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On a coupled Caputo conformable system of pantograph problems

Sabri T. M. THABET[1], **Sina ETEMAD**[2], **Shahram REZAPOUR**[2,3,*]
[1]Department of Mathematics, University of Aden, Aden, Yemen
[2]Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
[3]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan

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Abstract: Our fundamental purpose in the present manuscript is to explore existence and uniqueness criteria for a new coupled Caputo conformable system of pantograph problems in which for the first time, the given boundary conditions are formulated in the Riemann–Liouville conformable framework. To reach the mentioned aims, we utilize different analytical techniques in which some fixed point results play a vital role. In the final part, a simulative example is designed to cover the applicability aspects of theoretical findings available in this research manuscript from a numerical point of view.

Key words: Coupled system, fixed point theory, pantograph boundary problem, Caputo conformable derivative

1. Introduction

Investigation and analysis of different mathematical models for numerous natural dynamical processes are an interesting and important branches of the applied mathematics in which the researchers focus on dynamics of newly formulated systems by means of the existing computational tools. In such way, there exists a vast range of classical and modern fractional operators which play a vital role to model different natural phenomena and processes. Indeed during past decades, mathematicians have introduced various operators. But to describe natural processes, fractional order modelings are more accurate than integer order ones, thus new fractional operators have been provided.

In most published literatures, one can simply observe numerous structures of fractional modelings in which one of the Riemann–Liouville and the Caputo fractional operators have been utilized (see for instance, [5, 10–12, 16, 18, 22, 24, 31, 41, 43, 50, 52, 53, 63]). Besides, several applied generalizations of mentioned operators such as the Hadamard, Caputo–Hadamard and Hilfer fractional operators are employed by other mathematicians in the subsequent time periods and different fractional modelings are designed using these extended operators (see for example, [2, 8, 19, 20, 26–28, 34, 44, 51, 57]).

On the other side, coupled systems of fractional differential equations and inclusions are another part of the vast domain of mathematical modelings. These fractional coupled systems usually arise from different areas of applied sciences and biological processes. The study of the existence, uniqueness, stability and other dynamical behaviors of solutions related to such fractional systems plays a vital role in this regard. For example, Jin and Sun [37] studied the existence of solutions for a coupled system of fractional compartmental biological

*Correspondence: rezapourshahram@yahoo.ca
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models as differential inclusions. In [58], Wang, Shah and Ali investigated the existence and Hyers–Ulam stability of solutions to fractional nonlinear impulsive switched coupled system of evolution equations. Recently, Xu, Ma and Xing [59] studied the existence and asymptotic behavior of vector solutions for a new version of the linearly coupled Choquard type system by means of variational methods. In this direction, one can see many other works on the fractional coupled systems of boundary value problems such as ([3, 4, 13, 15, 36, 42, 62]).

Five years ago, Caputo and Fabrizio [29] formulated an extended normal structure of new fractional differentiation operator in which the kernel function has no singularity in any point. This novel operator is called the fractional Caputo–Fabrizio operator. Immediately after them, Losada and Nieto [40] focused on some pure aspects of this new nonsingular operator. Some flexible properties of the Caputo–Fabrizio integro-derivative operators led to publish research papers on the different fractional modelings in this subject (see for instance, [7, 9, 21, 23, 25, 47]).

After Caputo–Fabrizio operator, Abdeljawad [1] generalized some concepts given in [39] and investigated some pure and applied aspects of the well-behaved conformable differentiation operator of arbitrary order. After that, Jarad, Ugurlu, Abdeljawad and Baleanu [35] generalized the well-known standard fractional Riemann–Liouville integral provided that a unification to other differentiation operators including Riemann–Liouville, Caputo, Hadamard, Caputo–Hadamard and other derivatives are obtained [38]. In this way, they tried to derive two integration and differentiation operators of arbitrary order based on the conformable operators. They first constructed new functional spaces and in the sequel, they proved some basic applied aspects of two newly defined combined operators entitled the Reimann–Liouville conformable integral and the Caputo conformable derivative.

The pantograph differential equation can be considered one of the most important class of differential equations in the applied sciences. This type of differential equation is regarded as a kind of delay differential equations and has many applications in both applied and pure mathematics. For the first time, Balachandran, Kiruthika and Trujillo [14] introduced the pantograph equation of arbitrary order and established the existence and also uniqueness criteria for the given problem. Next, they expressed neutral version of the fractional pantograph equation and extracted existence criteria of its solutions [14]. After them, many other researchers began to study fractional pantograph equations by the aid of various numerical techniques including the spectral collocation method, Hermite wavelet method, the operational method and etc. (refer to [30, 48, 60, 61]). Besides, other mathematicians worked on various classes of fractional pantograph equations with the help of analytical methods (see for example, [54–56]).

In 2019, Rabiei and Ordokhani [46] derived some numerical results on the fractional pantograph delay differential equations in any arbitrary interval as

$$\begin{cases} {}^{\mathcal{C}}\mathfrak{D}_0^{\eta^*}\varpi(z) = a(z)\varpi(z) + \sum_{j=1}^{l} b_j(z)\,{}^{\mathcal{C}}\mathfrak{D}_0^{\eta_j^*}\varpi(q_j z), \qquad (z \in [0,h],\ h \le 1) \\ \varpi^{(i)}(0) = \mu_i, \qquad (i = 0,1,2,\ldots,n-1, \quad \mu_i \in \mathbb{R}), \end{cases}$$

where $n-1 < \eta^* \le n$, $0 \le \eta_j^* < \eta^* \le n$, $0 < q_j < 1$, ${}^{\mathcal{C}}\mathfrak{D}_0^{(\cdot)}$ is the Caputo fractional derivative and functions $b_j(z)$ and $a(z)$ are known and determined in $[0,h]$. In that work, the authors introduced fractional order Boubaker polynomials in relation to the Boubaker polynomials and then constructed pantograph operational matrices and by means of these matrices, they implemented Newton's iterative method to extract approximate solutions [46]. In the same year, Iqbal, Shah and Khan [33])established conditions for obtaining mild solutions to

the coupled system of multipoint boundary value problems of nonlinear fractional hybrid pantograph differential equations given by

$$\begin{cases} {}^{\mathcal{C}}\mathfrak{D}_0^{\eta^*}\left[\varpi(z)-\varphi(z,\varpi(z))\right]=\psi(z,\varrho(z),\varrho(\lambda_1 z)), & (z\in[0,1]),\\ {}^{\mathcal{C}}\mathfrak{D}_0^{\eta^*}\left[\varrho(z)-\varphi(z,\varrho(z))\right]=\psi(z,\varpi(z),\varpi(\lambda_2 z)), & (z\in[0,1]),\\ {}^{\mathcal{C}}\mathfrak{D}_0^{p}\varpi(0)=\kappa_1\varpi(\xi_1), \quad \varpi'(0)=0,\ \ldots\ ,\varpi^{n-2}(0)=0, \quad {}^{\mathcal{C}}\mathfrak{D}_0^{p}\varpi(1)=\kappa_2\varpi(\xi_2),\\ {}^{\mathcal{C}}\mathfrak{D}_0^{p}\varrho(0)=\kappa_1\varrho(\xi_1), \quad \varrho'(0)=0,\ \ldots\ ,\varrho^{n-2}(0)=0, \quad {}^{\mathcal{C}}\mathfrak{D}_0^{p}\varrho(1)=\kappa_2\varrho(\xi_2), \end{cases}$$

where $\eta^*\in(n-1,n]$, $n\in\mathbb{N}$, $p,\xi_1,\xi_2\in(0,1)$, $\lambda_1,\lambda_2\in(0,1)$, $\kappa_1,\kappa_2\in\mathbb{R}\setminus\{0\}$, ${}^{\mathcal{C}}\mathfrak{D}_0^{(\cdot)}$ denotes the Caputo fractional derivative and the nonlinear functions φ and ψ are continuous. In the mentioned paper, the authors proved the existence results by using Burton and coupled type fixed point theorems ([33]).

In 2020, Alrabaiah, Ahmad, Shah and Rahman [6] developed a qualitative analysis to the class of nonlinear coupled system of fractional pantograph delay differential equations with integral boundary conditions as follows:

$$\begin{cases} {}^{\mathcal{R}}\mathfrak{D}_0^{\eta_1^*}\varpi(z)=-f_1(z,\varpi(\lambda z),\varrho(z),{}^{\mathcal{R}}\mathfrak{D}_0^{p_1}\varrho(z)), & (z\in[0,1]),\\ {}^{\mathcal{R}}\mathfrak{D}_0^{\eta_2^*}\varrho(z)=-f_2(z,\varpi(z),\varrho(\lambda z),{}^{\mathcal{R}}\mathfrak{D}_0^{p_2}\varpi(z)),\\ \varpi(0)=0, \qquad \varpi(1)=\displaystyle\int_0^1\varphi(q)\varpi(q)\,\mathrm{d}q,\\ \varrho(0)=0, \qquad \varrho(1)=\displaystyle\int_0^1\varphi(q)\varrho(q)\,\mathrm{d}q, \end{cases}$$

where $\eta_1^*,\eta_2^*\in(1,2]$, $\lambda,p_1,p_2\in(0,1)$ and ${}^{\mathcal{R}}\mathfrak{D}_0^{(\cdot)}$ denotes the Riemann–Liouville fractional derivative. The nonlinear functions $f_1,f_2:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ are continuous and $\varphi:(0,1)\to[0,\infty)$ is a bounded function. The authors studied the existence and uniqueness of the solution related to this coupled system with the help of Leray–Schauder and Banach fixed point theorem and then investigated a kind of stability of solution named Ulam–Hyers for the system [6].

By taking into account aforementioned new operators introduced by Jarad et al. [35] and inspired by some existing primitive ideas in above articles, in the current manuscript for the first time, we formulate the following coupled system of the Caputo conformable fractional version of the pantograph differential equations as follows:

$$\begin{cases} {}^{\mathcal{CC}}\mathfrak{D}_{z_0}^{\sigma,\eta_1^*}\varpi(z)=\tilde{\mathcal{O}}_1(z,\varrho(z),\varrho(\lambda^* z)), \qquad (z\in[z_0,\tilde{T}],\ z_0\geq 0)\\ {}^{\mathcal{CC}}\mathfrak{D}_{z_0}^{\sigma,\eta_2^*}\varrho(z)=\tilde{\mathcal{O}}_2(z,\varpi(z),\varpi(\lambda^* z)) \end{cases} \tag{1.1}$$

subject to three-point Riemann–Liouville conformable integral conditions

$$\begin{cases} \varpi(z_0)=0, \qquad \mu_1^*\varpi(\tilde{T})+\mu_2^*{}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\theta^*}\varpi(\delta)=\xi_1^*,\\ \varrho(z_0)=0, \qquad \gamma_1^*\varrho(\tilde{T})+\gamma_2^*{}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\theta^*}\varrho(\nu)=\xi_2^* \end{cases} \tag{1.2}$$

so that ${}^{\mathcal{CC}}\mathfrak{D}_{z_0}^{\sigma,\eta_j^*}$ stands for the Caputo conformable derivatives of fractional order $\eta_j^*\in(1,2)$ with $\sigma\in(0,1]$ for $j=1,2$ and also ${}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\theta^*}$ illustrates the Riemann–Liouville conformable integral of fractional order $\theta^*>0$.

Furthermore, $\delta, \nu \in (z_0, \tilde{T})$, $\mu_1^*, \mu_2^*, \gamma_1^*, \gamma_2^*, \xi_1^*, \xi_2^* \in \mathbb{R}$, $\lambda^* \in (0,1)$ and $\tilde{\mathcal{O}}_j : [z_0, \tilde{T}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are supposed to be continuous maps for $j = 1, 2$.

It is notable that until now, limited researchers have been worked on the newly formulated Caputo conformable and the Riemann–Liouville conformable operators. It is necessary that researchers pay attention to this fact that this structure of a coupled system of pantograph boundary problem with respect to the Caputo conformable operators is a new version and this type of construction has not been studied in any literature yet. In fact, we extend the well-known pantograph differential equation to a coupled system by utilizing newly defined conformable operators in both Caputo and Riemann–Liouville settings simultaneously for the first time. The boundary integral conditions given in our coupled system involve a vast range of many simple boundary conditions defined before by other researchers.

We demonstrate the contents of the current research manuscript as follows. In Section 2, we briefly review some fundamental and auxiliary concepts and notions. Section 3 has two folds. First, we employ some well-known analytical techniques to establish existence and uniqueness criteria corresponding to the given pantograph coupled system problem (1.1)–(1.2). In the following of this section, some estimates for the solutions of the mentioned pantograph coupled system are investigated. The last part of the present research is devoted to propose a numerical example to demonstrates the applicability of our findings.

2. Preliminaries

In this moment, we are going to assemble some fundamental and basic concepts. It is well-known that the Riemann–Liouville integral for a function $\varpi : [0, +\infty) \to \mathbb{R}$ of order $\eta^* > 0$ is illustrated by

$$ {}^{\mathcal{R}}\mathcal{I}_0^{\eta^*} \varpi(z) = \int_0^z \frac{(z-q)^{\eta^*-1}}{\Gamma(\eta^*)} \varpi(q)\, \mathrm{d}q \tag{2.1} $$

so that the existing integral is finite valued ([45], [49]). Now we assume that $\eta^* \in (n-1, n)$ so that $n = [\eta^*] + 1$. About the given function $\varpi \in \mathcal{AC}_{\mathbb{R}}^{(n)}([0, +\infty))$, the fractional Caputo derivative of order η^* is formulated by

$$ {}^{\mathcal{C}}\mathfrak{D}_0^{\eta^*} \varpi(z) = \int_0^z \frac{(z-q)^{n-\eta^*-1}}{\Gamma(n-\eta^*)} \varpi^{(n)}(q)\, \mathrm{d}q \tag{2.2} $$

such that the existing integral has the finite value [45, 49]. In this place, we can define the left conformable derivative of a function $\varpi : [z_0, \infty) \to \mathbb{R}$ with $\sigma \in (0, 1]$ at the initial point z_0 by the following relation

$$ \mathfrak{D}_{z_0}^{\upsilon} \varpi(z) = \lim_{\upsilon \to 0} \frac{\varpi(z + \upsilon(z - z_0)^{1-\sigma}) - \varpi(z)}{\upsilon} $$

such that the limit exists [39]. It is necessary that you pay attention this interesting fact that an equality $\mathfrak{D}_{z_0}^{\sigma} \varpi(z_0) = \lim_{z \to z_0^+} \mathfrak{D}_{z_0}^{\sigma} \varpi(z)$ holds whenever $\mathfrak{D}_{z_0}^{\sigma} \varpi(z)$ exists on the interval (z_0, r) for $0 \leq z_0 < r$. Also, clearly the relation $\mathfrak{D}_{z_0}^{\sigma} \varpi(z) = (z - z_0)^{1-\sigma} \varpi'(z)$ is valid when ϖ is a differentiable function. In this way, one can indicate the definition of the left conformable integral about the function ϖ with $\sigma \in (0, 1]$ of the following form $\mathcal{I}_{z_0}^{\sigma} \varpi(z) = \displaystyle\int_{z_0}^z \varpi(q) \frac{\mathrm{d}q}{(q - z_0)^{1-\sigma}}$ when the existing integral is finite valued [39]. In the sequel, Jarad et al. [35] extended aforementioned conformable operators to arbitrary real orders in both the Caputo and the Riemann–Liouville settings as follows: Let $\eta^* \in \mathbb{C}$ be such that the real part of η^* is nonnegative. In this case,

the Riemann–Liouville conformable integral about a given function ϖ of order η^* with $\sigma \in (0,1]$ is formulated by

$$ {}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta^*}\varpi(z) = \frac{1}{\Gamma(\eta^*)}\int_{z_0}^{z}\left(\frac{(z-z_0)^\sigma-(q-z_0)^\sigma}{\sigma}\right)^{\eta^*-1}\varpi(q)\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}} \tag{2.3}$$

whenever the value of integral is finite ([35]). Notice that if we take $z_0 = 0$ and $\sigma = 1$, then ${}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta^*}\varpi(z)$ defined by (2.3) is transformed into the standard fractional integral ${}^{\mathcal{R}}\mathcal{I}_0^{\eta^*}\varpi(z)$ given by (2.1). In the similar manner, the Riemann–Liouville conformable derivative for a given function ϖ of order η^* with $\sigma \in (0,1]$ is introduced by

$$\begin{aligned} {}^{\mathcal{RC}}\mathfrak{D}_{z_0}^{\sigma,\eta^*}\varpi(z) &= \mathfrak{D}_{z_0}^{\sigma,n}\big({}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,n-\eta^*}\varpi\big)(z) \\ &= \frac{\mathfrak{D}_{z_0}^{\sigma,n}}{\Gamma(n-\eta^*)}\int_{z_0}^{z}\left(\frac{(z-z_0)^\sigma-(q-z_0)^\sigma}{\sigma}\right)^{n-\eta^*-1}\varpi(q)\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}} \end{aligned} \tag{2.4}$$

such that $n = [Re(\eta^*)]+1$ and $\mathfrak{D}_{z_0}^{\sigma,n} = \overbrace{\mathfrak{D}_{z_0}^{\sigma}\,\mathfrak{D}_{z_0}^{\sigma}\,\ldots\,\mathfrak{D}_{z_0}^{\sigma}}^{n \text{ times}}$ where $\mathfrak{D}_{z_0}^{\sigma}$ stands for the left conformable derivative with $\sigma \in (0,1]$ [35]. Again, it is notable that if we take $z_0 = 0$ and $\sigma = 1$, then ${}^{\mathcal{RC}}\mathfrak{D}_{z_0}^{\sigma,\eta^*}\varpi(z)$ defined by (2.4) is transformed into the standard Riemann-Liouville derivative ${}^{\mathcal{R}}\mathfrak{D}_0^{\eta^*}w(s)$. In the next step, to introduce the generalized notion of the conformable derivative in the Caputo settings, we construct

$$\mathcal{L}_\sigma(z_0) := \big\{\varphi : [z_0,r] \to \mathbb{R} : \quad \mathcal{I}_{z_0}^{\sigma}\varphi(z) \text{ exists for each } z \in [z_0,r]\,\big\}$$

for $\sigma \in (0,1]$ and we set

$$\mathbb{I}_\sigma([z_0,r]) := \big\{\varpi : [z_0,r] \to \mathbb{R} : \quad \varpi(z) = \mathcal{I}_{z_0}^{\sigma}\varphi(z) + \varpi(z_0), \text{ for some } \varphi \in \mathcal{L}_\sigma(z_0)\big\},$$

where $\mathcal{I}_{z_0}^{\sigma}\varphi(z)$ indicates the same left conformable integral of the given function φ at z_0 ([1]). Also, we make $\mathcal{C}_{z_0,\sigma}^{n}([z_0,r]) := \big\{\varpi : [z_0,r] \to \mathbb{R} : \quad \mathfrak{D}_{z_0}^{\sigma,n-1}\varpi \in \mathbb{I}_\sigma([z_0,r])\big\}$ for $n = 1,2,3,\ldots$. In consequence, the Caputo conformable derivative for a given function $\varpi \in \mathcal{C}_{z_0,\sigma}^{n}([z_0,r])$ of order η^* with $\sigma \in (0,1]$ is formulated by

$$\begin{aligned} {}^{\mathcal{C}\mathcal{C}}\mathfrak{D}_{z_0}^{\sigma,\eta^*}\varpi(z) &= {}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,n-\eta^*}\big(\mathfrak{D}_{z_0}^{\sigma,n}\varpi\big)(z) \\ &= \frac{1}{\Gamma(n-\eta^*)}\int_{z_0}^{z}\left(\frac{(z-z_0)^\sigma-(q-z_0)^\sigma}{\sigma}\right)^{n-\eta^*-1}\mathfrak{D}_{z_0}^{\sigma,n}\varpi(q)\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}} \end{aligned} \tag{2.5}$$

such that $n = [Re(\eta^*)]+1$ ([35]). It is known that the operator ${}^{\mathcal{C}\mathcal{C}}\mathfrak{D}_{z_0}^{\sigma,\eta^*}\varpi(z)$ given by (2.5) equals to ${}^{\mathcal{C}}\mathfrak{D}_0^{\eta^*}\varpi(z)$ given by (2.2) if we take $z_0 = 0$ and $\sigma = 1$. In some lemmas, we assemble several applied properties of both Caputo– and Riemann–Liouville fractional conformable operators which you can find it in the following.

Lemma 2.1 *([17, 35]) Let $Re(\eta^*) > 0$, $Re(\theta^*) > 0$ and $Re(\rho^*) > 0$. Then for $\sigma \in (0,1]$ and for any $z > z_0$, we have the following statements:*

(L1) ${}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta^*}\big({}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\theta^*}\varpi\big)(z) = \big({}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta^*+\theta^*}\varpi\big)(z)$,

(L2) $${}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta^*}(z-z_0)^{\sigma(\rho^*-1)}(s)=\frac{1}{\sigma^{\eta^*}}\frac{\Gamma(\rho^*)}{\Gamma(\rho^*+\eta^*)}(s-z_0)^{\sigma(\rho^*+\eta^*-1)},$$

(L3) $${}^{\mathcal{RC}}\mathfrak{D}_{z_0}^{\sigma,\eta^*}(z-z_0)^{\sigma(\rho^*-1)}(s)=\sigma^{\eta^*}\frac{\Gamma(\rho^*)}{\Gamma(\rho^*-\eta^*)}(s-z_0)^{\sigma(\rho^*-\eta^*-1)},$$

(L4) $${}^{\mathcal{RC}}\mathfrak{D}_{z_0}^{\sigma,\eta^*}\big({}^{RC}\mathcal{I}_{z_0}^{\sigma,\theta^*}\varpi\big)(z)=\big({}^{RC}\mathcal{I}_{z_0}^{\sigma,\theta^*-\eta^*}\varpi\big)(z),\qquad \big(Re(\eta^*)<Re(\theta^*)\big).$$

Lemma 2.2 *([17, 35]) Let $n-1<Re(\eta^*)<n$ and $\varpi\in\mathcal{C}_{z_0,\sigma}^{n}([z_0,r])$. Then for $\sigma\in(0,1]$, the following identity holds*

$$ {}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta^*}\big({}^{\mathcal{CC}}\mathfrak{D}_{z_0}^{\sigma,\eta^*}\varpi\big)(z)=\varpi(z)-\sum_{l=0}^{n-1}\frac{\mathfrak{D}_{z_0}^{\sigma,l}\varpi(z_0)}{\sigma^l l!}(z-z_0)^{l\sigma}. \tag{2.6}$$

In the light of the identity (2.6), it is concluded that the series solution of the homogeneous equation $({}^{\mathcal{CC}}\mathfrak{D}_{z_0}^{\sigma,\eta^*}\varpi)(z)=0$ is obtained by

$$\varpi(z)=\sum_{l=0}^{n-1}b_l^*(z-z_0)^{l\sigma}=b_0^*+b_1^*(z-z_0)^{\sigma}+b_2^*(z-z_0)^{2\sigma}+\cdots+b_{n-1}^*(z-z_0)^{(n-1)\sigma},$$

so that $n-1<Re(\eta^*)<n$ and $b_0^*,b_1^*,\ldots,b_{n-1}^*\in\mathbb{R}$. We utilize both following theorems to conclude our main results.

Theorem 2.3 *[32] (Leray–Schauder degree theorem) The space $\mathfrak{M}$ is supposed to be a Banach space. In addition, let $\mathbb{W}$ be an open bounded subset of $\mathfrak{M}$ with $0\in\mathbb{W}$ and $\mathfrak{P}:\overline{\mathbb{W}}\to\mathfrak{M}$ be a an operator having the complete continuity property. Also, let $\alpha\mathfrak{P}\varpi-\varpi\neq 0$ for all $\varpi\in\partial\mathbb{W}$ and for any $\alpha\in[0,1]$. Then $\mathfrak{deg}(I-\alpha\mathfrak{P},\mathbb{W},0)=1$ and $\mathfrak{P}$ has a fixed point in $\overline{\mathbb{W}}$.*

3. Main results

After assembling required auxiliary notions in the previous section, we are ready to establish the desired existence theorems. For this reason, we regard the Banach space $\mathfrak{M}=\{\varpi:\ \varpi\in\mathcal{C}_{\mathbb{R}}[z_0,\tilde{T}]\}$ furnished with the norm $\|\varpi\|_{\mathfrak{M}}=\sup_{z\in[z_0,\tilde{T}]}|\varpi(z)|$. One can deduce that the product space $\mathfrak{M}\times\mathfrak{M}$ supplemented with the norm $\|(\varpi,\varrho)\|_{\mathfrak{M}\times\mathfrak{M}}=\|\varpi\|_{\mathfrak{M}}+\|\varrho\|_{\mathfrak{M}}$ for $(\varpi,\varrho)\in\mathfrak{M}\times\mathfrak{M}$ is a Banach space. Further, for the sake of simplicity in writing, we consider the following constants:

$$\tilde{\Omega}_{1*}=\mu_1^*(\tilde{T}-z_0)^{\sigma}+\mu_2^*\frac{(\delta-z_0)^{\sigma(1+\theta^*)}}{\sigma^{\theta^*}\Gamma(2+\theta^*)}\neq 0,$$

$$\tilde{\Omega}_{2*}=\gamma_1^*(\tilde{T}-z_0)^{\sigma}+\gamma_2^*\frac{(\nu-z_0)^{\sigma(1+\theta^*)}}{\sigma^{\theta^*}\Gamma(2+\theta^*)}\neq 0. \tag{3.1}$$

In the next lemma, we derive an equivalent integral structure for the solution of the three-point Caputo conformable pantograph boundary problem given in the coupled system (1.1)–(1.2).

Lemma 3.1 *Let $\breve{h}_* \in \mathfrak{M}$ be an arbitrary function. Then the function ϖ_0^* satisfies the given Caputo conformable differential equation*

$$ {}^{\mathcal{CC}}\mathfrak{D}_{z_0}^{\sigma,\eta_1^*}\varpi(z) = \breve{h}_*(z), \qquad (z \in [z_0, \tilde{T}],\ z_0 \geq 0) \tag{3.2}$$

furnished with three-point Riemann–Liouville conformable integral boundary conditions

$$\varpi(z_0) = 0, \qquad \mu_1^*\varpi(\tilde{T}) + \mu_2^{*\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\theta^*}\varpi(\delta) = \xi_1^*, \tag{3.3}$$

if and only if ϖ_0^* is a solution function for the Riemann–Liouville conformable integral equation

$$\begin{aligned}\varpi(z) = {} & \frac{1}{\Gamma(\eta_1^*)}\int_{z_0}^{z}\left(\frac{(z-z_0)^\sigma - (q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*-1}\breve{h}_*(q)\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}} \\ & + \frac{(z-z_0)^\sigma}{\tilde{\Omega}_{1*}}\left[\xi_1^* - \frac{\mu_1^*}{\Gamma(\eta_1^*)}\int_{z_0}^{\tilde{T}}\left(\frac{(\tilde{T}-z_0)^\sigma - (q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*-1}\breve{h}_*(q)\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}}\right. \\ & \left. - \frac{\mu_2^*}{\Gamma(\eta_1^*+\theta^*)}\int_{z_0}^{\delta}\left(\frac{(\delta-z_0)^\sigma - (q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*+\theta^*-1}\breve{h}_*(q)\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}}\right],\end{aligned} \tag{3.4}$$

so that a nonzero constant $\tilde{\Omega}_{1*}$ is illustrated by (3.1).

Proof In the first stage, the function ϖ_0^* is supposed to be satisfied the Caputo conformable equation (3.2). Simply, we see that ${}^{\mathcal{CC}}\mathfrak{D}_{z_0}^{\sigma,\eta_1^*}\varpi_0^*(z) = \breve{h}_*(z)$. Now, we integrate the last equality in the Riemann–Liouville conformable settings of order η_1^* and so we obtain

$$\varpi_0^*(z) = \frac{1}{\Gamma(\eta_1^*)}\int_{z_0}^{z}\left(\frac{(z-z_0)^\sigma - (q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*-1}\breve{h}_*(q)\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}} + b_0^* + b_1^*(z-z_0)^\sigma, \tag{3.5}$$

where we intend to find two unknown constants $b_0^*, b_1^* \in \mathbb{R}$. On the other hand, we integrate from both sides of Equation (3.5) in the Riemann–Liouville conformable settings of order θ^* with respect to z. Then we get

$$\begin{aligned}{}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\theta^*}\varpi_0^*(z) = {} & \frac{1}{\Gamma(\eta_1^*+\theta^*)}\int_{z_0}^{z}\left(\frac{(z-z_0)^\sigma - (q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*+\theta^*-1}\breve{h}_*(q)\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}} \\ & + b_0^*\frac{(z-z_0)^{\sigma\theta^*}}{\sigma^{\theta^*}\Gamma(1+\theta^*)} + b_1^*\frac{(z-z_0)^{\sigma(1+\theta^*)}}{\sigma^{\theta^*}\Gamma(2+\theta^*)}.\end{aligned}$$

From the first boundary condition, we find that $b_0^* = 0$. Later, in the light of the second integral boundary condition, we reach

$$\begin{aligned}b_1^* = {} & \frac{1}{\tilde{\Omega}_{1*}}\left[\xi_1^* - \frac{\mu_1^*}{\Gamma(\eta_1^*)}\int_{z_0}^{\tilde{T}}\left(\frac{(\tilde{T}-z_0)^\sigma - (q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*-1}\breve{h}_*(q)\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}}\right. \\ & \left. - \frac{\mu_2^*}{\Gamma(\eta_1^*+\theta^*)}\int_{z_0}^{\delta}\left(\frac{(\delta-z_0)^\sigma - (q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*+\theta^*-1}\breve{h}_*(q)\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}}\right].\end{aligned}$$

In this position, we insert both obtained values b_0^* and b_1^* into Equation (3.5) and hence we get

$$\varpi_0^*(z) = \frac{1}{\Gamma(\eta_1^*)}\int_{z_0}^{z}\left(\frac{(z-z_0)^\sigma-(q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*-1}\breve{h}_*(q)\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}}$$

$$+\frac{(z-z_0)^\sigma}{\tilde{\Omega}_{1*}}\left[\xi_1^*-\frac{\mu_1^*}{\Gamma(\eta_1^*)}\int_{z_0}^{\tilde{T}}\left(\frac{(\tilde{T}-z_0)^\sigma-(q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*-1}\breve{h}_*(q)\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}}\right.$$

$$\left.-\frac{\mu_2^*}{\Gamma(\eta_1^*+\theta^*)}\int_{z_0}^{\delta}\left(\frac{(\delta-z_0)^\sigma-(q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*+\theta^*-1}\breve{h}_*(q)\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}}\right]$$

which demonstrates that ϖ_0^* is a solution for the Riemann–Liouville conformable integral equation (3.4). On contrary, we can simply conclude that ϖ_0^* is a solution for given three-point Caputo conformable boundary problem (3.2)–(3.3) whenever ϖ_0^* is regarded as a solution for the Riemann–Liouville conformable integral equation (3.4) and this ends the proof. □

In the light of Lemma 3.1, we can define a new operator $\mathfrak{P}^*:\mathfrak{M}\times\mathfrak{M}\to\mathfrak{M}\times\mathfrak{M}$ by

$$\mathfrak{P}^*(\varpi,\varrho)(z) = (\mathfrak{P}_1^*\varpi(z),\mathfrak{P}_2^*\varrho(z)), \tag{3.6}$$

where two operators $\mathfrak{P}_1^*:\mathfrak{M}\to\mathfrak{M}$ and $\mathfrak{P}_2^*:\mathfrak{M}\to\mathfrak{M}$ are given by follows:

$$\mathfrak{P}_1^*\varpi(z) = \frac{1}{\Gamma(\eta_1^*)}\int_{z_0}^{z}\left(\frac{(z-z_0)^\sigma-(q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*-1}\tilde{\mathcal{O}}_1(q,\varrho(q),\varrho(\lambda^*q))\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}} \tag{3.7}$$

$$+\frac{(z-z_0)^\sigma}{\tilde{\Omega}_{1*}}\left[\xi_1^*-\frac{\mu_1^*}{\Gamma(\eta_1^*)}\int_{z_0}^{\tilde{T}}\left(\frac{(\tilde{T}-z_0)^\sigma-(q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*-1}\tilde{\mathcal{O}}_1(q,\varrho(q),\varrho(\lambda^*q))\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}}\right.$$

$$\left.-\frac{\mu_2^*}{\Gamma(\eta_1^*+\theta^*)}\int_{z_0}^{\delta}\left(\frac{(\delta-z_0)^\sigma-(q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*+\theta^*-1}\tilde{\mathcal{O}}_1(q,\varrho(q),\varrho(\lambda^*q))\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}}\right],$$

and

$$\mathfrak{P}_2^*\varrho(z) = \frac{1}{\Gamma(\eta_2^*)}\int_{z_0}^{z}\left(\frac{(z-z_0)^\sigma-(q-z_0)^\sigma}{\sigma}\right)^{\eta_2^*-1}\tilde{\mathcal{O}}_2(q,\varpi(q),\varpi(\lambda^*q))\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}} \tag{3.8}$$

$$+\frac{(z-z_0)^\sigma}{\tilde{\Omega}_{2*}}\left[\xi_2^*-\frac{\gamma_1^*}{\Gamma(\eta_2^*)}\int_{z_0}^{\tilde{T}}\left(\frac{(\tilde{T}-z_0)^\sigma-(q-z_0)^\sigma}{\sigma}\right)^{\eta_2^*-1}\tilde{\mathcal{O}}_2(q,\varpi(q),\varpi(\lambda^*q))\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}}\right.$$

$$\left.-\frac{\gamma_2^*}{\Gamma(\eta_2^*+\theta^*)}\int_{z_0}^{\nu}\left(\frac{(\nu-z_0)^\sigma-(q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*+\theta^*-1}\tilde{\mathcal{O}}_2(q,\varpi(q),\varpi(\lambda^*q))\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}}\right],$$

for each $z\in[z_0,\tilde{T}]$ and the nonzero constants $\tilde{\Omega}_{1*}$ and $\tilde{\Omega}_{2*}$ are illustrated by (3.1).

In the subsequent, we shall discuss the existence criteria for the coupled system of the Caputo conformable fractional pantograph boundary problems (1.1)–(1.2). It is an evident fact that the fixed point of the operator $\mathfrak{P}^*$ is considered the same solution for such coupled system. In the sequel, we utilize the following notations:

$$\Delta_1 = \frac{(\tilde{T}-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)} + \frac{(\tilde{T}-z_0)^{\sigma}}{|\tilde{\Omega}_{1*}|}\left[\frac{|\mu_1^*|(\tilde{T}-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)} + \frac{|\mu_2^*|(\delta-z_0)^{\sigma(\eta_1^*+\theta^*)}}{\sigma^{(\eta_1^*+\theta^*)}\Gamma(\eta_1^*+\theta^*+1)}\right], \tag{3.9}$$

and

$$\Delta_2 = \frac{(\tilde{T}-z_0)^{\sigma\eta_2^*}}{\sigma^{\eta_2^*}\Gamma(\eta_2^*+1)} + \frac{(\tilde{T}-z_0)^{\sigma}}{|\tilde{\Omega}_{2*}|}\left[\frac{|\gamma_1^*|(\tilde{T}-z_0)^{\sigma\eta_2^*}}{\sigma^{\eta_2^*}\Gamma(\eta_2^*+1)} + \frac{|\gamma_2^*|(\nu-z_0)^{\sigma(\eta_2^*+\theta^*)}}{\sigma^{(\eta_2^*+\theta^*)}\Gamma(\eta_2^*+\theta^*+1)}\right]. \tag{3.10}$$

The first existence criterion demonstrated below is deduced by means of the well-known result attributed to Banach.

Theorem 3.2 *Let $\tilde{\mathcal{O}}_1, \tilde{\mathcal{O}}_2 : [z_0,\tilde{T}]\times\mathfrak{M}\times\mathfrak{M}\to\mathfrak{M}$ be two continuous functions. Moreover, assume that*

(PH1) *The real constants $m_1, m_2 > 0$ exists so that*

$$|\tilde{\mathcal{O}}_j(z,u,v) - \tilde{\mathcal{O}}_j(z,\tilde{u},\tilde{v})| \le m_j\big(|u-\tilde{u}| + |v-\tilde{v}|\big)$$

for any $z\in[z_0,\tilde{T}]$ and $u,v,\tilde{u},\tilde{v}\in\mathfrak{M}$ and $j=1,2$.

Then the coupled system of the Caputo conformable pantograph fractional boundary problems (1.1)–(1.2) has a solution on $[z_0,\tilde{T}]$ uniquely, whenever

$$2m\Delta < 1 \quad \text{such that} \quad m=\max\{m_1,m_2\} \quad \text{and} \quad \Delta=\max\{\Delta_1,\Delta_2\}$$

and also $\mathcal{M}_j = \sup_{z\in[z_0,\tilde{T}]}|\tilde{\mathcal{O}}_j(z,0,0)|, (j=1,2)$ and the constants Δ_1,Δ_2 are illustrated by (3.9) and (3.10).

Proof We regard the operator $\mathfrak{P}^*:\mathfrak{M}\times\mathfrak{M}\to\mathfrak{M}\times\mathfrak{M}$ by $\mathfrak{P}^*(\varpi,\varrho) = (\mathfrak{P}_1^*\varpi, \mathfrak{P}_2^*\varrho)$ so that both operators $\mathfrak{P}_1^*:\mathfrak{M}\to\mathfrak{M}$ and $\mathfrak{P}_2^*:\mathfrak{M}\to\mathfrak{M}$ are formulated by (3.7) and (3.8). In addition to this, we regard an open bounded ball

$$\mathbb{W}_\epsilon = \{(\varpi,\varrho)\in\mathfrak{M}\times\mathfrak{M}: \quad \|(\varpi(z),\varrho(z))\|_{\mathfrak{M}\times\mathfrak{M}} < \epsilon,\ z\in[z_0,\tilde{T}]\},$$

where

$$\epsilon \ge \frac{1}{(1-2m\Delta)}\Big[\Delta(\mathcal{M}_1+\mathcal{M}_2) + \frac{|\xi_1^*|(\tilde{T}-z_0)^{\sigma}}{|\tilde{\Omega}_{1*}|} + \frac{|\xi_2^*|(\tilde{T}-z_0)^{\sigma}}{|\tilde{\Omega}_{2*}|}\Big]. \tag{3.11}$$

Now, we verify that $\mathfrak{P}^*(\mathbb{W}_\epsilon)\subset\mathbb{W}_\epsilon$, by using hypothesis (PH1). For $(\varpi,\varrho)\in\mathbb{W}_\epsilon$ and $z\in[z_0,\tilde{T}]$, we get

$$\begin{aligned}|\tilde{\mathcal{O}}_1(z,\varrho(z),\varrho(\lambda^*z))| &\le |\tilde{\mathcal{O}}_1(z,\varrho(z),\varrho(\lambda^*z)) - \tilde{\mathcal{O}}_1(z,0,0)| + |\tilde{\mathcal{O}}_1(z,0,0)|\\ &\le m_1\left(|\varrho(z)| + |\varrho(\lambda^*z)|\right) + \mathcal{M}_1\\ &\le 2m_1\|\varrho\|_{\mathfrak{M}} + \mathcal{M}_1,\end{aligned}$$

and similarly $|\tilde{\mathcal{O}}_2(z,\varpi(z),\varpi(\lambda^* z))| \leq 2m_2\|\varpi\|_{\mathfrak{M}} + \mathcal{M}_2$, which implies

$$|\mathfrak{P}_1^*\varpi(z)| \leq \frac{1}{\Gamma(\eta_1^*)}\int_{z_0}^{z}\left(\frac{(z-z_0)^\sigma-(q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*-1}|\tilde{\mathcal{O}}_1(q,\varrho(q),\varrho(\lambda^* q))|\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}}$$

$$+\frac{(z-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}\left[|\xi_1^*|+\frac{|\mu_1^*|}{\Gamma(\eta_1^*)}\int_{z_0}^{\tilde{T}}\left(\frac{(\tilde{T}-z_0)^\sigma-(q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*-1}|\tilde{\mathcal{O}}_1(q,\varrho(q),\varrho(\lambda^* q))|\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}}\right.$$

$$\left.+\frac{|\mu_2^*|}{\Gamma(\eta_1^*+\theta^*)}\int_{z_0}^{\delta}\left(\frac{(\delta-z_0)^\sigma-(q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*+\theta^*-1}|\tilde{\mathcal{O}}_1(q,\varrho(q),\varrho(\lambda^* q))|\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}}\right]$$

$$\leq \frac{(2m_1\|\varrho\|_{\mathfrak{M}}+\mathcal{M}_1)\,(\tilde{T}-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)}+\frac{(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}\,(2m_1\|\varrho\|_{\mathfrak{M}}+\mathcal{M}_1)$$

$$\times\left[\frac{|\mu_1^*|(\tilde{T}-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)}+\frac{|\mu_2^*|(\delta-z_0)^{\sigma(\eta_1^*+\theta^*)}}{\sigma^{(\eta_1^*+\theta^*)}\Gamma(\eta_1^*+\theta^*+1)}\right]+\frac{|\xi_1^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}$$

$$\leq \Delta_1\,(2m_1\|\varrho\|_{\mathfrak{M}}+\mathcal{M}_1)+\frac{|\xi_1^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}.$$

Above inequalities yield that

$$\|\mathfrak{P}_1^*\varpi\|_{\mathfrak{M}} \leq \Delta_1\,(2m_1\|\varrho\|_{\mathfrak{M}}+\mathcal{M}_1)+\frac{|\xi_1^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|},$$

where Δ_1 is illustrated by (3.9). In the similar manner, we have an inequality

$$\|\mathfrak{P}_2^*\varrho\|_{\mathfrak{M}} \leq \Delta_2\,(2m_2\|\varpi\|_{\mathfrak{M}}+\mathcal{M}_2)+\frac{|\xi_2^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{2*}|}$$

so that Δ_2 is illustrated by (3.10). Hence, in view of the above inequalities we realize that $\mathfrak{P}^*(\mathbb{W}_\epsilon)\subset\mathbb{W}_\epsilon$, since we have

$$\|\mathfrak{P}^*(\varpi,\varrho)\|_{\mathfrak{M}\times\mathfrak{M}} \leq \|\mathfrak{P}_1^*\varpi\|_{\mathfrak{M}}+\|\mathfrak{P}_2^*\varrho\|_{\mathfrak{M}}$$

$$\leq \Delta_1\,(2m_1\|\varrho\|_{\mathfrak{M}}+\mathcal{M}_1)+\frac{|\xi_1^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}+\Delta_2\,(2m_2\|\varpi\|_{\mathfrak{M}}+\mathcal{M}_2)+\frac{|\xi_2^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{2*}|}$$

$$\leq 2m\Delta\|(\varpi,\varrho)\|_{\mathfrak{M}\times\mathfrak{M}}+\Delta(\mathcal{M}_1+\mathcal{M}_2)+\frac{|\xi_1^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}+\frac{|\xi_2^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{2*}|}$$

$$\leq 2m\Delta\,\epsilon+\Delta(\mathcal{M}_1+\mathcal{M}_2)+\frac{|\xi_1^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}+\frac{|\xi_2^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{2*}|} < \epsilon.$$

Next, to verify this fact that $\mathfrak{P}^*$ is a contraction, for $\varpi, \tilde{\varpi} \in \mathfrak{M}$ and $z \in [z_0, \tilde{T}]$, we have

$$\begin{aligned}
\left|\mathfrak{P}_1^*\varpi(z) - \mathfrak{P}_1\tilde{\varpi}(z)\right| &\leq {}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta_1^*}\left|\tilde{\mathcal{O}}_1(z, \varrho(z), \varrho(\lambda^* z)) - \tilde{\mathcal{O}}_1(z, \tilde{\varrho}(z), \tilde{\varrho}(\lambda^* z))\right| \\
&\quad + \frac{(z-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}\Bigg[|\mu_1^*|^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta_1^*}\left|\tilde{\mathcal{O}}_1(\tilde{T}, \varrho(\tilde{T}), \varrho(\lambda^*\tilde{T})) - \tilde{\mathcal{O}}_1(\tilde{T}, \tilde{\varrho}(\tilde{T}), \tilde{\varrho}(\lambda^*\tilde{T}))\right| \\
&\quad + |\mu_2^*|^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,(\eta_1^*+\theta^*)}\left|\tilde{\mathcal{O}}_1(\delta, \varrho(\delta), \varrho(\lambda^*\delta)) - \tilde{\mathcal{O}}_1(\delta, \tilde{\varrho}(\delta), \tilde{\varrho}(\lambda^*\delta))\right|\Bigg] \\
&\leq \frac{2m_1\left\|\varrho - \tilde{\varrho}\right\|_{\mathfrak{M}}(\tilde{T}-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)} + \frac{(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}\Bigg[\frac{2m_1|\mu_1^*|\left\|\varrho - \tilde{\varrho}\right\|_{\mathfrak{M}}(\tilde{T}-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)} \\
&\quad + \frac{2m_1|\mu_2^*|\left\|\varrho - \tilde{\varrho}\right\|_{\mathfrak{M}}(\delta-z_0)^{\sigma(\eta_1^*+\theta^*)}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+\theta^*+1)}\Bigg] \\
&\leq 2m_1\Delta_1\left\|\varrho - \tilde{\varrho}\right\|_{\mathfrak{M}}.
\end{aligned}$$

Hence,

$$\left\|\mathfrak{P}_1^*\varpi - \mathfrak{P}_1^*\tilde{\varpi}\right\|_{\mathfrak{M}} \leq 2m_1\Delta_1\left\|\varrho - \tilde{\varrho}\right\|_{\mathfrak{M}}.$$

In the same way, we obtain

$$\left\|\mathfrak{P}_2^*\varrho - \mathfrak{P}_2^*\tilde{\varrho}\right\|_{\mathfrak{M}} \leq 2m_2\Delta_2\left\|\varpi - \tilde{\varpi}\right\|_{\mathfrak{M}}.$$

Consequently,

$$\begin{aligned}
\|\mathfrak{P}^*(\varpi, \varrho) - \mathfrak{P}^*(\tilde{\varpi}, \tilde{\varrho})\|_{\mathfrak{M}\times\mathfrak{M}} &\leq 2m_1\Delta_1\left\|\varrho - \tilde{\varrho}\right\|_{\mathfrak{M}} + 2m_2\Delta_2\left\|\varpi - \tilde{\varpi}\right\|_{\mathfrak{M}} \\
&\leq 2m\Delta\left(\left\|\varrho - \tilde{\varrho}\right\|_{\mathfrak{M}} + \left\|\varpi - \tilde{\varpi}\right\|_{\mathfrak{M}}\right).
\end{aligned}$$

Since $2m\Delta < 1$, so $\mathfrak{P}^*$ is a contraction. Thus, by virtue of the Banach's principle, the operator $\mathfrak{P}^*$ has a fixed point uniquely which is the unique solution of the coupled system of the Caputo conformable pantograph fractional boundary problems (1.1)–(1.2) and the proof is finished. □

Lemma 3.3 *Let $\tilde{\mathcal{O}}_1, \tilde{\mathcal{O}}_2 : [z_0, \tilde{T}] \times \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$ be two continuous functions. Moreover, assume that*

(PH2) *there exist constants $\mathcal{N}_1, \mathcal{N}_2 > 0$ so that $|\tilde{\mathcal{O}}_1(z, \varrho, \tilde{\varrho})| \leq \mathcal{N}_1$ and $|\tilde{\mathcal{O}}_2(z, \varpi, \tilde{\varpi})| \leq \mathcal{N}_2$ for any $z \in [z_0, \tilde{T}]$ and each $\varpi, \tilde{\varpi}, \varrho, \tilde{\varrho} \in \mathfrak{M}$.*

Then the operator $\mathfrak{P}^* : \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M} \times \mathfrak{M}$ illustrated in (3.6) has the complete continuity property.

Proof Consider $\mathfrak{P}^* : \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M} \times \mathfrak{M}$ formulated by (3.6). At first, with due attention to the continuity of the functions $\tilde{\mathcal{O}}_1$ and $\tilde{\mathcal{O}}_2$ and by considering the Lebesgue's dominated convergence theorem, we immediately

find that $\mathfrak{P}^*$ has the continuity property. Now, let $\mathbb{W} \subset \mathfrak{M} \times \mathfrak{M}$ be a bounded subset of the mentioned product space. Then, for each $z \in [z_0, \tilde{T}]$ and $(\varpi(z), \varrho(z)) \in \mathbb{W}$ and in view of hypothesis (PH2), we get

$$
\begin{aligned}
|\mathfrak{P}_1^*\varpi(z)| &\leq \frac{1}{\Gamma(\eta_1^*)} \int_{z_0}^{z} \left(\frac{(z-z_0)^\sigma - (q-z_0)^\sigma}{\sigma} \right)^{\eta_1^*-1} |\tilde{\mathcal{O}}_1(q, \varrho(q), \varrho(\lambda^* q))| \frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}} \\
&+ \frac{(z-z_0)^\sigma}{|\tilde{\Omega}_{1*}|} \left[|\xi_1^*| + \frac{|\mu_1^*|}{\Gamma(\eta_1^*)} \int_{z_0}^{\tilde{T}} \left(\frac{(\tilde{T}-z_0)^\sigma - (q-z_0)^\sigma}{\sigma} \right)^{\eta_1^*-1} |\tilde{\mathcal{O}}_1(q, \varrho(q), \varrho(\lambda^* q))| \frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}} \right. \\
&\left. + \frac{|\mu_2^*|}{\Gamma(\eta_1^*+\theta^*)} \int_{z_0}^{\delta} \left(\frac{(\delta-z_0)^\sigma - (q-z_0)^\sigma}{\sigma} \right)^{\eta_1^*+\theta^*-1} |\tilde{\mathcal{O}}_1(q, \varrho(q), \varrho(\lambda^* q))| \frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}} \right] \\
&\leq \frac{\mathcal{N}_1(\tilde{T}-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)} + \frac{(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|} \left[\frac{|\mu_1^*|\mathcal{N}_1(\tilde{T}-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)} + \frac{|\mu_2^*|\mathcal{N}_1(\delta-z_0)^{\sigma(\eta_1^*+\theta^*)}}{\sigma^{(\eta_1^*+\theta^*)}\Gamma(\eta_1^*+\theta^*+1)} \right] + \frac{|\xi_1^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|} \\
&\leq \Delta_1\mathcal{N}_1 + \frac{|\xi_1^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|},
\end{aligned}
$$

which yields that $\|\mathfrak{P}_1^*\varpi\|_{\mathfrak{M}} \leq \Delta_1\mathcal{N}_1 + \frac{|\xi_1^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}$, where Δ_1 is illustrated by (3.9). In the similar manner, we have an inequality $\|\mathfrak{P}_2^*\varrho\|_{\mathfrak{M}} \leq \Delta_2\mathcal{N}_2 + \frac{|\xi_2^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{2*}|}$ so that Δ_2 is illustrated by (3.10). Hence, in view of the above inequalities we realize that the operator $\mathfrak{P}^*$ is uniformly bounded since we have

$$
\begin{aligned}
\|\mathfrak{P}^*(\varpi, \varrho)\|_{\mathfrak{M}\times\mathfrak{M}} &\leq \|\mathfrak{P}_1^*\varpi\|_{\mathfrak{M}} + \|\mathfrak{P}_2^*\varrho\|_{\mathfrak{M}} \\
&\leq \Delta_1\mathcal{N}_1 + \frac{|\xi_1^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|} + \Delta_2\mathcal{N}_2 + \frac{|\xi_2^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{2*}|}.
\end{aligned}
$$

In the sequel, we intend to check the equi-continuity of $\mathfrak{P}^*(\varpi, \varrho) = (\mathfrak{P}_1^*\varpi, \mathfrak{P}_2^*\varrho)$. To reach this goal, for any

$z', z'' \in [z_0, \tilde{T}]$ with $z' < z''$, we have

$$|\mathfrak{P}_1^*\varpi(z'') - \mathfrak{P}_1^*\varpi(z')| \leq \left| \frac{1}{\Gamma(\eta_1^*)} \int_{z_0}^{z''} \left(\frac{(z''-z_0)^\sigma - (q-z_0)^\sigma}{\sigma} \right)^{\eta_1^*-1} \tilde{\mathcal{O}}_1(q, \varrho(q), \varrho(\lambda^* q)) \frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}} \right.$$

$$\left. - \frac{1}{\Gamma(\eta_1^*)} \int_{z_0}^{z'} \left(\frac{(z'-z_0)^\sigma - (q-z_0)^\sigma}{\sigma} \right)^{\eta_1^*-1} \tilde{\mathcal{O}}_1(q, \varrho(q), \varrho(\lambda^* q)) \frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}} \right|$$

$$+ \frac{(z''-z_0)^\sigma - (z'-z_0)^\sigma}{|\tilde{\Omega}_{1*}|} \left[|\xi_1^*| + \frac{|\mu_1^*|}{\Gamma(\eta_1^*)} \int_{z_0}^{\tilde{T}} \left(\frac{(\tilde{T}-z_0)^\sigma - (q-z_0)^\sigma}{\sigma} \right)^{\eta_1^*-1} \right.$$

$$\left. \times |\tilde{\mathcal{O}}_1(q, \varrho(q), \varrho(\lambda^* q))| \frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}} \right]$$

$$\leq \left| \frac{\mathcal{N}_1(z''-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)} - \frac{\mathcal{N}_1(z'-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)} \right| + \frac{(z''-z_0)^\sigma - (z'-z_0)^\sigma}{|\tilde{\Omega}_{1*}|} \left[|\xi_1^*| + \frac{|\mu_1^*|\mathcal{N}_1(\tilde{T}-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)} \right].$$

We find that the right-hand side of the obtained inequality is not dependent of ϖ and also approaches to 0 when z' tends to z''. In consequence, $\mathfrak{P}_1^*$ is equi-continuous and hence it is confirmed the complete continuity of $\mathfrak{P}_1^*$ by the Arzelá–Ascoli theorem. Analogously, by similar reason, we realize that $\mathfrak{P}_2^*$ is equi-continuous and hence completely continuous too. Therefore it is conclude that $\mathfrak{P}^* = (\mathfrak{P}_1^*, \mathfrak{P}_2^*)$ is an operator with complete continuity property and the proof is finished. □

In this position, the next existence criterion depends on the Leray–Schauder degree theory indicated in Theorem 2.3.

Theorem 3.4 *Assume that* $\tilde{\mathcal{O}}_1, \tilde{\mathcal{O}}_2 : [z_0, \tilde{T}] \times \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$ *are continuous functions and there exist* $\mathcal{K}_j(z), \mathcal{L}_j(z) \in \mathcal{C}_{\mathbb{R}^+}[z_0, \tilde{T}]$ *so that we have* $|\tilde{\mathcal{O}}_j(z,u,v)| \leq \mathcal{K}_j(z)(|u|+|v|) + \mathcal{L}_j(z)$ *for* $j = 1, 2$. *Then the coupled system of the Caputo conformable pantograph fractional boundary problems* (1.1)–(1.2) *has at least one solution on* $[z_0, \tilde{T}]$ *whenever*

$$0 \leq \Delta\mathcal{K}^* < \frac{1}{2} \quad \text{such that} \quad \mathcal{K}^* = \max\{\mathcal{K}_1^*, \mathcal{K}_2^*\} \quad \text{and} \quad \Delta = \max\{\Delta_1, \Delta_2\},$$

where $\mathcal{K}_j^* = \sup_{z\in[z_0,\tilde{T}]} |\mathcal{K}_j(z)|$, $\mathcal{L}_j^* = \sup_{z\in[z_0,\tilde{T}]} |\mathcal{L}_j(z)|$ *and the constants* Δ_1, Δ_2 *are illustrated by* (3.9) *and* (3.10).

Proof To begin the proof, we regard the fixed point problem $\chi = \mathfrak{P}^*\chi$ where $\chi = (\varpi, \varrho)$ and $\mathfrak{P}^* = (\mathfrak{P}_1^*, \mathfrak{P}_2^*)$ is formulated by (3.6). Then we need to check that there exists at least one solution $\chi = (\varpi, \varrho) \in \mathfrak{M} \times \mathfrak{M}$ which satisfies $\chi = \mathfrak{P}^*\chi$. In this direction, we construct the following ball

$$\mathbb{W}_{\tilde{\epsilon}} = \{\chi \in \mathfrak{M} \times \mathfrak{M} : \ \|\chi(z)\|_{\mathfrak{M}\times\mathfrak{M}} < \tilde{\epsilon}, \ \forall \tilde{\epsilon} > 0, \ z \in [z_0, \tilde{T}]\}.$$

our target in this moment is to confirm that $\mathfrak{P}^* : \overline{\mathbb{W}}_{\tilde{\epsilon}} \to \mathfrak{M} \times \mathfrak{M}$ satisfies the condition

$$\chi \neq \alpha\mathfrak{P}^*\chi \tag{3.12}$$

for all $\chi \in \partial \mathbb{W}_{\tilde{\epsilon}}$ and for any $\alpha \in [0,1]$. Set $\mathcal{H}(\alpha,\chi) = \alpha\mathfrak{P}^*\chi$ for all $\chi \in \mathfrak{M}\times\mathfrak{M}$ and $\alpha \in [0,1]$. By taking into account Lemma 3.3, it is well known that the operator $\mathfrak{P}^*$ is continuous, uniformly bounded and equi-continuous. Hence, by invoking the Arzelá–Ascoli theorem, we realize that a continuous map $\mathfrak{h}_\alpha : \mathfrak{M}\times\mathfrak{M} \to \mathfrak{M}\times\mathfrak{M}$ formulated by $\mathfrak{h}_\alpha(\chi) = \chi - \mathcal{H}(\alpha,\chi) = \chi - \alpha\mathfrak{P}^*\chi$ is an operator with the complete continuity property. If (3.12) is valid, then Leray–Schauder degrees are well defined and in the light of the homotopy invariance of topological degree, we may write

$$\mathfrak{deg}(\mathfrak{h}_\alpha, \mathbb{W}_{\tilde{\epsilon}}, \tilde{0}) = \mathfrak{deg}(\tilde{I} - \alpha\mathfrak{P}^*, \mathbb{W}_{\tilde{\epsilon}}, \tilde{0}) = \mathfrak{deg}(\mathfrak{h}_1, \mathbb{W}_{\tilde{\epsilon}}, \tilde{0})$$

$$= \mathfrak{deg}(\mathfrak{h}_0, \mathbb{W}_{\tilde{\epsilon}}, \tilde{0}) = \mathfrak{deg}(\tilde{I}, \mathbb{W}_{\tilde{\epsilon}}, \tilde{0}) = \tilde{1} \neq \tilde{0}$$

so that $\tilde{I} = I \times I$ and I stands for the unit operator and $\tilde{1} = (1,1)$ and $\tilde{0} = (0,0) \in \mathbb{W}_{\tilde{\epsilon}}$. The nonzero property of Leray–Schauder degree implies that we have $\mathfrak{h}_1(\chi) = \chi - \mathfrak{P}^*\chi = \tilde{0}$ for at least one $\chi \in \mathbb{W}_{\tilde{\epsilon}}$. Now, to show that (3.12) holds, we assume that $\chi = \alpha\mathfrak{P}^*\chi$ for some $\alpha \in [0,1]$. Then for each $z \in [z_0, \tilde{T}]$, we get

$$\varpi(z) = \alpha(\mathfrak{P}_1^*\varpi)(z) \quad \text{and} \quad \varrho(z) = \alpha(\mathfrak{P}_2^*\varrho)(z).$$

Hence one can write

$$\begin{aligned}
|\varpi(z)| &= |\alpha(\mathfrak{P}_1^*\varpi)(z)| \\
&\leq \frac{1}{\Gamma(\eta_1^*)}\int_{z_0}^{z}\left(\frac{(z-z_0)^\sigma - (q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*-1}|\tilde{\mathcal{O}}_1(q,\varrho(q),\varrho(\lambda^*q))|\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}} \\
&+ \frac{(z-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}\left[|\xi_1^*| + \frac{|\mu_1^*|}{\Gamma(\eta_1^*)}\int_{z_0}^{\tilde{T}}\left(\frac{(\tilde{T}-z_0)^\sigma - (q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*-1}|\tilde{\mathcal{O}}_1(q,\varrho(q),\varrho(\lambda^*q))|\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}}\right. \\
&\left. + \frac{|\mu_2^*|}{\Gamma(\eta_1^*+\theta^*)}\int_{z_0}^{\delta}\left(\frac{(\delta-z_0)^\sigma - (q-z_0)^\sigma}{\sigma}\right)^{\eta_1^*+\theta^*-1}|\tilde{\mathcal{O}}_1(q,\varrho(q),\varrho(\lambda^*q))|\frac{\mathrm{d}q}{(q-z_0)^{1-\sigma}}\right] \\
&\leq \frac{(2\mathcal{K}_1^*\|\varrho\|_{\mathfrak{M}} + \mathcal{L}_1^*)(\tilde{T}-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)} + \frac{(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}\left[\frac{|\mu_1^*|(2\mathcal{K}_1^*\|\varrho\|_{\mathfrak{M}} + \mathcal{L}_1^*)(\tilde{T}-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)}\right. \\
&\left. + \frac{|\mu_2^*|(2\mathcal{K}_1^*\|\varrho\|_{\mathfrak{M}} + \mathcal{L}_1^*)(\delta-z_0)^{\sigma(\eta_1^*+\theta^*)}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+\theta^*+1)}\right] + \frac{|\xi_1^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|} \\
&\leq \Delta_1(2\mathcal{K}_1^*\|\varrho\|_{\mathfrak{M}} + \mathcal{L}_1^*) + \frac{|\xi_1^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}.
\end{aligned}$$

Thus

$$\|\varpi\|_{\mathfrak{M}} \leq \Delta_1(2\,\mathcal{K}_1^*\|\varrho\|_{\mathfrak{M}} + \mathcal{L}_1^*) + \frac{|\xi_1^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}.$$

By similar calculations, we get

$$\|\varrho\|_{\mathfrak{M}} \leq \Delta_2(2\mathcal{K}_2^*\|\varpi\|_{\mathfrak{M}} + \mathcal{L}_2^*) + \frac{|\xi_2^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{2*}|}.$$

Hence, we reach the following inequalities:

$$\begin{aligned}
\|(\varpi,\varrho)\|_{\mathfrak{M}\times\mathfrak{M}} &= \|\varpi\|_{\mathfrak{M}} + \|\varrho\|_{\mathfrak{M}} \\
&\leq \Delta_1(2\mathcal{K}_1^*\|\varrho\|_{\mathfrak{M}} + \mathcal{L}_1^*) + \frac{|\xi_1^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|} + \Delta_2(2\mathcal{K}_2^*\|\varpi\|_{\mathfrak{M}} + \mathcal{L}_2^*) + \frac{|\xi_2^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{2*}|} \\
&\leq 2\Delta_1\mathcal{K}_1^*\|\varrho\|_{\mathfrak{M}} + 2\Delta_2\mathcal{K}_2^*\|\varpi\|_{\mathfrak{M}} + \Delta_1\mathcal{L}_1^* + \Delta_2\mathcal{L}_2^* + \frac{|\xi_1^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|} + \frac{|\xi_2^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{2*}|} \\
&\leq 2\Delta\mathcal{K}^*(\|\varrho\|_{\mathfrak{M}} + \|\varpi\|_{\mathfrak{M}}) + \Delta_1\mathcal{L}_1^* + \Delta_2\mathcal{L}_2^* + \frac{|\xi_1^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|} + \frac{|\xi_2^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{2*}|} \\
&\leq 2\Delta\mathcal{K}^*\|(\varpi,\varrho)\|_{\mathfrak{M}\times\mathfrak{M}} + \Delta_1\mathcal{L}_1^* + \Delta_2\mathcal{L}_2^* + \frac{|\xi_1^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|} + \frac{|\xi_2^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{2*}|}.
\end{aligned}$$

This yields

$$\|(\varpi,\varrho)\|_{\mathfrak{M}\times\mathfrak{M}} \leq \Big(\frac{1}{1-2\Delta\mathcal{K}^*}\Big)\Big(\Delta_1\mathcal{L}_1^* + \Delta_2\mathcal{L}_2^* + \frac{|\xi_1^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|} + \frac{|\xi_2^*|(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{2*}|}\Big) := \Upsilon.$$

By setting $\tilde{\epsilon} = \Upsilon + 1$, we observe that 3.12 holds. Since $0 \leq \Delta\mathcal{K}^* < \frac{1}{2}$, thus by Theorem 2.3, we find that the coupled system of the Caputo conformable pantograph fractional boundary problems (1.1)–(1.2) has at least one solution on $[z_0, \tilde{T}]$ and this ends the proof. □

Here, some estimates for solutions of the given pantograph coupled system are investigated in two folds. The first estimate for solutions of the coupled system of the Caputo conformable pantograph fractional boundary problems (1.1)–(1.2) is established due to small changes occurred in the right-hand side of equations (1.1).

Theorem 3.5 *Let hypothesis of Theorem 3.2 are hold. In addition, assume that* (ϖ,ϱ) *and* $(\tilde{\varpi},\tilde{\varrho})$ *are solutions of the proposed Caputo conformable pantograph coupled system* (1.1)–(1.2) *and*

$$\begin{cases}
{}^{\mathcal{C}\mathcal{C}}\mathfrak{D}_{z_0}^{\sigma,\eta_1^*}\tilde{\varpi}(z) = \tilde{\mathcal{O}}_1(z,\tilde{\varrho}(z),\tilde{\varrho}(\lambda^* z)) + \varepsilon\tilde{\mathcal{F}}_1(z,\tilde{\varrho}(z),\tilde{\varrho}(\lambda^* z)), \\
{}^{\mathcal{C}\mathcal{C}}\mathfrak{D}_{z_0}^{\sigma,\eta_2^*}\tilde{\varrho}(z) = \tilde{\mathcal{O}}_2(z,\tilde{\varpi}(z),\tilde{\varpi}(\lambda^* z)) + \varepsilon\tilde{\mathcal{F}}_2(z,\tilde{\varpi}(z),\tilde{\varpi}(\lambda^* z)), \\
\tilde{\varpi}(z_0) = 0, \qquad \mu_1^*\tilde{\varpi}(\tilde{T}) + \mu_2^{*\mathcal{R}\mathcal{C}}\mathcal{I}_{z_0}^{\sigma,\theta^*}\tilde{\varpi}(\delta) = \xi_1^*, \\
\tilde{\varrho}(z_0) = 0, \qquad \gamma_1^*\tilde{\varrho}(\tilde{T}) + \gamma_2^{*\mathcal{R}\mathcal{C}}\mathcal{I}_{z_0}^{\sigma,\theta^*}\tilde{\varrho}(\nu) = \xi_2^*.
\end{cases} \tag{3.13}$$

for $\varepsilon > 0$, *respectively. Then the following inequality*

$$\|(\varpi,\varrho) - (\tilde{\varpi},\tilde{\varrho})\|_{\mathfrak{M}\times\mathfrak{M}} \leq \frac{\varepsilon\ \Delta_1\|\mathcal{F}_1\|_{\mathfrak{M}} + \varepsilon\ \Delta_2\|\mathcal{F}_2\|_{\mathfrak{M}}}{1-2m\Delta},$$

is valid provided that $m\Delta < \frac{1}{2}$, *where* $m = max\{m_1, m_2\}$ *and* $\Delta = \max\{\Delta_1, \Delta_2\}$ *so that* Δ_1 *and* Δ_2 *are illustrated by* (3.9) *and* (3.10) *and also we have* $\|\mathcal{F}_j\|_{\mathfrak{M}} = sup_{z\in[z_0,\tilde{T}]}|\mathcal{F}_j(z,u,v)|$ *for* $j = 1, 2$.

Proof First of all, in view of Lemma 3.1, we have

$$\tilde{\varpi}(z) = {}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta_1^*}\Big[\tilde{\mathcal{O}}_1(z,\tilde{\varrho}(z),\tilde{\varrho}(\lambda^* z)) + \varepsilon\ \tilde{\mathcal{F}}_1(z,\tilde{\varrho}(z),\tilde{\varrho}(\lambda^* z))\Big]$$

$$+\frac{(z-z_0)^\sigma}{\tilde{\Omega}_{1*}}\Bigg[\xi_1^* - \mu_1^{*\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta_1^*}\Big[\tilde{\mathcal{O}}_1(\tilde{T},\tilde{\varrho}(\tilde{T}),\tilde{\varrho}(\lambda^*\tilde{T})) + \varepsilon\ \tilde{\mathcal{F}}_1(\tilde{T},\tilde{\varrho}(\tilde{T}),\tilde{\varrho}(\lambda^*\tilde{T}))\Big]$$

$$-\mu_2^{*\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,(\eta_1^*+\theta^*)}\Big[\tilde{\mathcal{O}}_1(\delta,\tilde{\varrho}(\delta),\tilde{\varrho}(\lambda^*\delta)) + \varepsilon\ \tilde{\mathcal{F}}_1(\delta,\tilde{\varrho}(\delta),\tilde{\varrho}(\lambda^*\delta))\Big]\Bigg]$$

and

$$\tilde{\varrho}(z) = {}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta_2^*}\Big[\tilde{\mathcal{O}}_2(z,\tilde{\varpi}(z),\tilde{\varpi}(\lambda^* z)) + \varepsilon\ \tilde{\mathcal{F}}_2(z,\tilde{\varpi}(z),\tilde{\varpi}(\lambda^* z))\Big]$$

$$+\frac{(z-z_0)^\sigma}{\tilde{\Omega}_{2*}}\Bigg[\xi_2^* - \gamma_1^{*\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta_2^*}\Big[\tilde{\mathcal{O}}_2(\tilde{T},\tilde{\varpi}(\tilde{T}),\tilde{\varpi}(\lambda^*\tilde{T})) + \varepsilon\ \tilde{\mathcal{F}}_2(\tilde{T},\tilde{\varpi}(\tilde{T}),\tilde{\varpi}(\lambda^*\tilde{T}))\Big]$$

$$-\gamma_2^{*\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,(\eta_2^*+\theta^*)}\Big[\tilde{\mathcal{O}}_2(\delta,\tilde{\varpi}(\delta),\tilde{\varpi}(\lambda^*\delta)) + \varepsilon\ \tilde{\mathcal{F}}_2(\delta,\tilde{\varpi}(\delta),\tilde{\varpi}(\lambda^*\delta))\Big]\Bigg]$$

for any $z \in [z_0, \tilde{T}]$ where both nonzero constants $\tilde{\Omega}_{1*}$ and $\tilde{\Omega}_{2*}$ are illustrated by (3.1). Now, we obtain the

following estimates for $|\varpi - \tilde{\varpi}|$ as follows:

$$\begin{aligned}
\big|\varpi(z) - \tilde{\varpi}(z)\big| &\leq {}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta_1^*}\big|\tilde{\mathcal{O}}_1(z,\varrho(z),\varrho(\lambda^* z)) - \tilde{\mathcal{O}}_1(z,\tilde{\varrho}(z),\tilde{\varrho}(\lambda^* z))\big| \\
&+ \frac{(z-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}\Bigg[|\mu_1^*|{}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta_1^*}\big|\tilde{\mathcal{O}}_1(\tilde{T},\varrho(\tilde{T}),\varrho(\lambda^*\tilde{T})) - \tilde{\mathcal{O}}_1(\tilde{T},\tilde{\varrho}(\tilde{T}),\tilde{\varrho}(\lambda^*\tilde{T}))\big| \\
&+ |\mu_2^*|{}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,(\eta_1^*+\theta^*)}\big|\tilde{\mathcal{O}}_1(\delta,\varrho(\delta),\varrho(\lambda^*\delta)) - \tilde{\mathcal{O}}_1(\delta,\tilde{\varrho}(\delta),\tilde{\varrho}(\lambda^*\delta))\big|\Bigg] + \varepsilon\,{}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta_1^*}\big|\tilde{\mathcal{F}}_1(z,\tilde{\varrho}(z),\tilde{\varrho}(\lambda^* z))\big| \\
&+ \frac{(z-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}\Bigg[\varepsilon|\mu_1^*|{}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta_1^*}\big|\tilde{\mathcal{F}}_1(\tilde{T},\tilde{\varrho}(\tilde{T}),\tilde{\varrho}(\lambda^*\tilde{T}))\big| + \varepsilon|\mu_2^*|{}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,(\eta_1^*+\theta^*)}\big|\tilde{\mathcal{F}}_1(\delta,\tilde{\varrho}(\delta),\tilde{\varrho}(\lambda^*\delta))\big|\Bigg] \\
&\leq \frac{2m_1\big\|\varrho-\tilde{\varrho}\big\|_{\mathfrak{M}}(\tilde{T}-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)} + \frac{(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}\Bigg[\frac{2m_1|\mu_1^*|\big\|\varrho-\tilde{\varrho}\big\|_{\mathfrak{M}}(\tilde{T}-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)} \\
&+ \frac{2m_1|\mu_2^*|\big\|\varrho-\tilde{\varrho}\big\|_{\mathfrak{M}}(\delta-z_0)^{\sigma(\eta_1^*+\theta^*)}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+\theta^*+1)}\Bigg] + \frac{\varepsilon\|\mathcal{F}_1\|_{\mathfrak{M}}(\tilde{T}-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)} \\
&+ \frac{(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}\Bigg[\frac{\varepsilon|\mu_1^*|\|\mathcal{F}_1\|_{\mathfrak{M}}(\tilde{T}-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)} + \frac{\varepsilon\,|\mu_2^*|\|\mathcal{F}_1\|_{\mathfrak{M}}(\delta-z_0)^{\sigma(\eta_1^*+\theta^*)}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+\theta^*+1)}\Bigg] \\
&\leq 2m_1\Delta_1\big\|\varrho-\tilde{\varrho}\big\|_{\mathfrak{M}} + \varepsilon\Delta_1\|\mathcal{F}_1\|_{\mathfrak{M}}.
\end{aligned}$$

Hence, this yields

$$\big\|\varpi-\tilde{\varpi}\big\|_{\mathfrak{M}} \leq 2m_1\Delta_1\big\|\varrho-\tilde{\varrho}\big\|_{\mathfrak{M}} + \varepsilon\Delta_1\|\mathcal{F}_1\|_{\mathfrak{M}}. \tag{3.14}$$

In the same manner, we get

$$\big\|\varrho-\tilde{\varrho}\big\|_{\mathfrak{M}} \leq 2m_2\Delta_2\big\|\varpi-\tilde{\varpi}\big\|_{\mathfrak{M}} + \varepsilon\Delta_2\|\mathcal{F}_2\|_{\mathfrak{M}}. \tag{3.15}$$

By both inequalities (3.14) and (3.15), we deduce that

$$\|(\varpi,\varrho)-(\tilde{\varpi},\tilde{\varrho})\|_{\mathfrak{M}\times\mathfrak{M}} \leq \frac{\varepsilon\Delta_1\|\mathcal{F}_1\|_{\mathfrak{M}} + \varepsilon\Delta_2\|\mathcal{F}_2\|_{\mathfrak{M}}}{1-2m\Delta}$$

and the proof is completed. □

The second estimate for solutions of the given coupled system of the Caputo conformable pantograph fractional boundary problem (1.1)–(1.2) is established due to small changes occurred in the boundary conditions (1.2). Specifically, the time interval $[z_0,\tilde{T}]$ changes to $[z_0,\tilde{T}+\varepsilon]$ for $\varepsilon>0$.

Theorem 3.6 *Assume that all hypotheses of Theorem 3.5 are valid. Moreover, let* (ϖ,ϱ) *and* $(\tilde{\varpi},\tilde{\varrho})$ *be*

solutions of the pantograph coupled system (1.1)–(1.2) *and*

$$\begin{cases} {}^{\mathcal{C}}\mathcal{C}\mathcal{D}_{z_0}^{\sigma,\eta_1^*}\tilde{\varpi}(z) = \tilde{\mathcal{O}}_1(z,\tilde{\varrho}(z),\tilde{\varrho}(\lambda^* z)), \qquad z\in[z_0,\tilde{T}+\varepsilon], \\ {}^{\mathcal{C}}\mathcal{C}\mathcal{D}_{z_0}^{\sigma,\eta_2^*}\tilde{\varrho}(z) = \tilde{\mathcal{O}}_2(z,\tilde{\varpi}(z),\tilde{\varpi}(\lambda^* z)), \\ \tilde{\varpi}(z_0)=0, \qquad \mu_1^*\tilde{\varpi}(\tilde{T}+\varepsilon)+\mu_2^{*\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\theta^*}\tilde{\varpi}(\delta)=\xi_1^*, \\ \tilde{\varrho}(z_0)=0, \qquad \gamma_1^*\tilde{\varrho}(\tilde{T}+\varepsilon)+\gamma_2^{*\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\theta^*}\tilde{\varrho}(\nu)=\xi_2^* \end{cases} \tag{3.16}$$

for $\varepsilon>0$, *respectively. Then the following inequality*

$$\|(\varpi,\varrho)-(\tilde{\varpi},\tilde{\varrho})\|_{\mathfrak{M}\times\mathfrak{M}} \le \frac{\Xi}{1-2m\Delta}$$

holds provided that $m\Delta<\frac{1}{2}$, *where* $m=max\{m_1,m_2\}$, $\Delta=\max\{\Delta_1,\Delta_2\}$ *so that* Δ_1 *and* Δ_2 *are illustrated by* (3.9) *and* (3.10) *and* $\|\tilde{O}_j\|_{\mathfrak{M}}=sup_{z\in[\tilde{T},\tilde{T}+\varepsilon]}\{|\tilde{O}_j(z,u,v)|\}$ *for* $j=1,2$ *and*

$$\Xi=\frac{(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}\,\frac{|\mu_1^*|\,\left\|\tilde{O}_1\right\|_{\mathfrak{M}}\,\varepsilon^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)}+\frac{(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{2*}|}\,\frac{|\gamma_1^*|\,\left\|\tilde{O}_2\right\|_{\mathfrak{M}}\,\varepsilon^{\sigma\eta_2^*}}{\sigma^{\eta_2^*}\Gamma(\eta_2^*+1)}.$$

Proof first of all, with due attention to Lemma 3.1, we have

$$\begin{aligned}\tilde{\varpi}(z) &= {}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta_1^*}\tilde{\mathcal{O}}_1(z,\tilde{\varrho}(z),\tilde{\varrho}(\lambda^* z)) \\ &+\frac{(z-z_0)^\sigma}{\breve{\Omega}_{1*}}\Big[\xi_1^*-\mu_1^{*\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta_1^*}\tilde{\mathcal{O}}_1(\tilde{T}+\varepsilon,\tilde{\varrho}(\tilde{T}+\varepsilon),\tilde{\varrho}(\lambda^*(\tilde{T}+\varepsilon))) \\ &-\mu_2^{*\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,(\eta_1^*+\theta^*)}\tilde{\mathcal{O}}_1(\delta,\tilde{\varrho}(\delta),\tilde{\varrho}(\lambda^*\delta))\Big]\end{aligned}$$

and

$$\begin{aligned}\tilde{\varrho}(z) &= {}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta_2^*}\tilde{\mathcal{O}}_2(z,\tilde{\varpi}(z),\tilde{\varpi}(\lambda^* z)) \\ &+\frac{(z-z_0)^\sigma}{\breve{\Omega}_{2*}}\Big[\xi_2^*-\gamma_1^{*\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta_2^*}\tilde{\mathcal{O}}_2(\tilde{T}+\varepsilon,\tilde{\varpi}(\tilde{T}+\varepsilon),\tilde{\varpi}(\lambda^*(\tilde{T}+\varepsilon))) \\ &-\gamma_2^{*\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,(\eta_2^*+\theta^*)}\tilde{\mathcal{O}}_2(\delta,\tilde{\varpi}(\delta),\tilde{\varpi}(\lambda^*\delta))\Big]\end{aligned}$$

for all $z\in[z_0,\tilde{T}+\varepsilon]$ and both nonzero constants $\breve{\Omega}_{1*}$ and $\breve{\Omega}_{2*}$ are illustrated by

$$\breve{\Omega}_{1*}=\mu_1^*(\tilde{T}+\varepsilon-z_0)^\sigma+\mu_2^*\frac{(\delta-z_0)^{\sigma(1+\theta^*)}}{\sigma^{\theta^*}\Gamma(2+\theta^*)}\neq 0,$$

$$\breve{\Omega}_{2*}=\gamma_1^*(\tilde{T}+\varepsilon-z_0)^\sigma+\gamma_2^*\frac{(\nu-z_0)^{\sigma(1+\theta^*)}}{\sigma^{\theta^*}\Gamma(2+\theta^*)}\neq 0.$$

It is obvious that $\min\{\tilde{\Omega}_{1*}, \breve{\Omega}_{1*}\} = \tilde{\Omega}_{1*}$ and $\min\{\tilde{\Omega}_{2*}, \breve{\Omega}_{2*}\} = \tilde{\Omega}_{2*}$, where both nonzero constants $\tilde{\Omega}_{1*}$ and $\tilde{\Omega}_{2*}$ are illustrated by (3.1). Now, we have the following estimate

$$\begin{aligned}
\left|\varpi(z) - \tilde{\varpi}(z)\right| &\leq {}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta_1^*}\left|\tilde{\mathcal{O}}_1(z, \varrho(z), \varrho(\lambda^* z)) - \tilde{\mathcal{O}}_1(z, \tilde{\varrho}(z), \tilde{\varrho}(\lambda^* z))\right| \\
&+ \frac{(z-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}\Bigg[|\mu_1^*|{}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,\eta_1^*}\left|\tilde{\mathcal{O}}_1(\tilde{T}, \varrho(\tilde{T}), \varrho(\lambda^*\tilde{T})) - \tilde{\mathcal{O}}_1(\tilde{T}, \tilde{\varrho}(\tilde{T}), \tilde{\varrho}(\lambda^*\tilde{T}))\right| \\
&+ |\mu_2^*|{}^{\mathcal{RC}}\mathcal{I}_{z_0}^{\sigma,(\eta_1^*+\theta^*)}\left|\tilde{\mathcal{O}}_1(\delta, \varrho(\delta), \varrho(\lambda^*\delta)) - \tilde{\mathcal{O}}_1(\delta, \tilde{\varrho}(\delta), \tilde{\varrho}(\lambda^*\delta))\right|\Bigg] \\
&+ \frac{|\mu_1^*|(z-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}{}^{\mathcal{RC}}\mathcal{I}_{\tilde{T}}^{\sigma,\eta_1^*}\left|\tilde{\mathcal{O}}_1(\tilde{T}+\varepsilon, \varrho(\tilde{T}+\varepsilon), \varrho(\lambda^*(\tilde{T}+\varepsilon)))\right| \\
&\leq \frac{2m_1\left\|\varrho - \tilde{\varrho}\right\|_{\mathfrak{M}}(\tilde{T}-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)} + \frac{(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}\Bigg[\frac{2m_1|\mu_1^*|\left\|\varrho - \tilde{\varrho}\right\|_{\mathfrak{M}}(\tilde{T}-z_0)^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)} \\
&+ \frac{2m_1|\mu_2^*|\left\|\varrho - \tilde{\varrho}\right\|_{\mathfrak{M}}(\delta-z_0)^{\sigma(\eta_1^*+\theta^*)}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+\theta^*+1)}\Bigg] + \frac{(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}\ \frac{|\mu_1^*|\ \left\|\tilde{O}_1\right\|_{\mathfrak{M}}\ \varepsilon^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)} \\
&\leq 2m_1\Delta_1\left\|\varrho - \tilde{\varrho}\right\|_{\mathfrak{M}} + \frac{(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}\ \frac{|\mu_1^*|\ \left\|\tilde{O}_1\right\|_{\mathfrak{M}}\ \varepsilon^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)}.
\end{aligned}$$

Therefore,we obtain

$$\left\|\varpi - \tilde{\varpi}\right\|_{\mathfrak{M}} \leq 2m_1\Delta_1\left\|\varrho - \tilde{\varrho}\right\|_{\mathfrak{M}} + \frac{(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{1*}|}\ \frac{|\mu_1^*|\ \left\|\tilde{O}_1\right\|_{\mathfrak{M}}\ \varepsilon^{\sigma\eta_1^*}}{\sigma^{\eta_1^*}\Gamma(\eta_1^*+1)}. \tag{3.17}$$

In similar manner, we obtain

$$\left\|\varrho - \tilde{\varrho}\right\|_{\mathfrak{M}} \leq 2m_2\Delta_2\left\|\varpi - \tilde{\varpi}\right\|_{\mathfrak{M}} + \frac{(\tilde{T}-z_0)^\sigma}{|\tilde{\Omega}_{2*}|}\ \frac{|\gamma_1^*|\ \left\|\tilde{O}_2\right\|_{\mathfrak{M}}\ \varepsilon^{\sigma\eta_2^*}}{\sigma^{\eta_2^*}\Gamma(\eta_2^*+1)}. \tag{3.18}$$

Hence, according to both inequalities (3.17) and (3.18), we conclude that

$$\|(\varpi, \varrho) - (\tilde{\varpi}, \tilde{\varrho})\|_{\mathfrak{M}\times\mathfrak{M}} \leq \frac{\Xi}{1-2m\Delta},$$

and this ends the proof. □

4. Examples

In this part of the current research, we examine our obtained results by proposing an example to confirm the validity of the findings from a numerical point of view.

Example 4.1 *According to general structure (1.1)–(1.2), we design the following fractional coupled system of the Caputo conformable pantograph boundary problems*

$$\begin{cases} {}^{\mathcal{C}\mathcal{C}}\mathcal{D}_0^{0.5,1.27}\varpi(z) = \tilde{\mathcal{O}}_1(z,\varrho(z),\varrho(0.24z)), & (z\in[0,1]),\\ {}^{\mathcal{C}\mathcal{C}}\mathcal{D}_0^{0.5,1.09}\varrho(z) = \tilde{\mathcal{O}}_2(z,\varpi(z),\varpi(0.24z))\\ \varpi(0)=0, \qquad 0.11\varpi(1)+0.08^{\mathcal{R}\mathcal{C}}\mathcal{I}_0^{0.5,0.6}\varpi(0.2)=1,\\ \varrho(0)=0, \qquad 0.15\varrho(1)+0.04^{\mathcal{R}\mathcal{C}}\mathcal{I}_0^{0.5,0.6}\varrho(0.7)=1.5, \end{cases} \tag{4.1}$$

so that $\sigma=0.5$, $\eta_1^*=1.27$, $\eta_2^*=1.09$, $\theta^*=0.6$, $\mu_1^*=0.11$, $\mu_2^*=0.08$, $\gamma_1^*=0.15$, $\gamma_2^*=0.04$ $\lambda^*=0.24$, $\delta=0.2$, $\nu=0.7$, $\xi_1^*=1$, $\xi_2^*=1.5$, $z_0=0$ *and* $\tilde{T}=1$. *On the other hand, we formulate two continuous functions* $\tilde{\mathcal{O}}_1(z,\varrho(z),\varrho(0.24z))$ *and* $\tilde{\mathcal{O}}_2(z,\varpi(z),\varpi(0.24z))$ *by follows*

$$\tilde{\mathcal{O}}_1(z,\varrho,\tilde{\varrho}) = \frac{0.0001}{z+1}\big[\sin(\varrho)+\arcsin(\tilde{\varrho})\big]+\frac{z}{2}, \tag{4.2}$$

and

$$\tilde{\mathcal{O}}_2(z,\varpi,\tilde{\varpi}) = \frac{0.0008}{z+2}\big[\arctan(\varpi)+\tilde{\varpi}\big]+\frac{0.5z^2}{8}. \tag{4.3}$$

then one can write

$$\begin{aligned} |\tilde{\mathcal{O}}_1(z,\varrho,\tilde{\varrho})| &\le \frac{0.0001}{z+1}\big[\,|\sin(\varrho)|+|\arcsin(\tilde{\varrho})|\,\big]+\frac{z}{2}\\ &\le \frac{0.0001}{z+1}\big[|\varrho|+|\tilde{\varrho}|\,\big]+\frac{z}{2}\\ &\le \mathcal{K}_1(z)\big[\,|\varrho|+|\tilde{\varrho}|\,\big]+\mathcal{L}_1(z) \end{aligned}$$

for any $z\in[0,1]$. *In the similar manner, we have*

$$\begin{aligned} |\tilde{\mathcal{O}}_2(z,\varpi,\tilde{\varpi})| &\le \frac{0.0008}{z+2}\big[\,|\arctan(\varpi)|+|\tilde{\varpi}|\,\big]+\frac{0.5z^2}{8}\\ &\le \frac{0.0008}{z+2}\big[|\varpi|+|\tilde{\varpi}|\,\big]+\frac{0.5z^2}{8}\\ &\le \mathcal{K}_2(z)\big[\,|\varpi|+|\tilde{\varpi}|\,\big]+\mathcal{L}_2(z) \end{aligned}$$

for any $z\in[0,1]$. *In view of above estimates, we define functions* $\mathcal{K}_j(z),\mathcal{L}_j(z):[0,1]\to\mathbb{R}$ *by* $\mathcal{K}_1(z)=\dfrac{0.0001}{z+1}$, $\mathcal{K}_2(z)=\dfrac{0.0008}{z+2}$, $\mathcal{L}_1(z)=\dfrac{z}{2}$ *and* $\mathcal{L}_2(z)=\dfrac{0.5z^2}{8}$. *In this case, we have*

$$\mathcal{K}_1^*=\sup_{z\in[0,1]}|\mathcal{K}_1(z)|=0.0001, \qquad \mathcal{K}_2^*=\sup_{z\in[0,1]}|\mathcal{K}_2(z)|=0.0004,$$

$$\mathcal{L}_1^*=\sup_{z\in[0,1]}|\mathcal{L}_1(z)|=0.5, \qquad \mathcal{L}_2^*=\sup_{z\in[0,1]}|\mathcal{L}_2(z)|=0.0625.$$

In consequence, $\mathcal{K}^* = \max\{\mathcal{K}_1^*, \mathcal{K}_2^*\} = \max\{0.0001, 0.0004\} = 0.0004$. *On the other hand, by considering the existing parameters, we obtain the value of constants* $\tilde{\Omega}_{1*} = 0.13325$, $\tilde{\Omega}_{2*} = 0.18228$, $\Delta_1 = 4.1147$ *and* $\Delta_2 = 4.0678$ *and so* $\Delta = \max\{\Delta_1, \Delta_2\} = 4.1147$. *Hence since* $0 \leq \Delta\mathcal{K}^* = 0.0016459 < 0.5$, *thus we find that all hypotheses of Theorem 3.4 are valid. This causes that the fractional coupled system of the Caputo conformable pantograph boundary problems (4.1) along with two continuous functions* $\tilde{\mathcal{O}}_1$ *and* $\tilde{\mathcal{O}}_2$ *formulated by (4.2) and (4.3) has at least one solution on* $z \in [0, 1]$.

5. Conclusion

Over the decades, the human beings need to be familiar with structures and behaviors of different natural phenomena more and more. One of the possible ways to reach this goal is to apply techniques and existing tools in mathematics along with newly defined mathematical operators in modeling of various processes. In this paper, we used one of these newly introduced operators named as the Caputo conformable derivative to model a coupled system of pantograph equations. Further, the boundary value conditions of this coupled system are equipped with the Riemann–Liouville conformable integrals. After recalling some properties of these new fractional operators in Section 2, we applied the well-known Leray–Schauder degree theorem to obtain sufficient conditions for proving the existence result of solutions for the proposed coupled system of pantograph boundary value problems. Also, the uniqueness result is verified by applying the Banach fixed point theorem. By implementing small changes in the boundary conditions and the right-hand side of the equations, we estimated solutions of two existing coupled systems. Finally, we provided an example to check theoretical findings by a numerical point of view. The outcomes expressed in the current paper are unique and new and also will mainly contribute to the existing materials on the pantograph boundary value problems.

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