic graph requiring seven colors. Thus, they conjectured that $\chi_s^{'}(G) \leq 6$ for subcubic graphs.

Pradeep and Vijayalakshmi [8] proved that if G is a subcubic graph with maximum average degree, Mad$(G) < \frac{8}{3}$, then $\chi_s^{'}(G) \leq 6$. Bezegová et al. [1] obtained some tight bounds for the star chromatic index of trees and subcubic outerplanar graphs. Wang Y. et al. [10] showed that if G is a graph with $\Delta = 4$, then $\chi_s^{'}(G) \leq 14$ and if G is a bipartite graph with $\Delta = 4$, then $\chi_s^{'}(G) \leq 13$. Lei et al. [6] proved that it is NP-complete to determine whether $\chi_s^{'}(G) \leq 3$ for an arbitrary graph.

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The list version of star edge coloring is also studied in [3–5]. An edge list L for a graph G is a mapping that assigns a finite set of colors to each edge of G. Given an edge list L for a graph G, we say that G is L-star edge colorable if it has a star edge coloring c such that $c(e) \in L(e)$ for every edge e of G. The list star chromatic index of a graph G, denoted by $ch_s^{'}(G)$, is the smallest integer k such that for every edge list L for G with $|L(e)| = k$ for every edge $e \in E(G)$, G is L-star edge colorable. The bounds for list star chromatic index of a graph are given in terms of its maximum average degree. The maximum average degree of a graph G, denoted by Mad(G) is defined as $\mathrm{Mad}(G) = max_{H \subseteq G, |V(H)| \geq 1}\{\frac{2|E(H)|}{|V(H)|}\}$. Dvořák et al. [2] asked a question whether $ch_s^{'}(G) \leq 7$ for a subcubic graph G. Motivated by this Kerdjoudj et al. proved that for a subcubic graph G,

(i) if $\mathrm{Mad}(G) < \frac{7}{3}$, then $ch_s^{'}(G) \leq 5$,

(ii) if $\mathrm{Mad}(G) < \frac{5}{2}$, then $ch_s^{'}(G) \leq 6$,

(iii) if $\mathrm{Mad}(G) < \frac{30}{11}$, then $ch_s^{'}(G) \leq 7$.

Lužar et al. [7] have shown that seven colors suffice for the list star edge coloring of a subcubic graph G. Apart from subcubic graphs, the following results were proved in [4] and [5] for a graph G with maximum degree $\Delta \geq 4$,

(i) if $\mathrm{Mad}(G) < \frac{7}{3}$, then $ch_s^{'}(G) \leq 2\Delta - 1$,

(ii) if $\mathrm{Mad}(G) < \frac{5}{2}$, then $ch_s^{'}(G) \leq 2\Delta$,

(iii) if $\mathrm{Mad}(G) < \frac{8}{3}$, then $ch_s^{'}(G) \leq 2\Delta + 1$,

(iv) if $\mathrm{Mad}(G) < \frac{14}{5}$, then $ch_s^{'}(G) \leq 2\Delta + 2$,

(v) if $\mathrm{Mad}(G) < 3$, then $ch_s^{'}(G) \leq 2\Delta + 3$.

Using the concept of edge partitions, Wang et al. [9] showed that,

(i) for a planar graph G with maximum degree Δ, $\chi_s^{'}(G) \leq 2.75\Delta + 18$,

(ii) for a planar graph G of girth at least 8, $\chi_s^{'}(G) \leq \lfloor 1.5\Delta \rfloor + 3$.

In this paper, we consider the graphs of maximum degree $\Delta \geq 4$ with $\mathrm{Mad}(G) < \frac{14}{5}$ and improve the bound for the star chromatic index of G given in (iv) above and show that for such graphs $\chi_s^{'}(G) \leq 2\Delta + 1$.

2. Basic definitions and notations

For a graph G, let $d_G(v)$ denote the degree of a vertex v in G. If G is clear from the content, we may omit the subscript. Let $N(v)$ be the set of neighbors of v. A vertex of degree k is called a k-vertex. A k^+-vertex is a vertex of degree at least k. A k-vertex adjacent to a vertex v is a k-neighbor of v. A 3_k-vertex is a 3-vertex adjacent to exactly k $(0 \leq k \leq 3)$ 2-vertices. A 3_1-vertex adjacent to two 3-vertices is called a *bad* 3_1-vertex. A 2-vertex adjacent to a 2-vertex is called a *bad* 2-vertex. An edge incident to a vertex of degree one is called a *pendant* edge. For an edge coloring φ of the graph G, let $\varphi(v)$ denote the set of colors used on the edges incident with the vertex $v \in V(G)$, in the coloring φ. Similarly, for an edge $uv \in E(G)$, $\varphi(uv)$ denotes the color used on the edge uv. We say that a color c is an available color for an edge uv if c is not assigned to any of its neighbors and there is no bicolored path of length four or cycle of length four involving uv when uv is colored with c. Otherwise, it is said to be a forbidden color for the edge uv. The set of forbidden colors for a given edge uv is denoted by $F(uv)$.

3. Graphs G with Mad$(G) < \frac{14}{5}$

Theorem 3.1 *Let G be a graph with maximum degree $\Delta \geq 4$ and Mad(G) $< \frac{14}{5}$. Then $\chi_s^{'}(G) \leq 2\Delta + 1$.*

Proof The proof is by the method of contradiction. Let H be a counterexample to the theorem. First, we prove the nonexistence of some structures in H and then use the discharging technique to get a contradiction. For some integer k, let G_k be the class of graphs with maximum degree at most k and maximum average degree less than $\frac{14}{5}$. Let for the smallest k, $H \in G_k$ be a counterexample to this theorem minimizing $|E(H)|+|V(H)|$. That is, Mad$(H) < \frac{14}{5}$ and $\chi_s^{'}(H) > 2k+1$ and for any edge $e \in E(H)$, $\chi_s^{'}(H\setminus\{e\}) \leq 2k+1$. By minimality of H, we can assume that H is connected. Otherwise, we can star color independently the edges of each connected component of H with $2k+1$ colors.

Now, we claim some structures or set of some subgraphs do not exist in H. We prove all the claims by the method of contradiction. To prove each of the claims, we suppose that the described structure exists in H. Then, we remove a certain number of edges from H to form a graph $H^{'}$ which by minimality of H, is star edge colorable with $2k+1$ colors. Let φ be such a star edge coloring of $H^{'}$. We extend this star edge coloring φ of $H^{'}$ with $2k+1$ colors to a star edge coloring of H with $2k+1$ colors, which is a contradiction.

Structure of minimal counterexample

Claim 3.2 *H does not contain a vertex u adjacent to $d(u)$-1 vertices of degree 1.*

Suppose H contains a p-vertex u with $N(u) = \{u_1, ..., u_p\}$ and $d(u_i) = 1$ for $i \in \{1, ..., p-1\}$ $(p \leq k)$. Let $H^{'} = H \setminus \{uu_1\}$. By minimality of H, $H^{'}$ has a star edge coloring φ with $2k+1$ colors. As $d(u_p) \leq k$, $|\varphi(u_p)| \leq k$ and $|F(uu_1)| \leq |\varphi(u) \cup \varphi(u_p)| \leq (k-1) + (k-1) = 2k-2$. So, there are at least three colors available for uu_1. Hence, φ can be extended to H, a contradiction.

Now, consider $G^{'} = H \setminus \{v : v \in V(H), d_H(v) = 1\}$. If H does not contain a 1-vertex, then $G^{'} = H$. It can be observed that by Claim 3.2, $G^{'}$ does not contain 1-vertices. Since $G^{'} \subseteq H$, we have, Mad$(G^{'}) < \frac{14}{5}$.

Claim 3.3 *$G^{'}$ does not contain two adjacent 2-vertices.*

Suppose $G^{'}$ contains a 2-vertex u adjacent to another 2-vertex v. Let x and y be the neighbors of u and v respectively. If u and v have 1-neighbors in H let them be denoted by u_i and v_j respectively for $i, j \in \{1, ..., k-2\}$. Consider $H^{'} = H \setminus \{uu_1\}$. By minimality of H, $H^{'}$ has a star edge coloring φ with $2k+1$ colors. As $|F(uu_1)| \leq |\varphi(x) \cup \varphi(v) \cup \varphi(u)| \leq k + 2 + (k-3) = 2k-1$, the coloring φ can be extended to uu_1. Therefore, $d_H(u) = 2$. Similarly, $d_H(v) = 2$.

Now, consider $H^{'} = H \setminus \{uv\}$, which by minimality of H has a star edge coloring φ with $2k+1$ colors. As $|F(uv)| \leq |\varphi(x) \cup \varphi(y)| = k + k = 2k$, φ can be easily extended to H, a contradiction.

Claim 3.4 *If $G^{'}$ contains a 2-vertex u adjacent to a 3-vertex, then $d_H(u) = 2$.*

Suppose u is a 2-vertex adjacent to a 3-vertex v in $G^{'}$. Let w be a 3^{+}-neighbor of u. If u has 1-neighbors in H, let them be denoted by u_i for $i \in \{1, ..., k-2\}$. Consider $H^{'} = H \setminus \{uu_1\}$. By minimality of

H, $H^{'}$ has a star edge coloring φ with $2k+1$ colors. As $|F(uu_1)| \leq |\varphi(w) \cup \varphi(v) \cup \varphi(u)| \leq k+3+(k-3) = 2k$, there is at least one color available for the edge uu_1. The colors used on the pendant edges incident to v and w may be available for uu_1. Therefore, $d_H(u) = 2$.

Claim 3.5 *If $G^{'}$ contains a 3_2-vertex u adjacent to a p-vertex v, then $d_H(u) = 3$ and $d_H(v) = p = k$.*

Suppose u is a 3_2-vertex with $N(u) = \{v, w, x\}$ in $G^{'}$, where w and x are 2-vertices and v is a vertex of degree $p \leq k$. By Claim 3.4, $d_H(w) = d_H(x) = 2$. If u have 1-neighbors in H, denote them by u_i for $i \in \{1, ..., k-3\}$. Let $H^{'} = H \setminus \{uu_1\}$. By minimality of H, $H^{'}$ has a star edge coloring φ with $2k+1$ colors. As $|F(uu_1)| \leq |\varphi(v) \cup \varphi(w) \cup \varphi(x) \cup \varphi(u)| \leq p + 2 + 2 + (k-4) \leq 2k$, $(p \leq k)$, there is at least one color available for uu_1. Therefore, $d_H(u) = 3$.

Now, let $d_H(v) < k$. It may have 1-neighbors in H. Let y and z be the vertices adjacent to w and x respectively other than u. Consider $H^{'} = H \setminus \{uw, ux\}$ which by minimality of H, has a star edge coloring φ with $2k+1$ colors. First, we color the edge uw with a color, say, c_1 such that $c_1 \notin \varphi(v) \cup \varphi(y) \cup \{\varphi(xz)\}$. As $d_H(v) < k$, there is at least one such color for the edge uw. Now, as $c_1 \notin \varphi(y)$, $|F(ux)| \leq |\varphi(v) \cup \varphi(z) \cup \{c_1\}| \leq (k-1) + k + 1 = 2k$. The colors used on the pendant edges incident to v, y and z may be available for uw and ux. This is possible due to our assumption that $d_H(v) < k$. Hence, $d_H(v) = k$.

Claim 3.6 *$G^{'}$ does not contain a path $uvwxy$, where v, w and x are 3_1-vertices.*

Suppose $G^{'}$ contains a path $uvwxy$, where v, w and x are 3_1-vertices. Let v_1, w_1 and x_1 be the 2-neighbors of v, w and x respectively. Let v_2, w_2 and x_2 be the other neighbors of v_1, w_1 and x_1 other than v, w and x respectively. By Claim 3.4, $d_H(v_1) = d_H(w_1) = d_H(x_1) = 2$. It is easy to see that, all the vertices v_1, w_1 and x_1 are distinct and by Claim 3.3, any two of the vertices v_1, w_1 and x_1 are not adjacent.

The vertices v, w and x may have 1-neighbors in H. Let them be $v_i^{'}$, $w_j^{'}$ and $x_l^{'}$ respectively for $i, j, l \in \{1, ..., k-3\}$. Consider $H^{'} = H \setminus \{ww_1^{'}\}$. By minimality of H, $H^{'}$ has a star edge coloring φ with $2k+1$ colors. As $|F(ww_1^{'})| \leq |\varphi(v) \cup \varphi(w_1) \cup \varphi(x) \cup \varphi(w)| \leq 3 + 2 + 3 + (k-4) = k+4$ and $k \geq 4$, there is a color available for $ww_1^{'}$. The colors used on the pendant edges incident to v and x may also be available for $ww_1^{'}$. Therefore, $d_H(w) = 3$.

Now, consider the graph $H^{'} = H \setminus \{vw, vv_1, vv_i^{'}, wx, ww_1, xx_1, xx_l^{'}\}$ for $i, l \in \{1, ..., k-3\}$ as shown in Figure 1. By minimality of H, the graph $H^{'}$ has a star edge coloring φ with $2k+1$ colors. Let $\varphi(uv) = a$ and $\varphi(xy) = b$. Color the edge vv_1 with a color, say, c_1 such that $c_1 \notin \varphi(u) \cup \varphi(v_2)$ and the edge xx_1 with a color, say, c_2 such that $c_2 \notin \varphi(y) \cup \varphi(x_2)$. Then, color the edge ww_1 with a color, say, c_3 such that $c_3 \notin \varphi(w_2) \cup \{c_1, c_2, a, b\}$. Since $|\varphi(w_2) \cup \{c_1, c_2, a, b\}| \leq k+4$ and $k \geq 4$, there is at least one color available for ww_1. Next, color the edge vw with a color, say, c_4 such that $c_4 \notin \varphi(u) \cup \{c_1, c_2, c_3, b\}$. Now, it can be observed that $|F(wx)| \leq |\varphi(y) \cup \{c_1, c_2, c_3, c_4\}| \leq k+4$. So, we have at least one color available for wx, say, c_5.

Finally, to color the edges $vv_i^{'}$ and $xx_l^{'}$, for $i, l \in \{1, ..., k-3\}$, we choose two sets of colors A_1 and A_2 such that each color in A_1 is not in $\varphi(u) \cup \{c_1, c_3, c_4, c_5\}$ and each color in A_2 is not in $\varphi(y) \cup \{c_2, c_3, c_4, c_5\}$. As $|\varphi(u) \cup \{c_1, c_3, c_4, c_5\}| \leq k+4$ and $|\varphi(y) \cup \{c_2, c_3, c_4, c_5\}| \leq k+4$, there are at least $2k+1-(k+4) = k-3$

colors in each of the sets A_1 and A_2. So, we color the edges $vv_i^{'}$ with colors from A_1 and the edges $xx_l^{'}$ with colors from A_2. Therefore, φ can be extended to H, a contradiction.

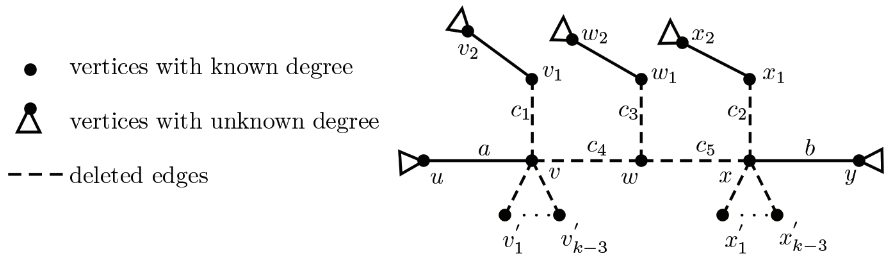

Figure 1. Illustration of Claim 3.6.

Claim 3.7 *$G^{'}$ does not contain a 3-vertex adjacent to two bad 3_1-vertices.*

Suppose $G^{'}$ contains a 3-vertex v adjacent to two bad 3_1-vertices u and w with $N(v) = \{u, w, z\}$. Let x and y be the 3-neighbors and $u^{'}$ and $w^{'}$ be the 2-neighbors of u and w respectively. Let $N(x) = \{x_1, x_2, u\}$ and $N(y) = \{y_1, y_2, w\}$. Let $u^{''}$ and $w^{''}$ be the neighbors of $u^{'}$ and $w^{'}$ respectively other than u and w. By Claim 3.4, $d_H(u^{'}) = d_H(w^{'}) = 2$. The 3-vertices u, v, w, x and y may have 1-neighbors in H. Using similar arguments as in Claim 3.6 for the vertex w, we can show that $d_H(w) = 3$. Similarly, $d_H(u) = 3$. Let the 1-neighbors of v, x and y be denoted by v_j, x_m and y_n respectively for $j \in \{1, ..., k-3\}$ and $m, n \in \{3, ..., k-1\}$.

Now, consider $H^{'} = H \setminus \{uu^{'}, uv, vv_j, vw, ww^{'}\}$ for $j \in \{1, ..., k-3\}$ as shown in Figure 2. By minimality of H, $H^{'}$ has a star edge coloring φ with $2k+1$ colors. First, we recolor the edge ux such that $\varphi(ux)$ do not appear on $\varphi(x_1) \cup \varphi(x_2) \cup \varphi(xx_m)$, for $m \in \{3, ..., k-1\}$. As we have $2k+1$ colors, there is at least one color that do not appear on $\varphi(x_1) \cup \varphi(x_2)$. Let that color be $a^{'}$. If $a^{'}$ is not present on the pendant edges xx_m, for any m, we set $\varphi(ux) = a^{'}$. Otherwise, we swap the color of the edge ux with the color $a^{'}$. Similarly, we recolor the edge wy such that $\varphi(wy)$ do not appear on $\varphi(y_1) \cup \varphi(y_2) \cup \varphi(yy_n)$, for $n \in \{3, ..., k-1\}$. After this recoloring, let $\varphi(ux) = a$ and $\varphi(wy) = b$. Let $\varphi(u^{'}u^{''}) = d$ and $\varphi(w^{'}w^{''}) = f$. We color the remaining edges as follows.

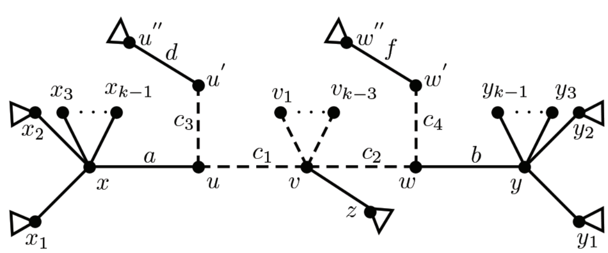

Figure 2. Configuration of Claim 3.7.

Color the edge uv with a color, say, c_1 such that $c_1 \notin \varphi(z) \cup \{a, b, d\}$, the edge vw with a color, say, c_2 such that $c_2 \notin \varphi(z) \cup \{a, b, f, c_1\}$, the edge $uu^{'}$ with color, say, c_3 such that $c_3 \notin \varphi(u^{''}) \cup \{a, c_1\}$ then color the edge $ww^{'}$ with a color, say, c_4 such that $c_4 \notin \varphi(w^{''}) \cup \{b, c_1, c_2\}$. As there are at most $k+4$ colors forbidden for each of the edge, we get at least one available color for every edge. Finally, to color the pendant edges vv_j, $j \in \{1, ..., k-3\}$, we choose a set of colors A such that each color in A is not in $\varphi(z) \cup \{a, b, c_1, c_2\}$. As $c_1 \neq d$ and $c_2 \neq f$, c_3 and c_4 can be in A. As $|A| \geq k-3$, we color all the pendant edges incident to v with colors from A. Hence, we can extend the coloring φ of $H^{'}$ to a star edge coloring of H.

Claim 3.8 *$G^{'}$ does not contain a 4-vertex adjacent to a 3_1-vertex and three 2-vertices.*

Suppose $G^{'}$ contains a 4-vertex u adjacent to a 3_1-vertex u_1 and three 2-vertices u_2, u_3 and u_4. Let $N(u_1) = \{u, v_1, v_2\}$, where v_1 is a 2-vertex. By Claim 3.4, $d_H(v_1) = 2$. For $i \in \{2, 3, 4\}$, let $N(u_i) = \{u, u_i^{'}\}$. If these u_i's have 1-neighbors in H, denote them by w_j^i, $j \in \{1, ..., k-2\}$. The vertices u and u_1 may also have 1-neighbors in H. Let them be u_m and x_l respectively for $m \in \{5, ..., k\}$ and $l \in \{1, ..., k-3\}$. It is easy to see that each 2-vertex u_i is distinct from the 2-vertex v_1 and by Claim 3.3, any two of the vertices u_2, u_3, u_4 and v_1 are not adjacent. Let $H^{'} = H \setminus \{uu_1, uu_i, u_iw_j^i, u_1x_l\}$, for $i \in \{2, 3, 4\}$, $j \in \{1, ..., k-2\}$ and $l \in \{1, ..., k-3\}$. By minimality of H, $H^{'}$ has a star edge coloring φ with $2k+1$ colors. Let $\varphi(u_2u_2^{'}) = a$, $\varphi(u_3u_3^{'}) = b$ and $\varphi(u_4u_4^{'}) = d$. We extend this coloring to H by coloring the following edges in order.

Color the edge uu_1 with a color, say, c_1 such that $c_1 \notin \varphi(v_1) \cup \varphi(v_2) \cup \{a, b\}$, the edge uu_2 with a color, say, c_2 such that $c_2 \notin \varphi(u_2^{'}) \cup \varphi(u_1) \cup \{d\}$ and the edge uu_3 with a color, say, c_3 such that $c_3 \notin \varphi(u_3^{'}) \cup \{c_1, c_2, a, d\}$. When $c_1 = d$, color the edge uu_4 with a color, say, c_4 such that $c_4 \notin \varphi(u_4^{'}) \cup \varphi(u_1) \cup \{c_2, c_3\}$. Otherwise, color the edge uu_4 with color c_4 such that $c_4 \notin \varphi(u_4^{'}) \cup \{c_1, c_2, c_3, b\}$. It can be observed that there are at most $k+4$ colors used on the colored edges which are forbidden for each of the above mentioned edges at each step hence, we have at least one color available for each of them. This coloring is shown in Figure 3.

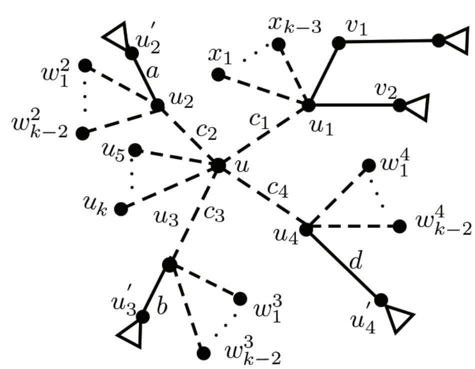

Figure 3. Configuration of Claim 3.8.

Now, we color the pendant edges incident to each u_i for $i \in \{1, ..., 4\}$. We choose a set of colors A_1 such that each color in A_1 is not in $\varphi(v_1) \cup \varphi(v_2) \cup \{c_1, c_4\}$. As $c_1 \notin \{a, b\}$, c_2 and c_3 can be in A_1. Therefore, $|A_1| \geq 2k+1-(k+4) = k-3$, so we color the edges u_1x_l for $l \in \{1, ..., k-3\}$ with colors from A_1. To color the edges $u_iw_j^i$ for $i \in \{2, 3, 4\}$ and $j \in \{1, ..., k-2\}$, we choose the three sets of colors A_2, A_3 and A_4 such that each color in A_2 is not in $\varphi(u_2^{'}) \cup \{c_1, c_2, c_3\}$ and each color in A_3 is not in $\varphi(u_3^{'}) \cup \{c_1, c_3, c_4\}$. If $c_1 = d$, we choose A_4 such that each color in A_4 is not in $\varphi(u_4^{'}) \cup \{c_2, c_3, c_4\}$. Otherwise, each color in A_4 is not in

$\varphi(u_4^{'}) \cup \{c_1, c_2, c_4\}$. As $c_2 \neq d$, c_4 can be in A_2 and $c_3 \neq a$, c_2 can be in A_3. When $c_1 \neq d$, $c_4 \neq b$ so c_3 can be in A_4. When $c_1 = d$, all the colors c_1, c_2, c_3, c_4 are not in A_4. Therefore, in every set A_i, $i \in \{2,3,4\}$, at most $k+3$ colors are forbidden. This gives, $|A_i| \geq k-2$, for $i \in \{2,3,4\}$. So, we color the pendant edges $u_i w_j^i$ with colors from A_i for $i \in \{2,3,4\}$ and $j \in \{1,...,k-2\}$. Finally, the edges uu_i, for $i \in \{5,...,k\}$ can be easily colored as only four colors c_1, c_2, c_3 and c_4 are forbidden for them. Hence, φ can be extended to H, a contradiction.

Claim 3.9 *$G^{'}$ does not contain **(i)** a 4-vertex adjacent to two 3_2-vertices or **(ii)** a k-vertex, $k \geq 5$, adjacent to three 3_2-vertices.*

(i) Suppose $G^{'}$ contains a 4-vertex u with $N(u) = \{u_i\}$ for $i \in \{1,...,4\}$, where u_1 and u_2 are 3_2-vertices. By Claim 3.5, $d_H(u) = 4 = k$ and $d_H(u_1) = d_H(u_2) = 3$. Let v_1 and v_2 be the 2-neighbors of u_1 and let v_3 and v_4 be the 2-neighbors of u_2. For $i \in \{1,...,4\}$, let $v_i^{'}$ be the neighbors of v_i other than v. By Claim 3.4, $d_H(v_i) = 2$, for all i. By Claim 3.3, any of the two v_i's are not adjacent. If $v_1 = v_3$, let $H^{'} = H \setminus \{u_1v_1\}$. By minimality of H, $H^{'}$ has a star edge coloring φ with $2k+1$ colors. As $|F(u_1v_1)| \leq \varphi(v_2) \cup \varphi(u) \cup \varphi(u_2)| = 2+4+1 = 7$, there are at least two colors available for u_1v_1. Therefore, we can easily extend this coloring to H. Hence, $v_1 \neq v_3$. Using similar arguments we can show that v_1 is distinct from v_4 and the vertex v_2 is distinct from v_3 and v_4.

Now, consider $H^{'} = H \setminus \{u_1v_1, u_1v_2\}$ as shown in Figure 4b. By minimality of H, $H^{'}$ has a star edge coloring φ with $2k+1$ colors. To extend this coloring, first we color the edge u_1v_1 with a color not in $\varphi(v_1^{'}) \cup \varphi(u)$. There is at least one such color say, c_1, available for u_1v_1. Next, if there is an available color for the edge u_1v_2, we are done. Otherwise, as all the colors are forbidden for u_1v_2, $|\varphi(v_2^{'}) \cap \varphi(u)| = 0$, $\varphi(v_1^{'}) = \varphi(v_2^{'})$ and the color of the edge uu_1, say, b appears on all the vertices u_i, for $i \in \{2,3,4\}$ forming bi-colored paths of length three. So, we uncolor the edge uu_1 and try to recolor it with a color other than b. If we get such a color, then it is easy to see that b is an available color for u_1v_2 and we are done.

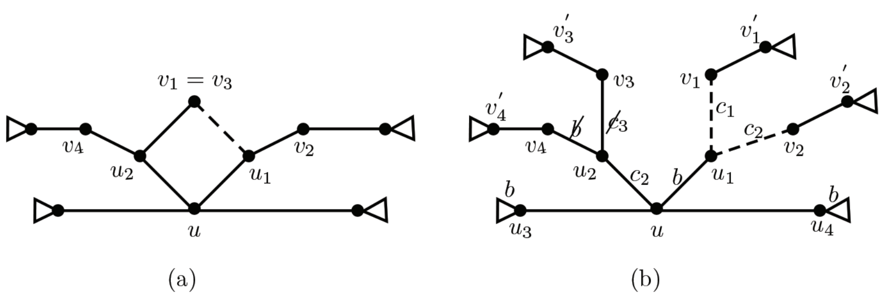

Figure 4. Configuration of Claim 3.9(i).

Otherwise, $|\varphi(u_2) \cap \varphi(u_3) \cap \varphi(u_4)| = 1$ and that color is b. So, we uncolor the edges u_2v_3 and u_2v_4 (shown with strike out colors on these edges). This makes the color $\varphi(uu_2) = c_2$ (say), an available color for the edge u_1v_2. We assign $\varphi(u_1v_2) = c_2$. Next, we color the edge u_2v_3 with a color, say, c_3 such that $c_3 \notin \varphi(v_3^{'}) \cup \varphi(u)$. There is at least one such color. Now, we can observe that there is at least one color available for the edge u_2v_4. As c_2 can be present on at most one of the vertices u_3 and u_4. If c_2 is on both u_3 and u_4, we could have recolored the edge uu_1. So, φ can be extended to H, a contradiction.

(ii) Suppose $G^{'}$ contains a k-vertex u with $N(u) = \{u_i\}$ for $i \in \{1, ..., k\}$, where u_1, u_2 and u_3 are 3_2-vertices. By Claim 3.5, $d_H(u) = k$ and $d_H(u_1) = d_H(u_2) = d_H(u_3) = 3$. Let v_1 and v_2 be the 2-neighbors of u_1, let v_3 and v_4 be the 2-neighbors of u_2 and let v_5 and v_6 be the 2-neighbors of u_3 in $G^{'}$. For $i \in \{1, ..., 6\}$, let $v_i^{'}$ be the neighbors of v_i other than v. By Claim 3.4, $d_H(v_i) = 2$, for all i. It is easy to see that all the 2-vertices v_i's are distinct.

Now, let $H^{'} = H \setminus \{u_1v_1, u_1v_2\}$. By minimality of H, $H^{'}$ has a star edge coloring φ with $2k+1$ colors. First, color the edge u_1v_1 with a color, say, c_1 as in Claim 3.9(i). If there is an available color for the edge u_1v_2, we are done. Otherwise, $|\varphi(v_2^{'}) \cap \varphi(u)| = 0$, $\varphi(v_1^{'}) = \varphi(v_2^{'})$ and $\varphi(uu_1) = b$ (say) appears on all the vertices u_i, for $i \in \{2, ..., k\}$ forming bicolored paths of length three. So, we uncolor the edges u_2v_3 and u_2v_4. This makes the color $\varphi(uu_2) = c_2$ (say) an available color for the edge u_1v_2. We assign $\varphi(u_1v_2) = c_2$.

Next, we color the edge u_2v_3 with a color, say, c_3 such that $c_3 \notin \varphi(v_3^{'}) \cup \varphi(u)$. If there is an available color for the edge u_2v_4, we are done. Otherwise, $|\varphi(v_4^{'}) \cap \varphi(u)| = 0$, $\varphi(v_3^{'}) = \varphi(v_4^{'})$ and $\varphi(uu_2) = c_2$ appears on all the vertices u_i, for $i \in \{1, ..., k\}$ forming bicolored paths of length three. So, we uncolor the edges u_3v_5 and u_3v_6. This makes the color $\varphi(uu_3) = c_4$ an available color for the edge u_2v_4. So, we assign $\varphi(u_2v_4) = c_4$ as shown in Figure 5. Finally, we color the edge u_3v_5 with a color, say, c_5 such that $c_5 \notin \varphi(v_5^{'}) \cup \varphi(u)$. There is at least one such color for u_3v_5. Now, observe that there is at least one color available for the edge u_3v_6 as $c_5 \notin \varphi(v_5^{'})$ and $c_4 \notin \varphi(u_1)$. So, φ can be extended to H, a contradiction.

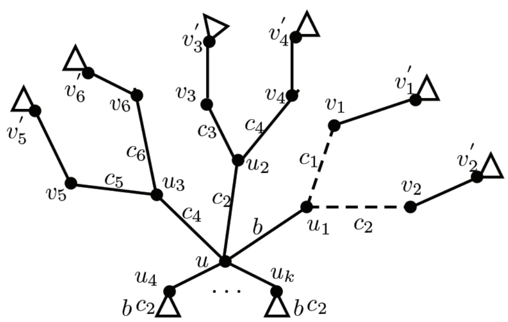

Figure 5. Configuration of Claim 3.9(ii).

Claim 3.10 $G^{'}$ *does not contain a 4-vertex* u *adjacent to a* 3_2*-vertex and a 2-vertex.*

Suppose there is a 4-vertex u in $G^{'}$ with $N(u) = \{u_i\}$ for $i \in \{1, ..., 4\}$, where u_1 is a 3_2-vertex and u_2 is a 2-vertex. By Claim 3.5, $d_H(u_1) = 3$ and $d_H(u) = 4 = k$. Let v_1 and v_2 be the 2-neighbors of u_1. By Claim 3.4, $d_H(v_1) = d_H(v_2) = 2$. Let $v_1^{'}$ and $v_2^{'}$ be the other neighbors of v_1 and v_2 respectively other than u_1. Let $u_2^{'}$ be a neighbor of u_2 (of degree greater than 2) other than u. The 2-vertex u_2 may have 1-neighbors in H. Let them be w_1 and w_2. Let $H^{'} = H \setminus \{u_1v_1, u_1v_2\}$. By minimality of H, the graph $H^{'}$ has a star edge coloring φ with $2k+1$ colors. We color the edge u_1v_1 with a color, say, c_1 such that $c_1 \notin \varphi(v_1^{'}) \cup \varphi(u)$. If we get an available color for the edge u_1v_2, we are done. Otherwise, $|\varphi(v_2^{'}) \cap \varphi(u)| = 0$, $\varphi(v_1^{'}) = \varphi(v_2^{'})$ and $\varphi(uu_1) = b$, (say) appears on the vertices u_i, for $i \in \{2, 3, 4\}$ forming bicolored paths of length three. So, we recolor uu_1 as follows.

• When $\varphi(u_2u_2^{'}) = b$. We uncolor u_1v_1 and choose a color, say, c_2 such that $c_2 \notin \varphi(u_4)\cup\varphi(u_3)\cup\varphi(u)$. As b is a common color at u_3 and u_4, there is at least one color for uu_1. Set $\varphi(uu_1) = c_2$. Then, we can extend this coloring to both the edges u_1v_1 and u_1v_2 easily, as there are at least two colors available for one edge and at least one for the other.

• When b is present on a pendant edge incident to u_2, let $\varphi(u_2w_1) = b$. We uncolor u_1v_1 and choose a color, say, c_3 such that $c_3 \notin \varphi(u_4)\cup\varphi(u_3)\cup\{\varphi(uu_2), \varphi(u_2u_2^{'})\}$. If such c_3 exists, we set $\varphi(uu_1) = c_3$. Then, we can extend the coloring to the edges u_1v_1 and u_1v_2 easily. Otherwise, we can swap the color b of the edge u_2w_1 with the color of the edge uu_2. Then, we get a color $\varphi(uu_2)$ available for uu_1. As this color do not appear on u_3 and u_4, we can easily color the edges u_1v_1 and u_1v_2. So, φ can be extended to H, a contradiction.

Claim 3.11 *For $k \geq 5$, $G^{'}$ does not contain a k-vertex u adjacent to two 3_2-vertices and $k-2$ vertices of degree 2.*

Suppose $G^{'}$ contains a k-vertex u with $N(u) = \{u_i\}$ for $i \in \{1, ..., k\}$, where u_1 and u_2 are 3_2-vertices and u_i, for $i \in \{3, ..., k\}$ are $k-2$ vertices of degree 2. By Claim 3.5, $d_H(u_1) = d_H(u_2) = 3$ and $d_{G'}(u) = d_H(u) = k$. Let v_1 and v_2 be the 2-neighbors of u_1 and let v_3 and v_4 be the 2-neighbors of u_2. By Claim 3.4, $d_H(v_i) = 2$, for all i. For $i \in \{1, ..., 4\}$, let $v_i^{'}$ be the other neighbors of v_i. The 2-vertices u_i, $i \in \{3, ..., k\}$ may have 1-neighbors in H. Let them be w_j^i for $j \in \{1, ..., k-2\}$.

Let $H^{'} = H \setminus \{u_1v_1, u_1v_2\}$ as shown in Figure 6. By minimality of H, $H^{'}$ has a star edge coloring φ with $2k+1$ colors. First, we color the edge u_1v_1 with a color, say, c_1 such that $c_1 \notin \varphi(v_1^{'})\cup\varphi(u)$. If we get an available color for the edge u_1v_2, we are done. Otherwise, $|\varphi(v_2^{'})\cap\varphi(u)| = 0$, $\varphi(v_1^{'}) = \varphi(v_2^{'})$ and $\varphi(uu_1) = b$ (say), appears on the vertices u_i, for each $i \in \{2, ..., k\}$ forming bi-colored paths of length three. So, we recolor uu_1 with a color other than b (shown with strike out color b on the edge). As at most two colors are forbidden for the edge uu_1 at each u_i, $i \in \{2, ..., k\}$, $|F(uu_1)| \leq |\bigcup_{i=2}^{k} \varphi(u_i)\cup\{c_1\}| \leq 2(k-1)+1 = 2k-1$. So, we get at least one color other than b for uu_1. Then, we have at least one available color for the remaining edge u_1v_2. Hence, φ can be extended to H, a contradiction.

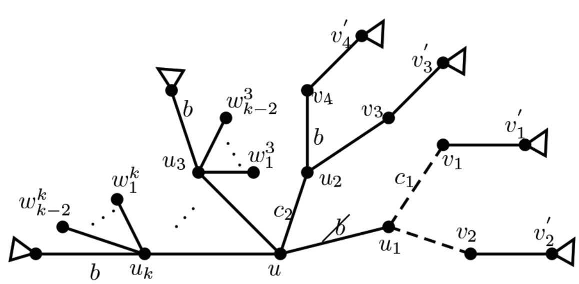

Figure 6. Configuration of Claim 3.11.

Discharging Next, we show that the counterexample H does not exist. We set a weight function $w : V(G^{'}) \to \mathbb{R}$ such that
$$w(v) = d(v) - \tfrac{14}{5},\ \forall\ v \in V(G^{'}).$$
As $\mathrm{Mad}(G^{'}) < \frac{14}{5}$, $\quad \sum_{v\in V(G^{'})} w(v) < 0.$ **(I)**

Then, we redistribute the weights among the vertices according to the discharging rules described below, to obtain a weight function $w^{'}$. During the discharging process, the total sum of weights is kept fixed.

Discharging rules:

R1: A 3_0-vertex sends $\frac{1}{5}$ to its adjacent *bad* 3_1-vertex.

R2: A 3_1-vertex sends $\frac{2}{5}$ to its adjacent 2-vertex.

R3: A 3_2-vertex sends $\frac{2}{5}$ to its adjacent 2-vertex.

R4: A k-vertex, $k \geq 4$ sends $\frac{2}{5}$ to each of its adjacent 2-vertex.

R5: A k-vertex, $k \geq 4$ sends $\frac{1}{5}$ to each of its adjacent 3_1-vertex.

R6: A k-vertex, $k \geq 4$ sends $\frac{3}{5}$ to each of its adjacent 3_2-vertex.

Let $v \in V(G^{'})$ be a k-vertex, $k \geq 2$.
Case 1: When $d(v) = 2$, $w(v) = -\frac{4}{5}$.
By Claim 3.3, v has 3^{+}-neighbors so, by R2, R3 or R4, v receives $\frac{2}{5}$ units from each neighbor. Therefore, $w^{'}(v) = -\frac{4}{5} + 2(\frac{2}{5}) = 0$.

Case 2: When $d(v) = 3$, $w(v) = \frac{1}{5}$.
• If v is a 3_0-vertex, then by Claim 3.5, it is not adjacent to any 3_2-vertices and by Claim 3.7, it can be adjacent to at most one bad 3_1-vertex. If v is adjacent to a bad 3_1-vertex, then by R1, v sends $\frac{1}{5}$ units to it. Therefore, $w^{'}(v) = \frac{1}{5} - \frac{1}{5} = 0$. Otherwise, the weight of v remains unchanged.
• If v is a 3_1-vertex which is not bad, then it is adjacent to p 4^{+}-vertices, $p \in \{1, 2\}$. By R5, v receives $\frac{1}{5}$ units from its 4^{+}-neighbor and by R2, v sends $\frac{2}{5}$ units to its 2-neighbor. Therefore, $w^{'}(v) = \frac{1}{5} + p(\frac{1}{5}) - \frac{2}{5} \geq 0$. If v is a bad 3_1-vertex, then by Claim 3.6, v can be adjacent to one 3_0 and one 3_1-vertex or two 3_0-vertices. By R1, v receives $\frac{1}{5}$ units from the 3_0-vertices and by R2, v sends $\frac{2}{5}$ units to its 2-neighbor. Therefore, $w^{'}(v) = \frac{1}{5} + p(\frac{1}{5}) - \frac{2}{5} \geq 0$, where $p \in \{1, 2\}$ is the number of 3_0-vertices adjacent to v.
• If v is a 3_2-vertex, then by Claim 3.5, it is adjacent to a 4^{+}-vertex. So, by R6, v receives $\frac{3}{5}$ units from its 4^{+}-neighbor and by R3, v sends $\frac{2}{5}$ units to each of its 2-neighbor. Therefore, $w^{'}(v) = \frac{1}{5} + \frac{3}{5} - 2(\frac{2}{5}) = 0$.

Case 3: When $d(v) = 4$, $w(v) = \frac{6}{5}$.
• If v is adjacent to 2-vertices, then by Claim 3.8, v can be adjacent to at most three 2-vertices. If v is adjacent to exactly three 2-vertices then it is not adjacent to a 3_1-vertex. So, by R4, v sends $\frac{2}{5}$ units to each of its 2-neighbors. Therefore, $w^{'}(v) = \frac{6}{5} - 3(\frac{2}{5}) = 0$.
• If v is adjacent to 3_1-vertices, then by Claim 3.8, v can be adjacent to at most two 2-vertices. Let v be adjacent to p 2-vertices and q 3_1-vertices, then by R4, v sends $\frac{2}{5}$ units to each of its 2-neighbors and by R5, v sends $\frac{1}{5}$ units to each of the adjacent 3_1-vertices. Therefore, $w^{'}(v) = \frac{6}{5} - p(\frac{2}{5}) - q(\frac{1}{5}) \geq 0$, where $p \in \{0, 1, 2\}$ and $q \in \{1, 2, 3, 4\}$ such that $p + q \leq 4$.

• If v is adjacent to 3_2-vertices, then by Claim 3.9, v can be adjacent to at most one 3_2-vertex and by Claim 3.10, such v is not adjacent to any 2-vertex. So, when v is adjacent to one 3_2-vertex and p 3_1-vertices, where $p \in \{0,1,2,3\}$, then by R5, v sends $\frac{3}{5}$ units to adjacent 3_2-vertex and by R6, v sends $\frac{1}{5}$ units to each of the adjacent 3_1-vertices. Therefore, $w^{'}(v) = \frac{6}{5} - \frac{3}{5} - p(\frac{1}{5}) \geq 0$.

Case 4: When $d(v) = k$, $k \geq 5$, $w(v) = k - \frac{14}{5}$.

• If v is adjacent to 2-vertices and 3_1 vertices, then by R4 and R5, v sends $\frac{2}{5}$ units to each of its 2-neighbors and $\frac{1}{5}$ units to each of the adjacent 3_1-vertices respectively. Let v be adjacent to p 2-vertices and q 3_1 vertices, then $w^{'}(v) = k - \frac{14}{5} - p(\frac{2}{5}) - q(\frac{1}{5}) = \frac{3k-14}{5} \geq 0$, where $p, q \in \{0, ..., k\}$ such that $p + q \leq k$.

• If v is adjacent to 2-vertices and 3_2 vertices, then by Claim 3.9, v can be adjacent to at most two 3_2-vertices and by Claim 3.11, such v can be adjacent to at most $k-3$ vertices of degree 2. Let v be adjacent to two 3_2-vertices, p 2-vertices and q 3_1-vertices, then by R4, R5 and R6, v sends $\frac{3}{5}$ units to each of the two 3_2 vertices, $\frac{1}{5}$ units to each of the 3_1-vertices and $\frac{2}{5}$ units to each of its 2-neighbors respectively. Therefore, $w^{'}(v) = k - \frac{14}{5} - 2(\frac{3}{5}) - p(\frac{2}{5}) - (q)(\frac{1}{5}) \geq k - \frac{14}{5} - 2(\frac{3}{5}) - (k-3)(\frac{2}{5}) - (\frac{1}{5}) \geq \frac{3k-15}{5} \geq 0$, where $p \in \{0, ..., k-3\}$ and $q \in \{0, ..., k-2\}$ such that $p + q \leq k - 2$.

Therefore, after discharging, in all the above cases, $w^{'}(v) \geq 0$ for every $v \in V(G^{'})$.

$\Rightarrow \sum_{v \in V(G')} w^{'}(v) \geq 0$. **(II)**

From (I) and (II), we get a contradiction. So, the subgraph $G^{'}$ does not exist. Hence, the minimal counterexample H cannot exist. This completes the proof. □

As every planar graph with girth g (length of the shortest cycle) satisfies (Folklore) $\text{Mad}(G) < \frac{2g}{g-2}$, the following corollary, can be easily derived from Theorem 3.1.

Corollary 3.12 *Let G be a planar graph with girth $g \geq 7$. Then $\chi^{'}_s(G) \leq 2\Delta + 1$.*

References

[1] Bezegová L, Lužar B, Mockovčiaková M, Soták R, Škrekovski R. Star edge coloring of some classes of graphs. Journal of Graph Theory 2016; 81 (1): 73-82. doi: 10.1002/jgt.21862.

[2] Dvořák Z, Mohar B, Šámal R. Star chromatic index. Journal of Graph Theory 2013; 72 (3): 313-326. doi: 10.1002/jgt.21644

[3] Kerdjoudj S, Kostochka AV, Raspaud A. List star edge coloring of subcubic graphs. Discussiones Mathematicae Graph Theory 2018; 38: 1037-1054. doi: 10.7151/dmgt.2037

[4] Kerdjoudj S, Pradeep K, Raspaud A. List star chromatic index of sparse graphs. Discrete Mathematics 2018; 341: 1835-1849. doi: 10.1016/j.disc.2018.03.012

[5] Kerdjoudj S, Raspaud A. List star edge coloring of sparse graphs. Discrete Applied Mathematics 2018; 238: 115-125. doi: 10.1016/j.dam.2017.12.010

[6] Lei H, Shi Y, Song Z-X. Star chromatic index of subcubic multigraphs. Journal of Graph Theory 2018; 88 (4): 566-576. doi: 10.1002/jgt.22230

[7] Lužar B, Mockovčiaková M, Soták R. Note on list star edge coloring of subcubic graphs. Journal of Graph Theory 2016; 81 (1): 73-82. doi: 10.1002/jgt.22402

[8] Pradeep K, Vijayalakshmi V. Star edge coloring of subcubic graphs. Applied Mathematics and Information Sciences 2019; 13 (2): 279-284. doi: 10.18576/amis/130216

[9] Wang Y, Wang W, Wang Y. Edge partition and star chromatic index. Applied Mathematics and Computation 2018; 333: 480-489. doi: 10.1016/j.amc.2018.03.079

[10] Wang Y, Wang Y, Wang W. Star edge coloring of graphs with maximum degree four. Applied Mathematics and Computation 2019; 340 (1): 268-275. doi: 10.1016/j.amc.2018.08.035

[11] Liu X, Deng K. An upper bound on the star chromatic index of graphs with $\delta \geq 7$. Journal of Lanzhou University (Natural Sciences) 2008; 44: 94-95.