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On some topological properties in the class of Alexandroff spaces

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Abstract: In the class of Alexandroff spaces we study the properties of being a submaximal, door, Whyburn and weakly Whyburn space. We provide characterizations in order theoretical terms. Connections with posets, counting formulas and numerical results in the class of finite spaces are also given.

Key words: Alexandroff space, submaximal space, door space, Whyburn space

1. Introduction

Alexandroff spaces are topological spaces in which any intersection of open sets is open. That is, a topological space is Alexandroff if and only if every point has a least neighborhood. Hence, any finite space is an Alexandroff space. These spaces were first introduced by Alexandroff [2] in 1937 under the name Diskrete Räume "discrete space". Alexandroff spaces have prominent aspects which are applied in several areas including geometry and digital topology, discretization space-time in theoretical physics and diverse branches of computer sciences. In fact, since the eighties of the 20th century, the interest in Alexandroff spaces was a consequence of the very expanding role of finite spaces in digital topology and its applications in biology, social and natural sciences, digital image processing etc. For some elementary properties of Alexandroff spaces we refer to [2, 21].

The most fundamental property of Alexandroff space is that the category of Alexandroff spaces **Alx** is isomorphic to the category of quasi-ordered sets **Qos**. More precisely, the specialization quasi-order is the canonical way to turn a topological space into a qoset and consequently establish a one-to-one correspondence between qosets and Alexandroff spaces. Let (X, τ) be a topological space and let $cl(A)$ denote the closure of the subset A of X. Then, the specialization quasi-order $\leq_\tau$ is defined by : $x \leq_\tau y \iff x \in cl(\{y\})$ is a quasi-order on X. We also check that $\leq_\tau$ is a partial order if and only if X is T_0 (Kolmogorov space). We note that some authors use the other direction of the order (that we can call it the dual specialization quasi-order).

Hence, the specialization functor $\Phi : \mathbf{Alx} \to \mathbf{Qos}$, which acts identically on the underlying set maps, induces a concrete isomorphism between the category **Alx** (with continuous maps) and the category of quasi-ordered sets **Qos** (with isotone maps).

A set O of a qoset $(X, \leq)$ is called upper set (resp. down set) if:

$$(x \in O \text{ and } x \leq y) \implies (y \in O)$$

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(resp. $y \leq x \implies y \in O$). So, for a topological space (X, τ) every open set (resp. closed) is an upper set in $(X, \leq_\tau)$ (resp. down set).

Let $A \subseteq X$, the upper hull of A, denoted by $\uparrow A$, is the smallest upper set containing A. Dually, the lower hull of A is the smallest down set containing A, denoted by $\downarrow A$. For each $x \in X$ we denote by $\uparrow x = \uparrow \{x\}$ and $\downarrow x = \downarrow \{x\}$. Then, $\forall x \in X$, we have:

$$\uparrow x = \{y \in X : x \leq_\tau y\} = \bigcap \{\text{open sets containing } x\}.$$

$$\downarrow x = \{y \in X : y \leq_\tau x\} = cl(\{x\}).$$

Then, the family $\mathcal{B} = \{\uparrow x : x \in X\}$ is a basis of an Alexandroff topology $\tau_\leq$ on X. The minimal neighborhood of $x \in X$ is denote by $\mathcal{V}(x) = \uparrow x$.

In this paper, we study some properties on Alexandroff spaces and for each property we derive its equivalent statement in order theoretical terms. Generally, each separation or connectedness property has its order theoretical counterpart. These correspondences have been widely studied and counting such topologies is still an open problem. For instance, the problem of counting the number of quasi-orders Q_n (or equivalently, the number of topologies on a n-set) and the number of partial orders P_n (or equivalently, the number of T_0 topologies on a n-set) remains unsolved. The numbers Q_n and P_n are known for only $n \leq 18$ (see [23]). In this direction, recall that Stine, in [24, Proposition 1.5.4], proved that the distinct pre Hausdorff topologies on a finite set X are in one to one correspondence with the partitions of X (for more information about pre-Hausdorff spaces, we refer the reader to [4]). For an excellent summary of the one-to-one correspondence between different properties of Alexandroff space and the corresponding properties of the specialization quasi-order, we refer to [11].

We are also interested with a recently introduced particular subclass of Alexandroff spaces. Let X be a set and for any morphism $f : X \longrightarrow X$ in the category **Set** of sets, we can define an Alexandroff topology denoted by $\mathcal{P}(f)$ on X by taking the family $\{A \subseteq X : f(A) \subseteq A\}$ (i.e. the f-invariant subsets of X) as the family of closed sets for this topology. Echi [9], called them primal spaces. In this paper, we follow the terminology used in [16] and we call the topological space $(X, \mathcal{P}(f))$ a functionally Alexandroff space. By the Alexandroff specialization theorem which characterizes an Alexandroff topology in terms of a quasi-order, the closure of $x \in (X, \mathcal{P}(f))$ is the lower set $\downarrow \{x\} = \{f^n(x), n \in \mathbb{N}\}$. The smallest open neighborhood of x, denoted by $\mathcal{V}_f(x)$ is given by the corresponding upper set, that is:

$$\mathcal{V}_f(x) = \uparrow x = \{y \in X : \exists n \in \mathbb{N},\ f^n(y) = x\}.$$

Alexandroff spaces which will be characterized throughout this paper are: submaximal, door and (weakly) Whyburn spaces. All results on functionally Alexandroff spaces are deduced. In the last section, we derive counting formulas for each corresponding finite space.

2. Characterization of submaximal, door, Whyburn and weakly Whyburn Alexandroff spaces

2.1. Submaximal Alexandroff spaces

A space X is called submaximal if every dense subspace of X is open in X [7, 17, 18]. Some authors add the condition that X has no isolated points. Hewitt [12] calls submaximal spaces without isolated points MI-spaces. The significance of considering submaximal spaces is provided by the theory of maximal spaces. A space X is called maximal if it is dense-in-itself and no larger topology on the set X is dense-in-itself. When the space is T_1, Theorem 1.2 of [3] gives many equivalent conditions so that the space is submaximal. As pointed in [5] when we remove the restriction of being T_1, we get the following equivalences.

Theorem 2.1 *[5, Theorem 3.1] Let X be a topological space. Then, the following statements are equivalent:*

1. *X is submaximal.*
2. *$\overline{S} \setminus S$ is closed, for each $S \subseteq X$.*
3. *$\overline{S} \setminus S$ is closed and discrete, for each $S \subseteq X$.*

Now, the first main result of this section characterize submaximal Alexandroff spaces as 3-avoiding posets, i.e. with no chain of length 3. We can call such posets as submaximal posets.

Theorem 2.2 *An Alexandroff space X is submaximal if and only if the specialization quasi-order is an order and the graph of this specialization order has no chains of length greater than 2.*

Proof Let X be an Alexandroff space. Suppose that the graph of the specialization order has no chains of length greater than 2. Let A be a subset of X and denote $A_c = \{a \in A : \{a\}$ is closed $\}$. Now, since there is no chain of length greater than 2, for any $a \notin A_c$ and for any $b \in \overline{\{a\}} \setminus \{a\}$, $\{b\}$ is closed. Finally using the fact that X is an Alexandroff space we have:

$$\overline{A} = \bigcup_{a \in A} \overline{\{a\}} = A \bigcup \left(\bigcup_{a \in A - A_c} (\overline{\{a\}} - \{a\} \right)$$

Now, $\overline{A} \setminus A$ is closed as desired.

Conversely, let $x, y \in X$, such that $x \leq y$ and $y \leq x$, then $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$. Thus, $\overline{\{y\}} = \overline{\{x\}}$, and since a submaximal space is T_0, then $x = y$. Therefore the specialization quasi-order is an order.

Now, let $x \leq y \leq z$ be a chain of distinct points. Now, for $A = \{x, z\}$ we have $y \in \overline{A} - A$ and $x \in \overline{\{y\}} \subseteq \overline{\overline{A} - A}$. So, the closure of $\overline{A} - A$ includes the point $x \notin \overline{A} - A$. Hence, $\overline{A} - A$ is not closed. □

Remark 2.3 *Then, as a consequence, a functionally Alexandroff space $(X, \mathcal{P}(f))$ is submaximal if and only if the map $f : X \to X$ is idempotent: $\forall x \in X, f(f(x)) = f(x)$. In fact, for all $x \in X$ we have $x \geq f(x) \geq f^2(x)$ but, since it is submaximal we do not have $x > f(x) > f^2(x)$. This result is already proven in [8] for submaximal functionally Alexandroff spaces.*

2.2. Door Alexandroff spaces

A topological space is a door space if and only if every set is either open or closed [13]. Check that a door space is submaximal. Considering Alexandroff door spaces, we prove the following theorem.

Theorem 2.4 *Let (X, τ) be an Alexandroff space. Then, (X, τ) is a door space if and only if the graph of the specialization order $\leq$ has no chains of length greater than 2 and all chains of length 2 contain a common point, which is necessarily a maximal point or a minimal point.*

Proof

($\Longleftarrow$) Suppose that the graph of the specialization order has no chains of length greater than 2 and all chains of length 2 contain a common minimal point m. If A is not closed in (X,τ) then it is not a down-set in $(X,\leq)$, so A contains points $x \in \uparrow m$, and $m \notin A$. But any set not containing m is an upper-set, and therefore it is open. If the common point M were maximal, the dual argument applies. (If A is not open, then it is not an upper-set, so A contains points $x \in \downarrow M$ and $M \notin A$. But any set not containing M is a down-set and therefore closed.)

($\Longrightarrow$) Suppose the condition on the specialization order fails. Then either there exists a chain with three distinct points $a < b < d$ or there exists chains $a < b$ and $c < d$ of length 2 with no common point. In either case, $\{a, d\}$ is neither increasing nor decreasing, and thus neither open not closed. Thus, (X,τ) is not a door space. □

2.3. Whyburn and weakly Whyburn Alexandroff spaces

In a topological space X a subset F is called almost closed if and only if $\overline{F}\backslash F = \{x\}$ for some $x \in X$, which can be abbreviated by the notation $F \longrightarrow x$. The topological space X is called a Whyburn space [20] if for every nonclosed subset A of X and for every $x \in \overline{A}\backslash A$, there exists $B \subseteq A$ such that $\overline{B}\backslash A = \{x\}$ or equivalently, there exists $B \subseteq A$ such that $B \longrightarrow x$. It is called weakly Whyburn [22] if for every nonclosed subset A of X there exists $B \subseteq A$ such that $\overline{B}\backslash A$ is a one point set. Clearly, every Whyburn space is weakly Whyburn. The class of Whyburn spaces and weakly Whyburn spaces were studied in [19, 25] and denoted respectively by AP spaces and WAP spaces.

Theorem 2.5 *Let (X,τ) be an Alexandroff space. Then the following statements are equivalent:*

1. *X is Whyburn;*
2. *X is weakly Whyburn;*
3. *The closure of each element of X has at most 2 points, that is, $|\downarrow x| \leq 2$.*

Proof

1. $[1 \Longrightarrow 3]$: Suppose that X is Whyburn and there exists $x \in X$ such that $(\downarrow x)\setminus\{x\}$ contains two distinct elements y and z. Since $\{x\}$ is not closed and $y \in \overline{\{x\}} \setminus \{x\}$ then there exists $B \subseteq \{x\}$ such that $\overline{B} \setminus \{x\} = \{y\}$. Yet, in that case $B = \{x\}$ which leads to a contradiction because $(\downarrow x) \setminus \{x\}$ contains also z.

2. $[3 \Longrightarrow 1]$: Suppose that the closure of each element of X has at most 2 points. Let A be a set of X which is not closed. Since X is an Alexandroff space, then:

$$\overline{A} = \cup_{t\in A}(\downarrow t).$$

Let $x \in \overline{A} \setminus A$ then there exists $t \in A$ such that $x \in \downarrow t$. Clearly $x \neq t$. So, the fact that $|\downarrow t| \leq 2$ implies that $\downarrow t = \{t, x\}$. Consider $B = \{t\}$ so we have $B \subseteq A$ and $\overline{B} \setminus A = (\downarrow t) \setminus A = \{t, x\} \setminus A = \{x\}$. Therefore, X is a Whyburn space.

3. $[2 \Longrightarrow 1]$: Let X be a weakly Whyburn space and suppose that there exists $x \in X$ such that $|\downarrow x| > 2$. Consider $A = \{x\}$. Since A is not closed then there exists $B \subseteq A$ such that $| \overline{B} \setminus A | = 1$. Thus $B = \{x\}$ and so $| \overline{B} \setminus A | = | (\downarrow x) \setminus \{x\} | > 1$ which is absurd.

□

Remark 2.6 *It follows that a functionally Alexandroff space $(X, \mathcal{P}(f))$ is Whyburn if and only if $f^2(x) \in \{x, f(x)\}$ for all $x \in X$.*

The following result states that Alexandroff spaces and functionally Alexandroff spaces are equal in the class of Whyburn spaces.

Proposition 2.7 *A Whyburn space is an Alexandroff space if and only if it is a functionally Alexandroff space.*

Proof It is enough to see that an Alexandroff Whyburn space is a functionally Alexandroff space. Hence by Theorem 2.5, $|\downarrow x| \leq 2$ for any $x \in X$. Two cases arise.

1. If $|\downarrow x| = 1$, we take $f(x) = x$ and thus $\overline{\{x\}} = \downarrow x = \{x\} = \{f^n(x), n \in \mathbb{N}\}$.

2. If $|\downarrow x| = 2$, then $\downarrow x = \{x, y\}$ and in this case we take $f(x) = y$. So, $\overline{\{x\}} = \downarrow x = \{x, y\} = \{x, f(x)\} = \{f^n(x), n \in \mathbb{N}\}$. Indeed, $f^2(x) \in \{x, f(x)\}$ since $f^2(x) = f(f(x)) = f(y)$ which is equivalent, by the construction of f, to write:

   (i) y if $|\downarrow y| = 1$ and thus $f^2(x) = f(x)$.

   (ii) z if $|\downarrow y| = 2$ and thus $z \in \downarrow y \setminus \{y\} \subseteq \downarrow x \setminus \{y\} = \{x\}$ and consequently $f^2(x) = x$.

□

3. Numerical and counting formulas in the class of finite spaces

As said in the introduction, there is no known formula that gives the number of topologies on a finite set. It is known that the number of topologies Q_n and the number of T_0 topologies P_n on a finite set with n elements are related by the formula:

$$Q_n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} P_k$$

where $\left\{ {n \atop k} \right\}$ is the Stirling number of the second kind. In [10] Erné show that:

$$\lim_{n \to \infty} \frac{Q_n}{P_n} = 1.$$

In the following, we gives formulas to count the number of submaximal, door and Whyburn finite spaces.

3.1. Finite submaximal spaces

Theorem 3.1 *let $S(n)$ (resp, $S_f(n)$) denote the number of submaximal (submaximal functionally Alexandroff) topologies on a finite set of order n. Then,*

$$S(n) = \sum_{k=1}^{n} \binom{n}{k} \left(2^k - 1\right)^{n-k}.$$

and

$$S_f(n) = \sum_{k=1}^{n} \binom{n}{k} k^{n-k}$$

Proof $S(n)$ is actually the number of labeled graded partially ordered sets with n elements of height at most 1. By [14], we have:

$$S(n) = \sum_{k=1}^{n} \binom{n}{k} \left(2^k - 1\right)^{n-k}.$$

In functionally Alexandroff space, a point $x \in (X, P(f))$ is called periodic of period $p \in \mathbb{N} - \{0\}$ if the points x, $f(x)$, $f^2(x)$, $\ldots$, $f^{p-1}(x)$ are distinct and $f^p(x) = x$. A point with period 1 is called a fixed point of f, and hence is a minimal point in the corresponding quasi-ordered set. Hence, in a submaximal functionally Alexandroff space every periodic point must be a fixed point. Using [15, Theorem 2.15], one can check that if f is an idempotent map, then there is no map $g : X \to X$ such that $g \neq f$ and the topology $\mathcal{P}(g)$ is equal to $\mathcal{P}(f)$. Thus, if $S_f(n)$ denote the number of submaximal functionally Alexandroff topologies on a finite set of order n, then $S_f(n)$ is equal to the number of idempotent mappings f from a set of n elements into itself. Let $H(n)$ denotes this number. This number is also known to be the number of forests with n nodes and height at most 1 [6, Exercice 43]. Thus,

$$S_f(n) = H(n) = \sum_{k=1}^{n} \binom{n}{k} k^{n-k}.$$

We note also that the number $H(n)$ can be calculated using the Taylor series of e^{xe^x}, that is:

$$H(n) = \left(e^{xe^x}\right)^{(n)}(0).$$

□

3.2. Finite door spaces

Theorem 3.2 *let $D(n)$ (resp, $D_f(n)$) denote the number of door (door functionally Alexandroff) topologies on a finite set of order n. Then,*

$$D(n) = n2^n - (n^2 + n - 1).$$

$$D_f(n) = n2^{n-1} - n + 1.$$

Proof By Theorem 2.4, all components of a door Alexandroff space are singleton points except only at most one component, which may be drawn with only two different shapes of Hasse diagram (Figure 1).

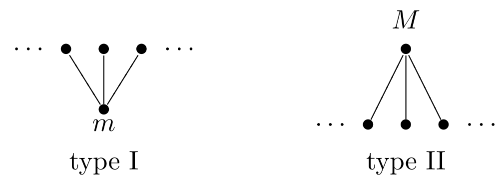

Figure 1. Hasse diagrams of type I and type II.

Then, it is clear that for functionally Alexandroff spaces, only type I shaped diagram is allowed. Thus, $(X, \mathcal{P}(f))$ is a door space if and only if $|f[Fix(f)^c]| \leq 1$. See also [8, Theorem 4.3]. Let denote by $D(n)$ (resp. $D_f(n)$) the number of finite door spaces with n points (resp. functionally door spaces with n points). Then, it is obvious that the number of diagrams of type I or II with $k \geq 2$ points is $k\binom{n}{k}$. Note that for $k = 2$, type I and type II are of the same shape. Hence we get the numbers:

$$D(n) = 1 + 2\binom{n}{2} + 2\sum_{k=3}^{n} k\binom{n}{k} = n2^n - (n^2 + n - 1).$$

$$D_f(n) = 1 + \sum_{k=2}^{n} k\binom{n}{k} = n2^{n-1} - n + 1.$$

□

3.3. Finite Whyburn spaces

Theorem 3.3 *let $W(n)$ denote the number of Whyburn topologies on a finite set of order n. Then,*

$$W(n) = \sum_{p=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2p} \frac{(2p)!}{2^p p!} H(n-2p)$$

Proof In the case of a Whyburn space, by Theorem 2.5, the graph of functionally Alexandroff space consists of some single points, some components of type I (Figure 1) and some cyclic components of length 2.

Without cyclic points, the Hasse diagram of Whyburn Alexandroff spaces with n points is exactly a forest of labeled rooted trees of height at most 1. So, their numbers is exactly the number $H(n)$. Let denotes by $W(n)$ the number of Whyburn finite topologies on a set with n points. We may first choose $2p$ points to form the p cycles of length 2, so we can form $\frac{(2p)!}{2^p p!}$ such different cycles. The remaining $n - 2p$ points form the forests of labeled rooted trees of height at most 1. Hence we get:

$$W(n) = \sum_{p=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2p} \frac{(2p)!}{2^p p!} H(n-2p)$$

□

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