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A study of impulsive discrete Dirac system with hyperbolic eigenparameter

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Abstract: Let L denote the discrete Dirac operator generated in $\ell_2\left(\mathbb{N},\mathbb{C}^2\right)$ by the difference operators of first order

$$\begin{cases} \triangle y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)} \\ \triangle y_{n-1}^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)}, \end{cases} \qquad n \in \mathbb{N} \setminus \{k-1, k, k+1\}$$

with boundary and impulsive conditions

$$y_0^{(1)} = 0 \ ,$$

$$\begin{pmatrix} y_{k+1}^{(1)} \\ y_{k+2}^{(2)} \end{pmatrix} = \theta \begin{pmatrix} y_{k-1}^{(2)} \\ y_{k-2}^{(1)} \end{pmatrix}; \ \theta = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix}, \ \ \{\theta_i\}_{i=1,2,3,4} \in \mathbb{R}$$

where $\{p_n\}_{n\in\mathbb{N}}$, $\{q_n\}_{n\in\mathbb{N}}$ are real sequences, $\lambda = 2\sinh\left(\frac{z}{2}\right)$ is a hyperbolic eigenparameter and $\triangle$ is forward operator. In this paper, the spectral properties of L such as the spectrum, the eigenvalues, the scattering function and their properties are given with an example in the special cases under the condition

$$\sum_{n=1}^{\infty} n\left(|p_n| + |q_n|\right) < \infty.$$

Key words: Discrete Dirac equations, impulsive condition, hyperbolic eigenparameter, scattering function, eigenvalues

1. Introduction

Difference equations appear as a mathematical model in many daily events about ecology, medicine, economics, control theory and mechanics. Because of that, spectral properties of these equations are attractive study topics for many mathematicians from the recent past to the present [2, 4, 6, 14, 16, 19, 21, 25]. In particular discrete Dirac equations have a wide place in these studies in terms of mathematical investigation of light theory in physics.

Consider the nonself-adjoint system of difference equations of first order

$$\begin{cases} a_{n+1} y_{n+1}^{(2)} + b_n y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)} \\ a_{n-1} y_{n-1}^{(1)} + b_n y_n^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)} \end{cases} \tag{1.1}$$

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where (a_n), (b_n), (p_n) and (q_n) are complex sequences for $n \in \mathbb{Z}$ and λ is a spectral parameter. The analytical properties of the Jost solutions of (1.1) have been studied in [10]. By using these properties, the authors have obtained eigenvalues and spectral singularities of (1.1) which are finite number with finite multiplicities. Before the study [10], the similar results included the properties of the principal vectors corresponding to the spectral singularities has been founded in [1] for $a_n \equiv 1$ and $b_n \equiv -1$. Some other studies related to discrete Dirac equation with different conditions can also be seen in [5, 9, 11, 17]. Moreover, the discrete impulsive equations have led to the rapid development of the spectral theory of difference equations. These equations mostly seen as a mathematical model in engineering, biological, physical problems, especially heat and mass transfer. Therefore, analyzing the spectral properties of such equations is crucial for understanding the events in these areas [3,7,8,12,15,18,20,22–24].

Let us consider the discrete Dirac operator L in Hilbert space $\ell_2\left(\mathbb{N}, \mathbb{C}^2\right)$ denoted by the first order difference operator system

$$\begin{cases} \triangle y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)} \\ \triangle y_{n-1}^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)} \end{cases} \quad , n \in \mathbb{N} \setminus \{k-1, k, k+1\} \tag{1.2}$$

with boundary and impulsive conditions

$$y_0^{(1)} = 0 \ , \tag{1.3}$$

$$\begin{pmatrix} y_{k+1}^{(1)} \\ y_{k+2}^{(2)} \end{pmatrix} = \theta \begin{pmatrix} y_{k-1}^{(2)} \\ y_{k-2}^{(1)} \end{pmatrix}; \ \theta = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix}, \ \{\theta_i\}_{i=1,2,3,4} \in \mathbb{R} \tag{1.4}$$

where $\{p_n\}_{n\in\mathbb{N}}$, $\{q_n\}_{n\in\mathbb{N}}$ are real sequences, $\det\theta \neq 0$, $\lambda = 2\sinh\left(\frac{z}{2}\right)$ is a hyperbolic eigenparameter and $\triangle$ is forward operator. The system (1.2) is the discrete analogue of the well-known Dirac system

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} p(x) & 0 \\ 0 & q(x) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

([19], Chap. 2). Therefore the systems (1.1) and (1.2) are called the discrete Dirac system. In this study, we analyze various spectral properties of L; i.e. the spectrum, the scattering function and their properties under the condition

$$\sum_{n=1}^{\infty} n\left(|p_n| + |q_n|\right) < \infty. \tag{1.5}$$

2. Scattering function of L

By following up [10], Eq. (1.2) has the bounded solutions

$$f_n(z) = \begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z) \end{pmatrix} = \left(I_2 + \sum_{m=1}^{\infty} A_{nm} e^{imz}\right)\begin{pmatrix} e^{\frac{z}{2}} \\ 1 \end{pmatrix} e^{nz}, \ n \in \mathbb{N} \tag{2.1}$$

and

$$f_0^{(1)}(z) = \left\{ e^{\frac{z}{2}} \left[1 + \sum_{m=1}^{\infty} A_{0m}^{11} e^{mz}\right] + \sum_{m=1}^{\infty} A_{0m}^{12} e^{mz} \right\} \tag{2.2}$$

under the condition (1.5) for $\lambda = 2\sinh\left(\frac{z}{2}\right)$ and $z \in \overline{\mathbb{C}}_{left} := \{z : z \in \mathbb{C},\ \mathcal{R}\rceil z \leq 0\}$ with $\lim\limits_{n\to\infty} e^{-\left(n+\frac{1}{2}\right)z} f_n^{(1)}(z) = \lim\limits_{n\to\infty} e^{-nz} f_n^{(2)}(z) = 1$, where I_2 is $2x2$ identity matrix and $A_{nm} = \begin{pmatrix} A_{nm}^{11} & A_{nm}^{12} \\ A_{nm}^{21} & A_{nm}^{22} \end{pmatrix}$ is expressed in terms of (p_n) and (q_n), $n \in \mathbb{N}$. Also,

$$\left|A_{nm}^{ij}\right| \leq C \sum_{k=n+\left[\left|\frac{m}{2}\right|\right]}^{\infty} \left(|p_k| + |q_k|\right) \tag{2.3}$$

holds for $i, j = 1, 2$, where $\left[\left|\frac{m}{2}\right|\right]$ is the integer part of $\frac{m}{2}$ and C is a positive constant. Therefore f_n is vector-valued analytic function in $\mathbb{C}_{left} := \{z : z \in \mathbb{C},\ \mathcal{R}\rceil z < 0\}$, continuous on $\mathbb{R}$ and $f_n(z) = f_n(z + 4\pi i)$. Let two semistrips $T_- = \{z \in \mathbb{C} : z = \xi + i\tau,\ \xi < 0,\ \tau \in [-\pi, 3\pi]\}$ and $T = T_- \cup T_0 = \{z \in \mathbb{C} : z = \xi + i\tau,\ \xi \leq 0,\ \tau \in [-\pi, 3\pi]\}$ are defined in $\overline{\mathbb{C}}_{left}$ where $T_0 = \{z \in \mathbb{C} : z = i\tau,\ \tau \in [-\pi, 3\pi]\}$.

After that, if we consider the fundamental solutions of (1.2) $\varphi_n(z) = \begin{pmatrix} \varphi_n^{(1)}(\lambda) \\ \varphi_n^{(2)}(\lambda) \end{pmatrix}$ and $\psi_n(z) = \begin{pmatrix} \psi_n^{(1)}(\lambda) \\ \psi_n^{(2)}(\lambda) \end{pmatrix}$, $n = 0, 1, \ldots, k-1$ for $z \in T$ subject to the initial conditions

$$\begin{array}{ll} \varphi_0^{(1)}(z) = 0 & ,\ \varphi_1^{(2)}(z) = 1\ , \\ \psi_0^{(1)}(z) = 1 & ,\ \psi_1^{(2)}(z) = 0\ , \end{array} \tag{2.4}$$

then $\varphi_n(z)$ and $\psi_n(z)$ are entire for $z \in \mathbb{C}$ and the wronskian of these equations is

$$\begin{aligned} W\left[\varphi_n(z), \psi_n(z)\right] &= \left[\varphi_n^{(1)}(z)\psi_{n+1}^{(2)}(z) - \varphi_{n+1}^{(2)}(z)\psi_n^{(1)}(z)\right] \\ &= \left[\varphi_0^{(1)}(z)\psi_1^{(2)}(z) - \varphi_1^{(2)}(z)\psi_0^{(1)}(z)\right] \\ &= -1. \end{aligned}$$

Furthermore,

$$J_n(z) = \begin{cases} p^{(1)}(z)\varphi_n(z) + p^{(2)}(z)\psi_n(z) & ;\ n = 0, 1, \ldots, k-1 \\ f_n(z) & ;\ n = k+1, k+2, \ldots \end{cases} \tag{2.5}$$

is the Jost solution of L for $z \in T$ with

$$p^{(1)}(z) = -\frac{\det B}{\det \theta} \quad , \quad p^{(2)}(z) = \frac{\det C}{\det \theta}$$

and

$$J_0^{(1)}(z) = \ p^{(2)}(z)$$

where

$$\begin{aligned} B &= \begin{pmatrix} f_{k+1}^{(1)}(z) & f_{k+2}^{(2)}(z) \\ \theta_1 \psi_{k-1}^{(2)}(z) + \theta_2 \psi_{k-2}^{(1)}(z) & \theta_3 \psi_{k-1}^{(2)}(z) + \theta_4 \psi_{k-2}^{(1)}(z) \end{pmatrix}, \\ C &= \begin{pmatrix} f_{k+1}^{(1)}(z) & f_{k+2}^{(2)}(z) \\ \theta_1 \varphi_{k-1}^{(2)}(z) + \theta_2 \varphi_{k-2}^{(1)}(z) & \theta_3 \varphi_{k-1}^{(2)}(z) + \theta_4 \varphi_{k-2}^{(1)}(z) \end{pmatrix} \end{aligned}$$

achieved from the condition (1.4). In addition,

$$\begin{aligned} W\left[f_n(z),\overline{f_n(z)}\right] &= \lim_{n\to\infty}\left[f_n^{(1)}(z)\overline{f_{n+1}^{(2)}(z)} - f_{n+1}^{(2)}(z)\overline{f_n^{(1)}(z)}\right] \\ &= \lim_{n\to\infty}\left[e^{\left(n+\frac{1}{2}\right)z}\overline{e^{(n+1)z}} - e^{(n+1)z}\overline{e^{\left(n+\frac{1}{2}\right)z}}\right] \\ &= e^{\left(n+\frac{1}{2}\right)z}e^{-(n+1)z} - e^{(n+1)z}e^{-\left(n+\frac{1}{2}\right)z} \\ &= e^{-\frac{z}{2}} + e^{\frac{z}{2}} \\ &= 2\cosh\left(\frac{z}{2}\right) \end{aligned}$$

for $z \in T_0 \setminus \{\pi i\}$ and then

$$F_n(z) = \begin{cases} \varphi_n(z) & ;\ n = 0,1,\ldots,k-1 \\ q^{(1)}(z)f_n(z) + q^{(2)}(z)\overline{f_n(z)} & ;\ n = k+1,k+2,\ldots \end{cases} \tag{2.6}$$

be the another solution of L for $z \in T_0 \setminus \{\pi i\}$ where

$$q^{(1)}(z) = -\frac{\det D}{2\cosh\left(\frac{z}{2}\right)} \quad , \quad q^{(2)}(z) = \frac{\det C}{2\cosh\left(\frac{z}{2}\right)}$$

with

$$D = \begin{pmatrix} \overline{f_{k+1}^{(1)}(z)} & \overline{f_{k+2}^{(2)}(z)} \\ \theta_1\varphi_{k-1}^{(2)}(z) + \theta_2\varphi_{k-2}^{(1)}(z) & \theta_3\varphi_{k-1}^{(2)}(z) + \theta_4\varphi_{k-2}^{(1)}(z) \end{pmatrix}.$$

On this occasion, we can find by following (2.5) and (2.6)

$$W\left[J_n(z), F_n(z)\right] = \begin{cases} p^{(2)}(z) & ;\ n = 0,1,\ldots,k-1 \\ -p^{(2)}(z)\det\theta & ;\ n = k+1,k+2,\ldots \end{cases}$$

for $z \in T_0 \setminus \{\pi i\}$ because of $q^{(2)}(z) = -\frac{\det\theta}{2\cosh\left(\frac{z}{2}\right)}p^{(2)}(z)$.

On the other hand, if we consider the unbounded solution of Eq. (1.2) $\widehat{f}_n(z) = \begin{pmatrix} \widehat{f}_n^{(1)}(z) \\ \widehat{f}_n^{(2)}(z) \end{pmatrix}$ for $n = k+1, k+2, \ldots$ with $\lim\limits_{n\to\infty} e^{\left(n+\frac{1}{2}\right)z}\widehat{f}_n^{(1)}(z) = \lim\limits_{n\to\infty} e^{nz}\widehat{f}_n^{(2)}(z) = 1,\ z \in \overline{\mathbb{C}}_{left}$, then

$$W\left[f_n(z),\widehat{f}_n(z)\right] = 2i\cosh\left(\frac{z}{2}\right)$$

for $T \setminus \{\pi i\}$. So we can write the unbounded solution of L is

$$G_n(z) = \begin{cases} \varphi_n(z) & ;\ n = 0,1,\ldots,k-1 \\ r^{(1)}(z)f_n(z) + r^{(2)}(z)\widehat{f}_n(z) & ;\ n = k+1,k+2,\ldots \end{cases} \tag{2.7}$$

for $z \in T$ with

$$r^{(1)}(z) = -\frac{\det E}{2\cosh\left(\frac{z}{2}\right)} \quad , \quad r^{(2)}(z) = \frac{\det C}{2\cosh\left(\frac{z}{2}\right)}$$

where

$$E=\begin{pmatrix} \widehat{f}^{(1)}_{k+1}(z) & \widehat{f}^{(2)}_{k+2}(z) \\ \theta_1\varphi^{(2)}_{k-1}(z)+\theta_2\varphi^{(1)}_{k-2}(z) & \theta_3\varphi^{(2)}_{k-1}(z)+\theta_4\varphi^{(1)}_{k-2}(z) \end{pmatrix}$$

and

$$r^{(2)}(z)=q^{(2)}(z)=-\frac{\det\theta}{2\cosh\left(\frac{z}{2}\right)}p^{(2)}(z) \tag{2.8}$$

for $z\in T_0\setminus\{\pi i\}$.

Theorem 2.1 *For all z in $T_0\setminus\{\pi i\}$, $p^{(2)}(z)\neq 0$.*

Proof Assume that $p^{(2)}(z_0)=0$ for $\exists z_0$ in $T_0\setminus\{\pi i\}$. From (2.6) and (2.8), we can find $F_n(z_0)=0$, $n\in\mathbb{N}\cup\{0\}$ by using the impulsive conditions (1.4). However, this is a contradiction since $F_n(z_0)$ cannot be a trivial solution of L. □

In this step, we define the function

$$S(z)=\frac{\overline{J_0^{(1)}(z)}}{J_0^{(1)}(z)}=\frac{\overline{p^{(2)}(z)}}{p^{(2)}(z)} \tag{2.9}$$

which is called the scattering function of L. Also, it can be written that

$$S(z)=\frac{\overline{f^{(2)}_{k+2}(z)}\left[\theta_1\overline{\varphi^{(2)}_{k-1}(z)}+\theta_2\overline{\varphi^{(1)}_{k-2}(z)}\right]-\overline{f^{(1)}_{k+1}(z)}\left[\theta_3\overline{\varphi^{(2)}_{k-1}(z)}+\theta_4\overline{\varphi^{(1)}_{k-2}(z)}\right]}{f^{(2)}_{k+2}(z)\left[\theta_1\varphi^{(2)}_{k-1}(z)+\theta_2\varphi^{(1)}_{k-2}(z)\right]-f^{(1)}_{k+1}(z)\left[\theta_3\varphi^{(2)}_{k-1}(z)+\theta_4\varphi^{(1)}_{k-2}(z)\right]}$$

and $|S(z)|=1$ because $S(z)=\left[\overline{S(z)}\right]^{-1}$ for $z\in T_0\setminus\{\pi i\}$ from (2.9).

Theorem 2.2

$$\begin{aligned}\sigma_d(L) &= \left\{\lambda\in\mathbb{C}:\lambda=2\sinh\left(\frac{z}{2}\right),\ z\in T_-,\ p^{(2)}(z)=0\right\},\\ \sigma_{ss}(L) &= \varnothing\end{aligned}$$

where $\sigma_d(L)$ and $\sigma_{ss}(L)$ are respectively the sets of eigenvalues and spectral singularities of L.

Proof The first part of the Jost solution $J_n(z)$ sets in finite number of elements and the second component $f_n(z)\in\ell_2\left(\mathbb{N},\mathbb{C}^2\right)$. Therefore, $J_n(z)$ is in $\ell_2\left(\mathbb{N},\mathbb{C}^2\right)$ from (2.5). Moreover

$$0=J_0^{(1)}(z)=p^{(1)}(z)\varphi_0^{(1)}(z)+p^{(2)}(z)\psi_0^{(1)}(z)=p^{(2)}(z)$$

by using the condition (1.3). Hence, we can obtain

$$\sigma_d(L)=\left\{\lambda\in\mathbb{C}:\lambda=2\sinh\left(\frac{z}{2}\right),\ z\in T_-,\ p^{(2)}(z)=0\right\}$$

and

$$\begin{aligned}\sigma_{ss}(L) &= \left\{\lambda \in \mathbb{C} : \lambda = 2\sinh\left(\frac{z}{2}\right),\ z \in T_0 \setminus \{\pi i\},\ p^{(2)}(z) = 0\right\} \\ &= \varnothing\end{aligned}$$

from the definition of spectral singularities and eigenvalues [13] and Theorem 2.1. □

In addition, we can write the Wronskian of $J_n(z)$ and $G_n(z)$ as

$$W\left[J_n(z), G_n(z)\right] = \begin{cases} p^{(2)}(z) & ;\ n = 0, 1, \ldots, k-1 \\ -p^{(2)}(z)\det\theta & ;\ n = k+1, k+2, \ldots \end{cases}$$

for $z \in T$, and in the light of the Theorem 2.2., we need to the quantitative properties of the zeros of $p^{(2)}(z)$ in T_- in order to investigate the quantitative properties of the eigenvalues of L.

3. Some special cases

Let M denote the operator in $\ell_2\left(\mathbb{N}, \mathbb{C}^2\right)$ generated by the unperturbed system

$$\begin{cases} y_{n+1}^{(2)} - y_n^{(2)} = \lambda y_n^{(1)} \\ y_n^{(1)} - y_{n-1}^{(1)} = \lambda y_n^{(2)} \end{cases} ,n \in \mathbb{N} \setminus \{2, 3, 4\} \tag{3.1}$$

and conditions

$$\begin{gathered} y_0^{(1)} = 0\ , \\ \begin{pmatrix} y_4^{(1)} \\ y_5^{(2)} \end{pmatrix} = \theta \begin{pmatrix} y_2^{(2)} \\ y_1^{(1)} \end{pmatrix},\ \theta = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix},\ \{\theta_i\}_{i=1,2,3,4} \in \mathbb{R}, \end{gathered} \tag{3.2}$$

with $\det\theta \neq 0$ and $\lambda = 2\sinh\left(\frac{z}{2}\right)$ is a hyperbolic eigenparameter. At the same time, if $\varphi_n(z) = \begin{pmatrix} \varphi_n^{(1)}(\lambda) \\ \varphi_n^{(2)}(\lambda) \end{pmatrix}$ and $\psi_n(z) = \begin{pmatrix} \psi_n^{(1)}(\lambda) \\ \psi_n^{(2)}(\lambda) \end{pmatrix}$, $n = 0, 1, 2, 3$ are the fundamental solutions of (3.1) for $z \in T$ subject to the initial conditions (2.4) which imply

$$\begin{matrix} \varphi_1^{(1)}(z) = 2\sinh\left(\frac{z}{2}\right) & ,\ \varphi_2^{(2)}(\lambda) = 2\cosh z - 1\ , \\ \psi_1^{(1)}(z) = 1 & ,\ \psi_2^{(2)}(\lambda) = 2\sinh\left(\frac{z}{2}\right) \end{matrix}$$

then

$$J_n(z) = \begin{cases} p^{(1)}(z)\varphi_n(z) + p^{(2)}(z)\psi_n(z) & ;\ n = 0, 1, 2 \\ f_n(z) & ;\ n = 4, 5, 6, \ldots \end{cases}$$

is the Jost solution of M where

$$\begin{aligned} p^{(1)}(z) &= -\frac{1}{\det\theta}\left\{f_5^{(2)}(z)\left[\theta_1\psi_2^{(2)}(z)+\theta_2\psi_1^{(1)}(z)\right]-f_4^{(1)}(z)\left[\theta_3\psi_2^{(2)}(z)+\theta_4\psi_1^{(1)}(z)\right]\right\} \\ &= -\frac{1}{\det\theta}e^{4z}\left[\theta_1 e^{\frac{3z}{2}}-(\theta_3-\theta_2)e^{z}-(\theta_1+\theta_4)e^{\frac{z}{2}}+\theta_3\right] \\ p^{(2)}(z) &= \frac{1}{\det\theta}\left\{f_5^{(2)}(z)\left[\theta_1\varphi_2^{(2)}(z)+\theta_2\varphi_1^{(1)}(z)\right]-f_4^{(1)}(z)\left[\theta_3\varphi_2^{(2)}(z)+\theta_4\varphi_1^{(1)}(z)\right]\right\} \\ &= \frac{1}{\det\theta}e^{\frac{7z}{2}}\left[\theta_1 e^{\frac{5z}{2}}-(\theta_3-\theta_2)e^{2z}-(\theta_1+\theta_4)e^{\frac{3z}{2}}+(\theta_3-\theta_2)e^{z}+(\theta_1+\theta_4)e^{\frac{z}{2}}-\theta_3\right] \end{aligned}$$

and also $f_n(z)=\begin{pmatrix} e^{\left(n+\frac{1}{2}\right)z} \\ e^{nz}\end{pmatrix}$. Moreover, the scattering function of M is

$$\begin{aligned} S(z) &= \frac{\overline{f_5^{(2)}(z)}\left[\theta_1\overline{\varphi_2^{(2)}(z)}+\theta_2\overline{\varphi_1^{(1)}(z)}\right]-\overline{f_4^{(1)}(z)}\left[\theta_3\overline{\varphi_2^{(2)}(z)}+\theta_4\overline{\varphi_1^{(1)}(z)}\right]}{f_5^{(2)}(z)\left[\theta_1\varphi_2^{(2)}(z)+\theta_2\varphi_1^{(1)}(z)\right]-f_4^{(1)}(z)\left[\theta_3\varphi_2^{(2)}(z)+\theta_4\varphi_1^{(1)}(z)\right]} \\ &= e^{-\frac{19z}{2}}\left[\frac{-\theta_3 e^{\frac{5z}{2}}+(\theta_1+\theta_4)e^{2z}+(\theta_3-\theta_2)e^{\frac{3z}{2}}-(\theta_1+\theta_4)e^{z}-(\theta_3-\theta_2)e^{\frac{z}{2}}+\theta_1}{\theta_1 e^{\frac{5z}{2}}-(\theta_3-\theta_2)e^{2z}-(\theta_1+\theta_4)e^{\frac{3z}{2}}+(\theta_3-\theta_2)e^{z}+(\theta_1+\theta_4)e^{\frac{z}{2}}-\theta_3}\right] \end{aligned}$$

can be written from (2.10) for $z\in T_0\setminus\{\pi i\}$, and the eigenvalues of M is

$$\sigma_d(M)=\left\{\lambda\in\mathbb{C}:\lambda=2\sinh\left(\frac{z}{2}\right),\ z\in T_-,\ p^{(2)}(z)=0\right\}.$$

So, we can obtain that

$$\theta_1 e^{\frac{5z}{2}}-(\theta_3-\theta_2)e^{2z}-(\theta_1+\theta_4)e^{\frac{3z}{2}}+(\theta_3-\theta_2)e^{z}+(\theta_1+\theta_4)e^{\frac{z}{2}}-\theta_3=0 \tag{3.3}$$

because $\lambda=2\sinh\left(\frac{z}{2}\right)$ and $p^{(2)}(z)=0$ in $\sigma_d(M)$.

Case 1: Let $\theta=I_2$ where I_2 is $2x2$ identity matrix. From (3.3), we get

$$e^{\frac{5z}{2}}-2e^{\frac{3z}{2}}+2e^{\frac{z}{2}}=0$$

and

$$\begin{aligned} e^{z} &= 1-i\ , \\ e^{z} &= 1+i. \end{aligned}$$

However, there is no roots of these equations in T_-, so $\sigma_d(M)=\varnothing$.

Case 2: If $\theta=\begin{pmatrix} 1 & 0 \\ 1 & 1\end{pmatrix}$, then

$$e^{\frac{5z}{2}}-e^{2z}-2e^{\frac{3z}{2}}+e^{z}+2e^{\frac{z}{2}}-1=0.$$

From this equation,

$$\begin{array}{c} e^{z}=1\ , \\ e^{z}\approx 0.27551\ , \\ e^{z}\approx 2.22074 \end{array} \tag{3.4}$$

can be obtained. The phrase (3.4) gives us, $z_p \approx -1.28913 + i2p\pi$, $p = 0, 1$ which located in T_-. Hence, M has two eigenvalues such as $\lambda_1 = 2\sinh\left(\frac{z_0}{2}\right)$ and $\lambda_2 = 2\sinh\left(\frac{z_1}{2}\right)$.

Case 3: For $\theta = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, it can be found that

$$e^{\frac{5z}{2}} - e^{\frac{3z}{2}} + e^{\frac{z}{2}} - 1 = 0$$

and then

$$\begin{gathered} e^z = 1\ , \\ e^z \approx -0.17509 - 0.69182i\ , \\ e^z \approx -0.17509 + 0.69182i. \end{gathered} \tag{3.5}$$

There are four roots of (3.5) $z_{1,p} \approx -0.33738 - i\,(1.81868 - 2p\pi)$ and $z_{2,p} \approx -0.33738 + i\,(1.81868 + 2p\pi)$, $p = 0, 1$ in T_-, hence the eigenvalues of M is $\sigma_d(M) = \left\{\lambda \in \mathbb{C} : \lambda = 2\sinh\left(\frac{z_{m,p}}{2}\right);\ m = 1, 2;\ p = 0, 1\right\}$.

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