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An improved oscillation criteria for first order dynamic equations

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Abstract: In this work, we consider the first-order dynamic equations

$$x^{\Delta}(t) + p(t)x(\tau(t)) = 0,\ t \in [t_0, \infty)_{\mathbb{T}}$$

where $p \in C_{rd}\left([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^{+}\right)$, $\tau \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ and $\tau(t) \le t$, $\lim_{t\to\infty} \tau(t) = \infty$. When the delay term $\tau(t)$ is not necessarily monotone, we present a new sufficient condition for the oscillation of first-order delay dynamic equations on time scales.

Key words: Dynamic equations, nonmonotone, oscillation, time scales

1. Introduction

The oscillatory behavior of solutions of differential/difference and dynamic equations has been studied by many authors. See, for example, [1–34] and the references cited therein. Consider the first-order delay dynamic equations

$$x^{\Delta}(t) + p(t)x(\tau(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.1}$$

where $\mathbb{T}$ is a time scale unbounded above with $t_0 \in \mathbb{T}$, $p \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}_0^{+})$, $\tau \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ is not necessarily monotone such that

$$\tau(t) \le t \ \text{ for all } t \in \mathbb{T}, \quad \lim_{t\to\infty} \tau(t) = \infty. \tag{1.2}$$

A function $p : \mathbb{T} \to \mathbb{R}$ is called positively regressive (we write $p \in \mathcal{R}^{+}$) if it is rd-continuous and satisfies $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$, where $\mu : \mathbb{T} \to \mathbb{R}_0^{+}$ is the graininess function defined by $\mu(t) := \sigma(t) - t$ with the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ defined by $\sigma := \inf\{s \in \mathbb{T} : s > t\}$ for $t \in \mathbb{T}$. A point $t \in \mathbb{T}$ is called right-dense if $\sigma(t) = t$ and/or equivalently $\mu(t) = 0$ holds, otherwise it is called right-scattered. The readers are referred to Bohner and Peterson [2] for further details concerning the time scales calculus.
A function $x : \mathbb{T} \to \mathbb{R}$ is called a solution of Equation (1.1), if $x(t)$ is delta differentiable for $t \in \mathbb{T}^{\kappa}$ and satisfies Equation (1.1) for $t \in \mathbb{T}^{\kappa}$. We say that a solution x of Equation (1.1) has a generalized zero at t if $x(t) = 0$ or if $\mu(t) > 0$ and $x(t)x(\sigma(t)) < 0$. Let $\sup \mathbb{T} = \infty$ and then a nontrivial solution x of equation (1.1) is called oscillatory on $[t, \infty)$ if it has arbitrarily large generalized zeros in $[t, \infty)$.

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Now, we give some well-known tests on oscillation of (1.1).

In 2002, Zhang and Deng [32], using the cylinder transforms, proved that if $\tau(t)$ is eventually nondecreasing and

$$\limsup_{t\to\infty} \sup_{\lambda\in E} \{\lambda e_{-\lambda p}(t,\tau(t))\} < 1,$$

where $E = \{\lambda : \lambda > 0,\ 1 - \lambda p(t)\mu(t) > 0\}$ and in 2005, Bohner [4], using exponential functions notation, proved that if $\tau(t)$ is eventually nondecreasing and

$$\limsup_{t\to\infty} \sup_{-\lambda p\in\mathcal{R}^+} \{\lambda e_{-\lambda p}(t,\tau(t))\} < 1,$$

where

$$e_{-\lambda p}(t,\tau(t)) = \exp\left\{ \int_{\tau(t)}^{t} \xi_{\mu(s)}(-\lambda p(s))\Delta s \right\}$$

and

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h} & , \text{ if } h \neq 0 \\ z & , \text{ if } h = 0 \end{cases},$$

then all solutions of Equation (1.1) are oscillatory.

In 2005, Zhang et al. [33] and in 2006, Şahiner and Stavroulakis [27], using different technique, obtained that if $\tau(t)$ is eventually nondecreasing and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s > 1, \tag{1.3}$$

then all solutions of Equation (1.1) are oscillatory.

Zhang et al. [33] (See also Agarwal and Bohner [1, Theorem 1] gave the following result. Assume that $\tau(t)$ is eventually nondecreasing and

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s > \frac{1}{e}. \tag{1.4}$$

Then all solutions of (1.1) oscillate.

Şahiner and Stavroulakis [27] obtained that if $\tau(t)$ is eventually nondecreasing and

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s > c \quad \text{and} \quad \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s > 1 - \frac{c^2}{4}, \tag{1.5}$$

where $c \in (0,1)_{\mathbb{R}}$, then every solution of Equation (1.1) oscillates.

Agarwal and Bohner [1] improved the condition (1.5) by the following. If $\tau(t)$ is eventually nondecreasing and

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s > c \quad \text{and} \quad \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s > 1 - \left(1 - \sqrt{1-c}\right)^2, \tag{1.6}$$

where $c \in (0,1)_{\mathbb{R}}$, then every solution of Equation (1.1) oscillates.

In 2006, Karpuz and Öcalan [17] improved the condition (1.6) by extending the second integral condition to the larger interval $[\tau(t), t]_{\mathbb{T}}$ as the following. Assume that $\tau(t)$ is eventually nondecreasing and

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s > c \quad \text{and} \quad \limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s > 1 - \left(1 - \sqrt{1-c}\right)^2, \tag{1.7}$$

where $c \in (0,1)_{\mathbb{R}}$. Then every solution of Equation (1.1) oscillates.

Zhang et al. [33] established the following result. Assume that $\tau(t)$ is eventually nondecreasing and $\alpha \in [0, \frac{1}{e}]$ (where α is defined by (1.4)). Furthermore,

$$\limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s > \frac{1+\ln\lambda_1}{\lambda_1} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{1.8}$$

where $\lambda_1 \in [1, e]$ is the unique root of the equation $\lambda = e^{\alpha\lambda}$, then all solutions of Equation (1.1) are oscillatory.

It is clear that, since

$$\frac{1+\ln\lambda_1}{\lambda_1} \leq 1 \quad \text{for} \quad \lambda_1 \in [1, e],$$

the condition (1.8) implies that

$$\limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}. \tag{1.9}$$

Clearly, when $0 < c \leq \frac{1}{e}$, it is easy to verify that

$$\frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} > \left(1-\sqrt{1-\alpha}\right)^2 > \frac{\alpha^2}{4}$$

and therefore the condition (1.9) is weaker than the conditions (1.5) and (1.7).

Now, we assume that $\tau(t)$ is not necessarily monotone. Set

$$h(t) = \sup_{s\leq t} \tau(s), \quad t \in \mathbb{T}, \; t \geq 0. \tag{1.10}$$

Clearly, $h(t)$ is nondecreasing and $\tau(t) \leq h(t)$ for all $t \geq 0$.

In 2017, when $\tau(t)$ is not necessarily monotone, Öcalan, Özkan and Yıldız [24, Theorem 2.2] studied Equation (1.1) and obtained the following result. If

$$\limsup_{t\to\infty} \int_{h(t)}^{\sigma(t)} p(s)\Delta s > 1, \tag{1.11}$$

where $h(t)$ is defined by (1.10), then every solution of (1.1) is oscillatory.

Very recently, Öcalan [25, Corollary 2.4] established the following result when $\tau(t)$ is not necessarily monotone. If

$$\liminf_{t\to\infty}\int_{h(t)}^{t} p(s)\Delta s = \liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)\Delta s > \frac{1}{e}, \tag{1.12}$$

where $h(t)$ is defined by (1.10), then all solutions of (1.1) oscillate.

Finally, Öcalan [25] obtained the following result when $\tau(t)$ is not necessarily monotone.

Theorem 1.1 *Assume that* (1.2) *holds,* $-p \in \mathcal{R}^+$ *and*

$$\limsup_{t\to\infty}\int_{h(t)}^{\sigma(t)} \frac{p(s)}{e_{-p}\left(h(t),\tau(s)\right)}\Delta s > 1 \tag{1.13}$$

or

$$\liminf_{t\to\infty}\int_{h(t)}^{t} \frac{p(s)}{e_{-p}\left(h(s),\tau(s)\right)}\Delta s > \frac{1}{e}, \tag{1.14}$$

where $h(t)$ *is defined by* (1.10)*. Then all solutions of* (1.1) *oscillate.*

Lately, Kılıç and Öcalan [20] studied Equation (1.1) and obtained the following result, when $\tau(t)$ is not necessarily monotone.

Theorem 1.2 *Assume that* (1.2) *holds,* $-p \in \mathcal{R}^+$*. If*

$$\limsup_{t\to\infty}\int_{h(t)}^{\sigma(t)} p(s)\Delta s > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{1.15}$$

where $h(t)$ *is defined by* (1.10) *and* α *is defined by* (1.4)*, then all solutions of* (1.1) *oscillate.*

2. Main results

In this section, we present a new sufficient condition for the oscillation of all solutions of (1.1), under the assumption that the argument $\tau(t)$ is not necessarily monotone.

Now, we consider the case where $0 < \alpha \leq \frac{1}{e}$. So, we will obtain new oscillatory criteria for all solutions of (1.1). To establish our main results, we need the following lemmas. The following result was given in [4].

Lemma 2.1 *Suppose that* $-p \in \mathcal{R}^+$ *and* $s \in \mathbb{T}$*. If*

$$x^{\Delta}(t) + p(t)x(t) \leq 0 \quad \textit{for all } t \geq s,$$

then

$$x(t) \leq e_{-p}\left(t,s\right)x(s) \quad \textit{for all } t \geq s.$$

The proof of the next Lemma can be done in a similar way to the proof of Lemma 2.4 in [33]. (Also, see [20].)

Lemma 2.2 *Assume that $\tau(t)$ is not necessarily monotone. Let $0 \leq \alpha \leq \frac{1}{e}$ and $x(t)$ be an eventually positive solution of Equation (1.1). Then*

$$\liminf_{t\to\infty} \frac{x(\sigma(t))}{x(h(t))} \geq \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{2.1}$$

where $h(t)$ is defined by (1.10) and α is defined by (1.4).

Theorem 2.3 *Assume that (1.2) holds, $-p \in \mathcal{R}^+$ and*

$$\limsup_{t\to\infty} \int_{h(t)}^{\sigma(t)} \frac{p(s)}{e_{-p}(h(t),\tau(s))}\Delta s > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{2.2}$$

where $h(t)$ is defined by (1.10) and α is defined by (1.4). Then all solutions of (1.1) oscillate.

Proof Assume, for the sake of contradiction, that there exists a nonoscillatory solution $x(t)$ of (1.1). Since $-x(t)$ is also a solution of (1.1), we can confine our discussion only to the case where the solution $x(t)$ is eventually positive. Then, there exists $t_1 > t_0$ such that $x(t),\ x(\tau(t)),\ x(h(t)) > 0$, for all $t \geq t_1$. Thus, from (1.1) we have

$$x^{\Delta}(t) = -p(t)x(\tau(t)) \leq 0, \quad \text{for all } t \geq t_1,$$

which means that $x(t)$ is an eventually nonincreasing function. In view of this and taking into account that $\tau(t) \leq t$, (1.1) gives

$$x^{\Delta}(t) + x(t)\,p(t) \leq 0, \quad t \geq t_1$$

and so we have Lemma 2.1. Integrating (1.1) from $h(t)$ to $\sigma(t)$ and using Lemma 2.1, we obtain

$$\begin{aligned} x(\sigma(t)) - x(h(t)) + \int_{h(t)}^{\sigma(t)} p(s)x(\tau(s))\Delta s &= 0 \\ x(\sigma(t)) - x(h(t)) + \int_{h(t)}^{\sigma(t)} p(s)\frac{x(h(t))}{e_{-p}(h(t),\tau(s))}\Delta s &\leq 0 \\ x(\sigma(t)) - x(h(t)) + x(h(t))\int_{h(t)}^{\sigma(t)} \frac{p(s)}{e_{-p}(h(t),\tau(s))}\Delta s &\leq 0 \end{aligned}$$

or

$$\int_{h(t)}^{\sigma(t)} \frac{p(s)}{e_{-p}(h(t),\tau(s))}\Delta s \leq 1 - \frac{x(\sigma(t))}{x(h(t))}. \tag{2.3}$$

Consequently, from (2.3) we obtain

$$\limsup_{t\to\infty} \int_{h(t)}^{\sigma(t)} \frac{p(s)}{e_{-p}\left(h(t),\tau(s)\right)} \Delta s \le 1 - \liminf_{t\to\infty} \frac{x\left(\sigma(t)\right)}{x\left(h(t)\right)} \tag{2.4}$$

and by (2.1), the inequality (2.4) leads to

$$\limsup_{t\to\infty} \int_{h(t)}^{\sigma(t)} \frac{p(s)}{e_{-p}\left(h(t),\tau(s)\right)} \Delta s \le 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2},$$

which contradicts to (2.2). The proof of the theorem is completed. □

Example 2.4 *Let $h \in \mathbb{Z}$ and $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$. Then, we have for $t \in \mathbb{T}$*

$$\sigma(t) = t + h, \quad \mu(t) = h \quad and \quad x^{\Delta}(t) = \frac{x(t+h)-x(t)}{h}.$$

Thus, Equation (1.1) becomes

$$\frac{x(t+h)-x(t)}{h} + p(t)x\left(\tau(t)\right) = 0, \quad t \in \{hk : k \in \mathbb{Z}\}.$$

Let $\tau(t) = t-2$ and $h = 2$. Since $p(t) \in \{2k : k \in \mathbb{Z}\}$, we assume

$$p(2t) = 0.18 \; and \; p(2t+2) = 0.21, \quad t = 0, 2, 4, \ldots .$$

When $\mathbb{T} = h\mathbb{Z}$, from (iii) in Theorem 1.79 [2], we have the following formula.

$$\int_a^b f(t)\Delta t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h \quad for \; a < b. \tag{2.5}$$

Thus, by using (2.5) we obtain that for $\tau(t)$, $p(t) \in \{2k : k \in \mathbb{Z}\}$

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s = \liminf_{t\to\infty} \sum_{j=\frac{t-2}{2}}^{\frac{t}{2}-1} 2p(2j) = \liminf_{t\to\infty} 2p(t-2) = 0.36$$

and

$$M := \limsup_{t\to\infty} \int_{h(t)}^{\sigma(t)} \frac{p(s)}{e_{-p}\left(h(t),\tau(s)\right)} \Delta s = \limsup_{t\to\infty} \sum_{j=\frac{h(t)}{2}}^{\frac{\sigma(t)}{2}-1} \frac{2p(2j)}{e_{-p}\left(h(t),\tau(2j)\right)}.$$

Now, we observe that

$$
\begin{aligned}
e_{-p}\left(h(t), \tau(2j)\right) &= \exp\left\{\int_{\tau(2j)}^{h(t)} \xi_{\mu(u)}(-p(u))\Delta u\right\} = \exp\left\{\sum_{i=\frac{\tau(2j)}{2}}^{\frac{h(t)}{2}-1} \frac{2\log\left(1-\mu(2i)p(2i)\right)}{\mu(2i)}\right\} \\
&= \exp\left\{\sum_{i=\frac{\tau(2j)}{2}}^{\frac{h(t)}{2}-1} \log\left(1-2p(2i)\right)\right\} = \exp\left\{\log \prod_{i=\frac{\tau(2j)}{2}}^{\frac{h(t)}{2}-1}\left(1-2p(2i)\right)\right\} \\
&= \prod_{i=\frac{\tau(2j)}{2}}^{\frac{h(t)}{2}-1}\left(1-2p(2i)\right) = \prod_{i=j-1}^{\frac{t-2}{2}-1}\left(1-2p(2i)\right).
\end{aligned}
$$

So, we have

$$
\begin{aligned}
M \quad &: = \limsup_{t\to\infty} \int_{h(t)}^{\sigma(t)} \frac{p(s)}{e_{-p}\left(h(t), \tau(s)\right)}\Delta s = \limsup_{t\to\infty} \sum_{j=\frac{t-2}{2}}^{\frac{t+2}{2}-1} 2p(2j) \prod_{i=j-1}^{\frac{t-2}{2}-1} \frac{1}{(1-2p(2i))} \\
&= \limsup_{t\to\infty}\left[2p(t-2)\frac{1}{(1-2p(t-4))} + 2p(t)\right]
\end{aligned}
$$

and

$$M \cong 0.9825 \ngtr 1$$

shows that condition (1.13) *fails. However, since*

$$M \cong 0.9825 > 1 - \frac{1 - 0.36 - \sqrt{1-2(0.36)-(0.36)^2}}{2} \cong 0.87391$$

and therefore every solution oscillates by Theorem 2.3.

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