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Near-rings on nearness approximation spaces

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Abstract: In this study, nearness near-ring, subnearness near-ring, nearness M-group and nearness ideal are introduced. By considering operations on the set of all near left weak cosets, nearness near-ring of all near left weak cosets and nearness near-ring homomorphism are also presented. Moreover, some properties of these structures are investigated.

Key words: Nearness ring, nearness near-ring, nearness ideal, weak coset

1. Introduction

Nearness approximation spaces and near sets were introduced in 2007 as generalizations of rough sets [8, 13]. Near set theory begins with the selection of probe functions that provide a basis for describing and discerning affinities between objects in distinct perceptual granules. A probe function is a real-valued function representing a feature of objects such as images.

In the concept of ordinary algebraic structures, the sets consist of abstract points and the sets with binary operations must hold certain axioms given in [1]. In the nearness approximation space, however, the sets are composed of perceptual objects (nonabstract points) instead of abstract points. Perceptual objects are points that have features. These points are describable with feature vectors [8]. Upper approximation of a set is determined by matching descriptions of objects in the set of perceptual objects. In the algebraic structures on nearness approximation spaces, the basic tool is the consideration of upper approximations of the subsets of perceptual objects. In a nearness groupoid, the binary operation must be closed in upper approximation of the set instead of the set.

Near-rings were introduced in 1983 by Pilz as a generalization of rings. In these rings, the addition operation does not need to be commutative as only one distributive law is sufficient. [11].

In 2012, İnan and Öztürk [2, 3] investigated the concept of groups on nearness approximation spaces. In 2013, nearness group of weak cosets are introduced [7]. In 2015, İnan et al. [4] also investigated the nearness semigroups. In 2019, nearness rings are introduced as well [5].

The aim of this study is to introduce nearness near-ring, subnearness near-ring, nearness M-group and nearness ideal. By considering operations on the set of all near left weak cosets, nearness near-ring of all near left weak cosets and nearness near-ring homomorphism are also presented. Moreover, some properties of these structures are investigated.

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2. Preliminaries

Perceptual objects are points that are describable with feature vectors. Let $\mathcal{O}$ be a set of perceptual objects, $X \subseteq \mathcal{O}$, $\mathcal{F}$ be a set of probe functions and $\Phi : \mathcal{O} \longrightarrow \mathbb{R}^L$ be a mapping where the description length is $|\Phi| = L$.

$\Phi(x) = (\varphi_1(x), \varphi_2(x), \varphi_3(x), \cdots, \varphi_i(x), \cdots, \varphi_L(x))$ is an object description of $x \in X$ such that each $\varphi_i \in B \subseteq \mathcal{F}$ ($\varphi_i : \mathcal{O} \longrightarrow \mathbb{R}$) is a probe function that represents features of sample objects $X \subseteq \mathcal{O}$ [8].

Sample objects are near each other if and only if the objects have similar descriptions. Recall that each φ_i defines a description of an object. Δ_{φ_i} is defined by $\Delta_{\varphi_i} = |\varphi_i(x') - \varphi_i(x)|$, where $x, x' \in \mathcal{O}$.

Let $x, x' \in \mathcal{O}$ and $B \subseteq \mathcal{F}$.

$$\sim_B = \{(x, x') \in \mathcal{O} \times \mathcal{O} \mid \Delta_{\varphi_i} = 0 \text{ for all } \varphi_i \in B\}$$

is called the indiscernibility relation on $\mathcal{O}$, where description length is $i \leq |\Phi|$ [8].

Definition 2.1 *[6] Let $\mathcal{O}$ be a set of perceptual objects, Φ be an object description and $A \subseteq \mathcal{O}$. Then the set description of A is defined as*

$$Q(A) = \{\Phi(a) \mid a \in A\}.$$

Definition 2.2 *[6, 10] Let $\mathcal{O}$ be a set of perceptual objects and A, $B \subseteq \mathcal{O}$. Then the descriptive (set) intersection of A and B is defined as*

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B \mid \Phi(x) \in \mathcal{Q}(A) \text{ and } \Phi(x) \in \mathcal{Q}(B)\}.$$

If $Q(A) \cap Q(B) \neq \emptyset$, then A is called descriptively near B and denoted by $A\delta_\Phi B$. Also, $\xi_\Phi(A) = \{B \in \mathcal{P}(\mathcal{O}) \mid A\delta_\Phi B\}$ is a descriptive nearness collection [9].

Definition 2.3 *[8] Let $X \subseteq \mathcal{O}$ and $x \in X$.*

$$[x]_{B_r} = \{x' \in \mathcal{O} \mid x \sim_{B_r} x'\}$$

is called nearness class of $x \in X$.

Definition 2.4 *[8] Let $X \subseteq \mathcal{O}$.*

$$N_r(B)^* X = \bigcup_{[x]_{B_r} \cap X \neq \varnothing} [x]_{B_r}$$

is called upper approximation of X.

A nearness approximation space is $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ where $\mathcal{O}$ is a set of perceptual objects, $\mathcal{F}$ is a set of probe functions, "$\sim_{B_r}$" is an indiscernibility relation relative to $B_r \subseteq B \subseteq \mathcal{F}$, $N_r(B)$ is a collection of partitions and $\nu_{N_r} : \wp(\mathcal{O}) \times \wp(\mathcal{O}) \longrightarrow [0, 1]$ is an overlap function that maps a pair of sets to $[0, 1]$ representing the degree of nearness between sets. The subscript r denotes the cardinality of the restricted subset B_r.

Definition 2.5 *[2] Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ be a nearness approximation space and "$\cdot$" be a binary operation defined on $\mathcal{O}$. $G \subseteq \mathcal{O}$ is called a nearness group if the following properties are satisfied:*

(NG_1) *For all* $x, y \in G$, $x \cdot y \in N_r(B)^* G$,

(NG_2) *For all* $x, y, z \in G$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *property holds in* $N_r(B)^* G$,

(NG_3) *There exists* $e_G \in N_r(B)^* G$ *such that* $x \cdot e_G = e_G \cdot x = x$ *for all* $x \in G$ *(*e_G *is called the near identity element of* G*),*

(NG_4) *There exists* $y \in G$ *such that* $x \cdot y = y \cdot x = e_G$ *for all* $x \in G$ *(*y *is called the near inverse of* x *in* G *and denoted as* x^{-1}*).*

Additionally, if the property $x \cdot y = y \cdot x$ is satisfied in $N_r(B)^* G$ for all $x, y \in G$, then G is said to be a commutative nearness group.

Also, $S \subseteq \mathcal{O}$ is called a nearness semigroup if $x \cdot y \in N_r(B)^* S$ for all $x, y \in S$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property is satisfied in $N_r(B)^* (S)$ for all $x, y, z \in S$.

Theorem 2.6 *[3] Let* G *be a nearness group,* H *be a nonempty subset of* G *and* $N_r(B)^* H$ *be a groupoid. Then* $H \subseteq G$ *is a subnearness group of* G *if and only if* $x^{-1} \in H$ *for all* $x \in H$.

Definition 2.7 *[5] Let* $NAS = (\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ *be a nearness approximation space and "+" and "·" be binary operations defined on* $\mathcal{O}$. $R \subseteq \mathcal{O}$ *is called a nearness ring if the following properties are satisfied:*

(NR_1) R *is an commutative nearness group with binary operation "+",*

(NR_2) R *is a nearness semigroup with binary operation "·",*

(NR_3) *For all* $x, y, z \in R$,

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z) \text{ and } (x + y) \cdot z = (x \cdot z) + (y \cdot z)$$

properties hold in $N_r(B)^* R$.

In addition,

(NR_4) R *is said to be a commutative nearness ring if* $x \cdot y = y \cdot x$ *for all* $x, y \in R$,

(NR_5) R *is said to be a nearness ring with identity if* $N_r(B)^* R$ *contains an element* 1_R *such that* $1_R \cdot x = x \cdot 1_R = x$ *for all* $x \in R$.

Definition 2.8 *[11] Let* N *be a nonempty set and "+" and "·" be binary operations defined on* N. N *is called a (right) near-ring if the following properties are satisfied:*

(N_1) N *is a group with binary operation "+" (It does not need to be commutative),*

(N_2) N *is a semigroup with binary operation "·",*

(N_3) *For all* $x, y, z \in N$, $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ *properties hold in* $N_r(B)^* N$.

Theorem 2.9 *[7] Let* G *be a nearness group,* H *be a subnearness group of* G *and* $G/_{\sim_\ell}$ *be a set of all near left weak cosets of* G *determined by* H. *If* $\left(N_r(B)^* G\right)/_{\sim_\ell} \subseteq N_r(B)^* (G/_{\sim_\ell})$, *then* $G/_{\sim_\ell}$ *is a nearness group with the operation given by* $aH \odot bH = (a \cdot b) H$ *for all* $a, b \in G$.

3. Nearness near-rings

Definition 3.1 *Let* $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ *be a nearness approximation space, "+" and "·" be binary operations defined on* $\mathcal{O}$. $M \subseteq \mathcal{O}$ *is called a near-ring on nearness approximation spaces or shortly nearness near-ring if the following properties are satisfied:*

(NN_1) $(M,+)$ *is a nearness group (it does not need to be commutative),*

(NN_2) $(M,\cdot)$ *is a nearness semigroup,*

(NN_3) *For all* $x,y,z\in M$, $(x+y)\cdot z=(x\cdot z)+(y\cdot z)$ *property holds in* $N_r(B)^* M$.

In addition, M *is called a commutative nearness near-ring if* $x\cdot y=y\cdot x$ *for all* $x,y\in M$ *and* M *is called a nearness near-ring with identity if* $N_r(B)^* M$ *contains* 1_M *such that* $1_M\cdot x=x\cdot 1_M=x$ *for all* $x\in M$.

Since (NN_3), nearness right near-ring can be used instead of nearness near-ring. Furthermore, if we consider the condition $x\cdot(y+z)=(x\cdot y)+(x\cdot z)$ for all $x,y,z\in M$ instead of (NN_3), then it can be called as a nearness left near-ring. Throughout this study nearness near-ring will be used.

Generally, the identity element of the nearness group $(M,+)$ is defined as zero of the nearness near-ring M. Also, the set of all nearness near-rings is denoted by $\mathcal{M}$.

$(NN_1)-(NN_3)$ properties must be satisfied in $N_r(B)^* M$. Sometimes these properties may be hold in $\mathcal{O}\diagdown N_r(B)^* M$, in which case M is not a nearness near-ring. Addition or multiplying of finite number of elements in M may not always belong to $N_r(B)^* M$. As a result, we cannot always say that $nx\in N_r(B)^* M$ or $x^n\in N_r(B)^* M$ for all $x\in M$ and some $n\in\mathcal{Z}^+$.

If $\left(N_r(B)^* M,+\right)$ and $\left(N_r(B)^* M,\cdot\right)$ are groupoids, then we can say that $nx\in N_r(B)^* M$ for all $x\in M$ and all $n\in\mathcal{Z}$ or $x^n\in N_r(B)^* M$ for all $x\in M$ and all $n\in\mathcal{Z}^+$.

Let M be a nearness near-ring with identity. $x\in M$ is said to be a left (resp. right) near invertible if there exists $y\in N_r(B)^* M$ (resp. $z\in N_r(B)^* M$) such that $y\cdot x=1_M$ (resp. $x\cdot z=1_M$). y (resp. z) is called a left (resp. right) near inverse of x. If $x\in M$ is both a left and a right near invertible, then x is said to be a near invertible.

Example 3.2 *Let* $\mathcal{O}=\{a,b,c,d,e,f\}$ *be a set of perceptual objects and* $B=\{\varphi_1,\varphi_2\}\subseteq\mathcal{F}$ *be a set of probe functions. Probe functions*

$$\varphi_1:\mathcal{O}\longrightarrow V_1=\{\alpha_1,\alpha_2\},$$
$$\varphi_2:\mathcal{O}\longrightarrow V_2=\{\beta_1,\beta_2,\beta_3\}$$

are given in Table 1.

	a	b	c	d	e	f
φ_1	α_1	α_1	α_1	α_2	α_2	α_2
φ_2	β_1	β_2	β_2	β_3	β_3	β_3

Table 1.

Let "+" and "·" be binary operations on $\mathcal{O}$ *as in Tables 2 and 3.*

+	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	c	a	f	d	e
c	c	a	b	e	f	d
d	d	e	f	a	b	c
e	e	f	d	c	a	b
f	f	d	e	b	c	a

Table 2.

·	a	b	c	d	e	f
a	a	a	a	a	a	a
b	b	b	b	b	b	b
c	c	c	c	c	c	c
d	d	d	d	d	d	d
e	e	e	e	e	e	e
f	f	f	f	f	f	f

Table 3.

Let $M = \{a, d, e, f\} \subseteq \mathcal{O}$*, "+" and "·" be binary operations on* $M \subseteq \mathcal{O}$ *as in Tables 4 and 5. From Table 4, since* $d + e = b \notin M$ *for* $d, e \in M \subseteq \mathcal{O}$ *is not a group with binary operation "+" and so* M *is not a near-ring.*

+	a	d	e	f
a	a	d	e	f
d	d	a	b	c
e	e	c	a	b
f	f	b	c	a

Table 4.

·	a	d	e	f
a	a	a	a	a
d	d	d	d	d
e	e	e	e	e
f	f	f	f	f

Table 5.

$$\begin{aligned} [a]_{\varphi_1} &= \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_1(a) = \alpha_1\} \\ &= \{a, b, c\} = [b]_{\varphi_1} = [c]_{\varphi_1}, \end{aligned}$$

$$\begin{aligned} [d]_{\varphi_1} &= \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_1(d) = \alpha_2\} \\ &= \{d, e, f\} = [e]_{\varphi_1} = [f]_{\varphi_1}. \end{aligned}$$

Then $\xi_{\varphi_1} = \left\{[a]_{\varphi_1}, [d]_{\varphi_1}\right\}$.

$$\begin{aligned} [a]_{\varphi_2} &= \{x' \in \mathcal{O} \mid \varphi_2(x') = \varphi_2(a) = \beta_1\} \\ &= \{a\}, \end{aligned}$$

$$\begin{aligned} [b]_{\varphi_2} &= \{x' \in \mathcal{O} \mid \varphi_2(x') = \varphi_2(b) = \beta_2\} \\ &= \{b, c\} = [c]_{\varphi_2}, \end{aligned}$$

$$\begin{aligned} [d]_{\varphi_2} &= \{x' \in \mathcal{O} \mid \varphi_2(x') = \varphi_2(d) = \beta_3\} \\ &= \{d, e, f\} = [e]_{\varphi_2} = [f]_{\varphi_2}. \end{aligned}$$

Thus we obtain $\xi_{\varphi_2} = \left\{[a]_{\varphi_2}, [b]_{\varphi_2}, [d]_{\varphi_2}\right\}$*. Therefore the set of partitions of* $\mathcal{O}$ *is* $N_1(B) = \{\xi_{\varphi_1}, \xi_{\varphi_2}\}$ *for* $r = 1$.

In this case, we can write

$$\begin{aligned} N_1(B)^* M &= \bigcup_{[x]_{\varphi_i} \cap M \neq \varnothing} [x]_{\varphi_i} \\ &= \{a, b, c\} \cup \{d, e, f\} \cup \{a\} \\ &= \{a, b, c, d, e, f\} = \mathcal{O}. \end{aligned}$$

From Definition 3.1,

(NN_1) :

(NG_1) $x+y \in N_r(B)^* M$ *for all* $x,y \in M$.

(NG_2) $(x+y)+z = x+(y+z)$ *property holds in* $N_r(B)^* M$ *for all* $x,y,z \in M$.

(NG_3) *There exists* $e_M = a \in N_r(B)^* M$ *such that* $x+e_M = e_M + x = x$ *for all* $x \in M$.

(NG_4) *There exists* $y \in M$ *such that* $x+y = y+x = a$ *for all* $x \in M$, *i.e.* $-a = a$, $-d = d$, $-e = e$ *and* $-f = f$.

Also, since $e+f = b$, $f+e = c$ *and* $b \neq c$ *from Table 4,* $(M,+)$ *is a noncommutative nearness group.*

(NN_2) $(M,\cdot)$ *is a nearness semigroup.*

(NN_3) $(x+y)\cdot z = (x\cdot z)+(y\cdot z)$ *property holds in* $N_r(B)^* M$ *for all* $x,y,z \in M$.

As a result, M *is a nearness near-ring.*

Lemma 3.3 *Every near-ring is a nearness near-ring.*

Proof Let $M \subseteq \mathcal{O}$ be a near-ring. Since $M \subseteq N_r(B)^* M$, then the properties $(NN_1)-(NN_3)$ hold in $N_r(B)^* M$. Therefore M is a nearness near-ring. □

Lemma 3.4 *Every nearness ring is a nearness near-ring.*

Proof Let $M \subseteq \mathcal{O}$ be a nearness ring. From definition of nearness ring, it is easily shown that M is a nearness near-ring. □

Remark 3.5 *Nearness near-ring is not always near-ring, and also nearness near-ring is not always nearness ring.*

Examples 3.6 and 3.7 are show that the opposites of the Lemma 3.3 and Lemma 3.4 are not true.

Example 3.6 *From Example 3.2* M *is a nearness near-ring. But* M *is not a near-ring since* $d+e = b \notin M$ *for* $d,e \in M$.

Example 3.7 *From Example 3.2* M *is a nearness near-ring. But* M *is not a nearness ring since* $d\cdot(e+f) \neq (d\cdot e)+(d\cdot f)$ *for* $d,e,f \in M$.

Lemma 3.8 *Let* $M \subseteq \mathcal{O}$ *be a nearness near-ring and* $0_M \in M$. *If* $0_M \cdot x \in M$ *for all* $x \in M$, *then*

(i) $0_M \cdot x = 0_M$,

(ii) $(-x)\cdot y = -(x\cdot y)$

for all $x,y \in M$.

Proof (i) For all $x \in M$,

$$0_M \cdot x = (0_M + 0_M)\cdot x = 0_M \cdot x + 0_M \cdot x.$$

Since the near identity element is unique, $0_M \cdot x = 0_M$.

(ii) From (i), $0_M \cdot y = 0_M$ for all $y \in M$. Then

$$0_M = 0_M \cdot y = ((-x)+x)\cdot y = (-x)\cdot y + x\cdot y.$$

Since the near inverse element is unique, $(-x) \cdot y = -(x \cdot y)$. □

For all $x, y \in M$, the equalities $x \cdot 0_M = 0_M$ and $x \cdot (-y) = -(x \cdot y)$ may not be provided.

Definition 3.9 *Let M be a nearness near-ring. The set*

$$M_0 = \{x \in M \mid x \cdot 0_M = 0_M\}$$

is called a zero symmetric part of M and the set

$$M_c = \{x \in M \mid x \cdot 0_M = x\}$$

is called a constant part of M.

If $M = M_0$, then M is called a zero symmetric nearness near-ring. If $M = M_c$, then M is called a constant nearness near-ring. The set of all zero symmetric nearness near-rings is denoted by $\mathcal{M}_0$ and the set of all constant nearness near-rings is denoted by $\mathcal{M}_c$.

If the condition $d \cdot (x + y) = d \cdot x + d \cdot y$ holds in $N_r(B)^* M$ for all $x, y \in M$, then d is called a distributive element. Also, the set of all nearness near-ring with the identity is represented as $\mathcal{M}_1$ and the set of all distributive elements in M is represented as M_d. If $M = M_d$, then M is called a distributive nearness near-ring.

Definition 3.10 *Let $(G, +)$ be a nearness group, M be a nearness near-ring and*

$$\eta : N_r(B)^* M \times G \to N_r(B)^* G,\ \eta((x, g)) = xg.$$

(G, η) is called a nearness M-group if $(x + y)g = xg + yg$ and $(x \cdot y)g = x(yg)$ properties are satisfied in $N_r(B)^* G$ for all $g \in G$ and all $x, y \in M$. It is denoted by ${}_MG$ and the set of all nearness M-groups is denoted by ${}_M\mathcal{G}$.

Lemma 3.11 *Every nearness near-ring $(M, +, \cdot)$ is a nearness M-group.*

Definition 3.12 *Let $M \in \mathcal{M}_1$ and ${}_MG \in {}_M\mathcal{G}$. If $1_M g = g$ property holds in $N_r(B)^* G$ for all $g \in G$, then ${}_MG$ is called an unitary nearness M-group.*

Lemma 3.13 *Let M be a nearness near-ring and G be a nearness M-group. Then*

(i) $0_M g = 0_G$ for all $g \in G$.

(ii) $(-x)g = -xg$ for all $g \in G$ and all $x \in M$.

(iii) $x0_G = 0_G$ for all $x \in M_0$.

(iv) $xg = x0_G$ for all $g \in G$ and all $x \in M_c$.

Proof (i) For all $g \in G$,

$$0_M g = (0_M + 0_M)g = 0_M g + 0_M g.$$

Since the near identity element is unique, $0_M g = 0_G$.

(ii) From (i), $0_M g = 0_G$ for all $g \in G$. Then

$$0_G = 0_M g = ((-x) + x)g = (-x)g + xg.$$

Since the near inverse element is unique, $(-x)\,g = -xg$.

(iii) Since $x \cdot 0_M = 0_M$ for all $x \in M_0$,

$$x0_G = x\,(0_M g) = (x \cdot 0_M)\,g = 0_M g = 0_G$$

from (i).

(iv) Since $x \cdot 0_M = x$ for all $x \in M_c$,

$$xg = (x \cdot 0_M)\,g = x\,(0_M g) = x0_G$$

from (i). □

Definition 3.14 *Let M be a nearness near-ring and $(K,+)$ be a subnearness group of $(M,+)$. K is called a subnearness near-ring of M if $K \cdot K \subseteq N_r\,(B)^* K$.*

Example 3.15 *Let M be a nearness near-ring. Then M_0 and M_c are subnearness near-rings of M.*

Theorem 3.16 *Let M be a nearness near-ring, K be a nonempty subset of M and $(N_r\,(B)^* K,+)$, $(N_r\,(B)^* K,\cdot)$ be groupoids. Then K is a subnearness near-ring of M if and only if $-x \in K$ for all $x \in K$.*

Proof ($\Rightarrow$) Let K be a subnearness near-ring of M. Then $(K,+)$ is a nearness group and hence $-x \in K$ for all $x \in K$.

($\Leftarrow$) Let $-x \in K$ for all $x \in K$. Since $\left(N_r\,(B)^* K,+\right)$ is a groupoid, $(K,+)$ is a nearness group from Theorem 2.6. Therefore, since $(N_r\,(B)^* K,\cdot)$ is a groupoid and $K \subseteq M$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $N_r\,(B)^* K$ for all $x,y,z \in K$. Hence $(K,\cdot)$ is a nearness semigroup. Furthermore, since $\left(N_r\,(B)^* K,+\right)$, $\left(N_r\,(B)^* K,\cdot\right)$ are groupoids and M is a nearness near-ring, $(x+y)\cdot z = (x \cdot z) + (y \cdot z)$ property holds in $N_r\,(B)^* K$ for all $x,y,z \in K$. Consequently, K is a subnearness near-ring of M. □

Example 3.17 *From Example 3.2, let $K = \{a,f\} \subseteq M$ with the binary operations as in Tables 6 and 7.*

$+$	a	f
a	a	f
f	f	a

Table 6.

$\cdot$	a	f
a	a	a
f	f	f

Table 7.

For $r=1$, a classification of $\mathcal{O}$ is $N_1\,(B) = \{\xi_{\varphi_1}, \xi_{\varphi_2}\}$ by Example 3.2. Then $N_1\,(B)^* K = \{a,b,c,d,e,f\}$. Hence $(N_r\,(B)^* K,+)$, $(N_r\,(B)^* K,\cdot)$ are groupoids and $-a = a, -f = f \in K$. Therefore K is a subnearness near-ring of M from Theorem 3.16.

Definition 3.18 *Let M be a nearness near-ring, G be a nearness M-group and H be a subnearness group of $(G,+)$. Then H is called a subnearness M-group of G if $H \cdot M \subseteq N_r\,(B)^* H$.*

Definition 3.19 *Let M be a nearness near-ring, I be a subnearness group of $(M,+)$ and $\left(N_r\,(B)^* I,+\right)$, $\left(N_r\,(B)^* I,\cdot\right)$ be groupoids. Then I is called a nearness ideal of M if the following properties are satisfied:*

(1) $I \cdot M \subseteq N_r(B)^* I$,

(2) $x \cdot (y + a) - x \cdot y \in N_r(B)^* I$ *for all* $x, y \in M$ *and all* $a \in I$.

Furthermore, I *is called a right nearness ideal of* M *if only the condition (1) is satisfied. Also,* I *is called a left nearness ideal of* M *if only the condition (2) is satisfied.*

Example 3.20 *From Example 3.2, let we consider the nearness near-ring* M. *Let* $K = \{a, f\} \subseteq M$. *Then* K *is a nearness ideal of* M *with the binary operations "+" and "$\cdot$".*

Definition 3.21 *Let* M *be a nearness near-ring,* G *be a nearness* M*-group and* H *be a subnearness* M*-group of* G. *Then* H *is called a nearness ideal of* G *if* $x(g+h) - xg \in N_r(B)^* H$ *for all* $g \in G$, *all* $h \in H$ *and all* $x \in M$.

Theorem 3.22 *Let* M *be a nearness near-ring,* K_1 *and* K_2 *be two subnearness near-rings of* M *and* $N_r(B)^* K_1$, $N_r(B)^* K_2$ *be groupoids with the binary operations "+" and "$\cdot$". If*

$$\left(N_r(B)^* K_1\right) \cap \left(N_r(B)^* K_2\right) = N_r(B)^* (K_1 \cap K_2),$$

then $K_1 \cap K_2$ *is a subnearness near-ring of* M.

Proof It is straightforward. □

Corollary 3.23 *Let* M *be a nearness near-ring,* $\{K_i : i \in \Delta\}$ *be a nonempty family of subnearness near-rings of* M *and* $N_r(B)^* K_i$ *be groupoids with the binary operations "+" and "$\cdot$" for all* $i \in \Delta$. *If*

$$\bigcap_{i \in \Delta} N_r(B)^* K_i = N_r(B)^* \left(\bigcap_{i \in \Delta} K_i\right),$$

then $\bigcap_{i \in \Delta} K_i$ *is a subnearness near-ring of* M.

4. Nearness near-rings of weak cosets

Let M be a nearness near-ring and K be a subnearness near-ring of M. The relation "$\sim_r$" is defined as

$$x \sim_r y :\Leftrightarrow x + (-y) \in K \cup \{0_M\}$$

where $x, y \in M$.

Theorem 4.1 *Let* M *be a nearness near-ring. Then "$\sim_r$" is a right weak equivalence relation on* M.

Proof Since $(M,+)$ is a nearness group, $-x \in M$ for all $x \in M$. Due to $x + (-x) = 0_M \in K \cup \{0_M\}$, $x \sim_r x$. Let $x \sim_r y$ for all $x, y \in M$. Then $x + (-y) \in K \cup \{0_M\}$, that is $x + (-y) \in K$ or $x + (-y) \in \{0_M\}$. If $x + (-y) \in K$, since $(K,+)$ is a subnearness group, then $-(x + (-y)) = y + (-x) \in K$. Hence $y \sim_r x$. Also, if $x + (-y) \in \{0_M\}$, then $x + (-y) = 0_M$. Therefore $y + (-x) = -(x + (-y)) = -0_M = 0_M$ and so $y \sim_r x$. Consequently, "$\sim_r$" is a right weak equivalence relation on M. □

A class that contains the element $x \in M$ is

$$\tilde{x}_r = \{k + x | k \in K, x \in M, k + x \in M\} \cup \{x\}$$

that is determined by "$\sim_r$".

Definition 4.2 *Let M be a nearness near-ring. A weak class determined by right weak equivalence relation "$\sim_r$" is called a near right weak coset.*

Similarly, the relation "$\sim_\ell$" is defined as

$$x \sim_\ell y :\Leftrightarrow (-x) + y \in K \cup \{0_M\}$$

where $x, y \in M$.

Theorem 4.3 *Let M be a nearness near-ring. Then "$\sim_\ell$" is a left weak equivalence relation on M.*

Proof Since $(M, +)$ is a nearness group, $-x \in M$ for all $x \in M$. Due to $(-x) + x = 0_M \in K \cup \{0_M\}$, $x \sim_\ell x$. Let $x \sim_\ell y$ for all $x, y \in M$. Then $(-x) + y \in K \cup \{0_M\}$, that is, $(-x) + y \in K$ or $(-x) + y \in \{0_M\}$. If $(-x) + y \in K$, since $(K, +)$ is a subnearness group, then $-((-x) + y) = (-y) + x \in K$. Hence $y \sim_\ell x$. Also if $(-x) + y \in \{0_M\}$, then $(-x) + y = 0_M$. Therefore $(-y) + x = -((-x) + y) = -0_M = 0_M$ and so $y \sim_\ell x$. Consequently, "$\sim_\ell$" is a left weak equivalence relation on M. □

A class that contains the element $x \in M$ is

$$\tilde{x}_\ell = \{x + k | k \in K, x \in M, x + k \in M\} \cup \{x\}$$

that is determined by "$\sim_\ell$".

Definition 4.4 *Let M be a nearness near-ring. A class determined by left weak equivalence relation "$\sim_\ell$" is called a near left weak coset.*

We can easily show that $\tilde{x}_r = K + x$ and $\tilde{x}_\ell = x + K$. Nearness group $(K, +)$ may not always be commutative. If $(K, +)$ is a commutative nearness group, $\tilde{x}_r = \tilde{x}_\ell$. Otherwise $\tilde{x}_r \neq \tilde{x}_\ell$.

Let M be a nearness near-ring and K be a subnearness near-ring of M. Then

$$M/_{\sim_\ell} = \{x + K | x \in M\}$$

is a set of all near left weak cosets of M determined by K. If we consider $N_r(B)^* M$ instead of nearness near-ring M

$$\left(N_r(B)^* M\right)/_{\sim_\ell} = \left\{x + K | x \in N_r(B)^* M\right\}.$$

Hence

$$x + K = \left\{x + k | k \in K, x \in N_r(B)^* M, x + k \in M\right\} \cup \{x\}.$$

Definition 4.5 *Let M be a nearness near-ring and K be a subnearness near-ring of M. For $x, y \in M$, let $x + K$ and $y + K$ be two near left weak cosets determined the elements x and y, respectively. Then sum of two near left weak cosets determined by $x + y \in N_r(B)^* M$ can be defined as*

$$\left\{(x + y) + k | k \in K, x + y \in N_r(B)^* M, (x + y) + k \in M\right\} \cup \{x + y\}$$

and denoted by

$$(x+K)\oplus(y+K)=(x+y)+K.$$

Definition 4.6 *Let M be a nearness near-ring and K be a subnearness near-ring of R. For $x,y\in R$, let $x+K$ and $y+K$ be two near left weak cosets that determined the elements x and y, respectively. Then product of two near left weak cosets that determined by $x\cdot y\in N_r(B)^* M$ can be defined as*

$$\left\{(x\cdot y)+k|k\in K, x\cdot y\in N_r(B)^* M,(x\cdot y)+k\in M\right\}\cup\{x\cdot y\}$$

and denoted by

$$(x+K)\odot(y+K)=(x\cdot y)+K.$$

Definition 4.7 *Let $M/_{\sim_\ell}$ be a set of all near left weak cosets of M determined by K and $\xi_\Phi(A)$ be a descriptive nearness collection of $A\in P(\mathcal{O})$. Then*

$$N_r(B)^*(M/_{\sim_\ell})=\bigcup_{\xi_\Phi(A)\underset{\Phi}{\cap} M/_{\sim_\ell}\neq\emptyset}\xi_\Phi(A)$$

is called upper approximation of $M/_{\sim_\ell}$.

Theorem 4.8 *Let M be a nearness near-ring, K be a subnearness near-ring of M and $M/_{\sim_\ell}$ be a set of all near left weak cosets of M determined by K. If*

$$\left(N_r(B)^* M\right)/_{\sim_\ell}\subseteq N_r(B)^*(M/_{\sim_\ell}),$$

then $M/_{\sim_\ell}$ is a nearness near-ring with the operations given by

$$(x+K)\oplus(y+K)=(x+y)+K$$

and

$$(x+K)\odot(y+K)=(x\cdot y)+K$$

for all $x,y\in M$.

Proof (NN_1) Let $\left(N_r(B)^* M\right)/_{\sim_\ell}\subseteq N_r(B)^*(M/_{\sim_\ell})$. Since M is a nearness near-ring, $(M/_{\sim_\ell},\oplus)$ is a nearness group of all near left weak cosets of M determined by K from Theorem 2.9.

(NN_2)

(NS_1) Since $(M,\cdot)$ is a nearness semigroup, $x\cdot y\in N_r(B)^* M$ for all $x,y\in M$ and $(x+K)\odot(y+K)=(x\cdot y)+K\in\left(N_r(B)^* M\right)/_{\sim_\ell}$ for all $(x+K),(y+K)\in M/_{\sim_\ell}$. From the hypothesis, $(x+K)\odot(y+K)=(x\cdot y)+K\in N_r(B)^*(M/_{\sim_\ell})$ for all $(x+K),(y+K)\in M/_{\sim_\ell}$.

(NS_2) Since $(M,\cdot)$ is a nearness semigroup, associative property holds in $N_r(B)^* M$. Hence

$$\begin{aligned}
&((x+K)\odot(y+K))\odot(z+K)\\
=\;&((x\cdot y)+K)\odot(z+K)\\
=\;&((x\cdot y)\cdot z)+K\\
=\;&(x\cdot(y\cdot z))+K\\
=\;&(x+K)\odot((y\cdot z)+K)\\
=\;&(x+K)\odot((y+K)\odot(z+K))
\end{aligned}$$

holds in $\left(N_r(B)^* M\right)/_{\sim_\ell}$ for all $(x+K),(y+K),(z+K) \in M/_{\sim_\ell}$. From the hypothesis, associative property holds in $N_r(B)^*(M/_{\sim_\ell})$. So $(M/_{\sim_\ell}, \odot)$ is a nearness semigroup of all near left weak cosets of M determined by K.

(NN_3) Since M is a nearness near-ring, right distributive property holds in $N_r(B)^* M$ for all $x,y,z \in M$. Then

$$\begin{aligned}
& ((x+K) \oplus (y+K)) \odot (z+K) \\
= & ((x+y)+K) \odot (z+K) \\
= & ((x+y) \cdot z) + K \\
= & ((x \cdot z) + (y \cdot z)) + K \\
= & ((x \cdot z) + K) \oplus ((y \cdot z) + K) \\
= & ((x+K) \odot (z+K)) \oplus ((y+K) \odot (z+K))
\end{aligned}$$

for all $(x+K),(y+K),(z+K) \in M/_{\sim_\ell}$.

Hence, right distributive property holds in $N_r(B)^*(M/_{\sim_\ell})$ by the hypothesis.

Consequently, $M/_{\sim_\ell}$ is a nearness near-ring. □

Definition 4.9 *Let M be a nearness near-ring and K be a subnearness near-ring of M. The nearness near-ring $M/_{\sim_\ell}$ is called a nearness near-ring of all near left weak cosets of M determined by K and denoted by $M/_w K$.*

4.1. Nearness near-ring homomorphisms

Definition 4.10 *Let $M_1, M_2 \subseteq \mathcal{O}$ be two nearness near-rings and*

$$\psi : N_r(B)^* M_1 \to N_r(B)^* M_2$$

be a mapping. If

$$\psi(x+y) = \psi(x) + \psi(y)$$

and

$$\psi(x \cdot y) = \psi(x) \cdot \psi(y)$$

for all $x,y \in M_1$, then ψ is called a nearness near-ring homomorphism. Also, M_1 is called a nearness homomorphic to M_2 and denoted by $M_1 \simeq_n M_2$.

A nearness near-ring homomorphism $\psi : N_r(B)^* M_1 \to N_r(B)^* M_2$ is called

(1) a nearness near-ring monomorphism if η is one-one,

(2) a nearness near-ring epimorphism if η is onto,

(3) a nearness near-ring isomorphism if η is one-one and onto.

Set of all nearness near-ring homomorphisms from $N_r(B)^* M_1$ into $N_r(B)^* M_2$ is denoted by $Hom\left(N_r(B)^* M_1, N_r(B)^* M_2\right)$.

Definition 4.11 *Let $M \subseteq \mathcal{O}$ be a nearness near-ring and G_1, G_2 be two nearness M-groups. Let*

$$\mu : N_r(B)^* G_1 \to N_r(B)^* G_2$$

be a mapping such that

$$\mu(g+h) = \mu(g) + \mu(h)$$

and

$$\mu(xg) = x\mu(g)$$

for all $g, h \in G_1$ *and all* $x \in M$. *Then* μ *is called a nearness* M*-homomorphism. Also,* G_1 *is called near* M*-homomorphic to* G_2 *and denoted by* $G_1 \simeq_n G_2$.

A nearness M*-homomorphism* $\mu : N_r(B)^* G_1 \to N_r(B)^* G_2$ *is called*

(1) a nearness M*-monomorphism if* μ *is one-one,*

(2) a nearness M*-epimorphism if* μ *is onto,*

(3) a nearness M*-isomorphism if* μ *is one-one and onto.*

Set of all nearness M*-homomorphisms from* $N_r(B)^* G_1$ *into* $N_r(B)^* G_2$ *is denoted by* $Hom_M\left(N_r(B)^* G_1, N_r(B)^* G_2\right)$.

Example 4.12 *Let* $M \subseteq \mathcal{O}$ *be a nearness near-ring and* G *be a nearness* M*-group. Let us consider*

$$\begin{array}{rrl} \mu : & N_r(B)^* M \to & N_r(B)^* G \\ & x \mapsto & \mu(x) = xg \end{array}$$

where $g \in G$ *and* $x \in M$. *Then* μ *is a nearness* M*-homomorphism.*

Theorem 4.13 *Let* M_1, M_2 *be two nearness near-rings and* ψ *be a nearness near-ring homomorphism from* $N_r(B)^* M_1$ *into* $N_r(B)^* M_2$. *Then*

(i) $\psi(0_{M_1}) = 0_{M_2}$, *where* $0_{M_2} \in N_r(B)^* M_2$ *is the near zero of* M_2.

(ii) $\psi(-x) = -\psi(x)$ *for all* $x \in M_1$.

Proof (i) Since $0_{M_1} = 0_{M_1} + 0_{M_1}$ and ψ is a nearness near-ring homomorphism, $\psi(0_{M_1}) = \psi(0_{M_1} + 0_{M_1}) = \psi(0_{M_1}) + \psi(0_{M_1})$. Hence $\psi(0_{M_1}) = 0_{M_2}$ by the near identity element is unique.

(ii) $x + (-x) = 0_{M_1}$ for all $x \in M_1$. Then $0_{M_2} = \psi(0_{M_1}) = \psi(x + (-x)) = \psi(x) + \psi(-x)$ by (i). Similarly, $0_{M_2} = \psi(-x) + \psi(x)$ for all $x \in M_1$. Since $\psi(x)$ has a unique near inverse, $\psi(-x) = -\psi(x)$ for all $x \in M_1$. □

Definition 4.14 *Let* $M_1, M_2 \subseteq \mathcal{O}$ *be two nearness near-rings and* ψ *be a nearness near-ring homomorphism from* $N_r(B)^* M_1$ *to* $N_r(B)^* M_2$. *The set*

$$Ker\psi = \{x \in M_1 \mid \psi(x) = 0_{M_2}\}$$

is called a kernel of nearness near-ring homomorphism ψ.

Theorem 4.15 *Let* $M_1, M_2 \subseteq \mathcal{O}$ *be two nearness near-rings and* ψ *be a nearness near-ring homomorphism from* $N_r(B)^* M_1$ *to* $N_r(B)^* M_2$ *and* $\left(N_r(B)^* Ker\,\psi, +\right)$, $\left(N_r(B)^* Ker\,\psi, \cdot\right)$ *be groupoids. Then* $Ker\psi$ *is a subnearness near-ring of* M_1.

Proof Let $x \in Ker\psi$. Then $\psi(x) = 0_{M_2}$. Since $M_1, M_2 \subseteq \mathcal{O}$ are two nearness near-rings, $0_{M_1} \in N_r(B)^* M_1$ and $0_{M_2} \in N_r(B)^* M_2$, $\psi(0_{M_1}) = 0_{M_2}$ by Theorem 4.13 (i). Hence $0_{M_2} = \psi(0_{M_1}) = \psi(x + (-x)) = \psi(x) + \psi(-x)$ and so $\psi(-x) = 0_{M_2}$ from $\psi(x) = 0_{M_2}$. Thus from Definition 4.14, $-x \in Ker\,\psi$. Therefore $Ker\psi$ is a subnearness near-ring of M_1 from Theorem 3.16. □

Theorem 4.16 *Let $M_1, M_2 \subseteq \mathcal{O}$ be two nearness near-rings and ψ be a nearness near-ring homomorphism from $N_r(B)^* M_1$ to $N_r(B)^* M_2$ and $\left(N_r(B)^* Ker\,\psi, +\right)$, $\left(N_r(B)^* Ker\,\psi, \cdot\right)$ be groupoids. If K is a subnearness near-ring of M_1 and*

$$\psi\left(N_r(B)^* K\right) = N_r(B)^* \psi(K),$$

then $\psi(K) = \{\psi(x) \,|x \in K\}$ is a subnearness near-ring of M_2.

Proof Since $M_1, M_2 \subseteq \mathcal{O}$ are two nearness near-rings, $0_{M_1} \in N_r(B)^* M_1$ and $0_{M_2} \in N_r(B)^* M_2$, $\psi(0_{M_1}) = 0_{M_2}$ by Theorem 4.13 (i). Thus $0_{M_2} = \psi(0_{M_1}) \in \psi\left(N_r(B)^* K\right) = N_r(B)^* \psi(K)$. This means that $N_r(B)^* \psi(K) \neq \emptyset$, i.e. $\psi(K) \neq \emptyset$. Since K is a subnearness near-ring of M_1, $-x \in K$ for all $x \in K$ from Theorem 3.16. Therefore $-\psi(x) = \psi(-x) \in \psi(K)$ for all $\psi(x) \in \psi(K)$ by Theorem 4.13 (ii). Consequently, $\psi(K)$ is a subnearness near-ring of M_2 from Theorem 3.16. □

Theorem 4.17 *Let $M_1, M_2 \subseteq \mathcal{O}$ be two nearness near-rings and ψ be a nearness near-ring homomorphism from $N_r(B)^* M_1$ to $N_r(B)^* M_2$ and $\left(N_r(B)^* L, +\right)$, $\left(N_r(B)^* L, \cdot\right)$ be groupoids. If L is a subnearness near-ring of M_2, then $\psi^{-1}(L) = \{x \in M_1 | \psi(x) \in L\}$ is a subnearness near-ring of M_1.*

Proof Let $x \in \psi^{-1}(L)$. Then $\psi(x) \in L$. Since L is a subnearness near-ring of M_2, $-\psi(x) \in L$ from Theorem 3.16. Hence $\psi(-x) \in L$ and so $-x \in \psi^{-1}(L)$ by Theorem 4.13 (ii). Consequently, $\psi^{-1}(L)$ is a subnearness near-ring of M_1 from Theorem 3.16. □

Theorem 4.18 *Let M be a nearness near-ring and K be a subnearness near-ring of M. Then the mapping $\Pi : N_r(B)^* M \to N_r(B)^* (M/_w K)$ defined by $\Pi(x) = x + K$ for all $x \in N_r(B)^* M$ is a nearness near-ring homomorphism.*

Proof From the definition of Π, Definitions 4.5 and 4.6,

$$\Pi(x+y) = (x+y) + K = (x+K) \oplus (y+K) = \Pi(x) \oplus \Pi(y),$$

$$\Pi(x \cdot y) = (x \cdot y) + K = (x+K) \odot (y+K) = \Pi(x) \odot \Pi(y)$$

for all $x, y \in M$. Thus Π is a nearness near-ring homomorphism from Definition 4.10. □

Definition 4.19 *The nearness near-ring homomorphism Π is called a natural nearness near-ring homomorphism from $N_r(B)^* M$ into $N_r(B)^* (M/_w K)$.*

Definition 4.20 *Let $M_1, M_2 \subseteq \mathcal{O}$ be two nearness near-rings and $K \subseteq M_1$. Let*

$$\tau : N_r(B)^* M_1 \longrightarrow N_r(B)^* M_2$$

be a mapping and

$$\tau_{K} = {}^{\tau}\big|_{K} : K \longrightarrow N_r\left(B\right)^{*} M_2$$

be a restricted mapping. If

$$\tau\left(x+y\right) = \tau_{K}\left(x+y\right) = \tau_{K}\left(x\right) + \tau_{K}\left(y\right) = \tau\left(x\right) + \tau\left(y\right)$$

and

$$\tau\left(x\cdot y\right) = \tau_{K}\left(x\cdot y\right) = \tau_{K}\left(x\right)\cdot \tau_{K}\left(y\right) = \tau\left(x\right)\cdot \tau\left(y\right)$$

for all $x, y \in K$, then τ is called a restricted nearness near-ring homomorphism and also, M_1 is called restricted near homomorphic to M_2, denoted by $M_1 \simeq_{rn} M_2$.

Theorem 4.21 *Let $M_1, M_2 \subseteq \mathcal{O}$ be two nearness near-rings and ψ be a nearness near-ring homomorphism from $N_r\left(B\right)^{*} M_1$ to $N_r\left(B\right)^{*} M_2$. Let $\left(N_r\left(B\right)^{*} Ker\tau, +\right)$, $\left(N_r\left(B\right)^{*} Ker\tau, \cdot\right)$ be groupoids and $\left(N_r\left(B\right)^{*} M_1\right)/_{\sim_\ell}$ be a set of all near left weak cosets of $N_r\left(B\right)^{*} M_1$ determined by $Ker\tau$. If*

$$\left(N_r\left(B\right)^{*} M_1\right)/_{\sim_\ell} \subseteq N_r\left(B\right)^{*}\left(M_1/_{\sim_\ell}\right)$$

and

$$N_r\left(B\right)^{*} \tau\left(M_1\right) = \tau\left(N_r\left(B\right)^{*} M_1\right),$$

then

$$M_1/_{\sim_\ell} \simeq_{rn} \tau\left(M_1\right).$$

Proof Since $\left(N_r\left(B\right)^{*} Ker\tau, +\right)$ and $\left(N_r\left(B\right)^{*} Ker\tau, \cdot\right)$ are groupoids, $Ker\tau$ is a subnearness near-ring of M_1 from Theorem 4.15. Since $Ker\tau$ is a subnearness near-ring of M_1 and $\left(N_r\left(B\right)^{*} M_1\right)/_{\sim_\ell} \subseteq N_r\left(B\right)^{*}\left(M_1/_{\sim_\ell}\right)$, then $M_1/_{\sim_\ell}$ is a nearness near-ring of all near left weak cosets of M_1 determined by $Ker\tau$, from Theorem 4.8. Since $N_r\left(B\right)^{*} \tau\left(M_1\right) = \tau\left(N_r\left(B\right)^{*} M_1\right)$, $\tau\left(M_1\right)$ is a subnearness near-ring of M_2 from Theorem 4.16. Let

$$\begin{array}{rcl}\sigma : N_r\left(B\right)^{*}\left(M_1/_{\sim_\ell}\right) & \longrightarrow & N_r\left(B\right)^{*} \tau\left(M_1\right) \\ A & \longmapsto & \sigma(A) = \left\{\begin{array}{ll}\sigma_{M_1/_{\sim_\ell}}\left(A\right) & , A \in \left(N_r\left(B\right)^{*} M_1\right)/_{\sim_\ell} \\ 0_{\tau(M_1)} & , A \notin \left(N_r\left(B\right)^{*} M_1\right)/_{\sim_\ell}\end{array}\right.\end{array}$$

be a mapping where

$$\begin{array}{rcl}\sigma_{M_1/_{\sim_\ell}} = {}^{\sigma}\big|_{M_1/_{\sim_\ell}} : M_1/_{\sim_\ell} & \longrightarrow & N_r\left(B\right)^{*} \tau\left(M_1\right) \\ x + Ker\tau & \longmapsto & \sigma_{M_1/_{\sim_\ell}}\left(x + Ker\tau\right) = \tau\left(x\right)\end{array}$$

for all $x + Ker\tau \in M_1/_{\sim_\ell}$.

Since

$$x + Ker\tau = \left\{x + k \mid k \in Ker\tau, x + k \in M_1\right\} \cup \left\{x\right\},$$

$$y + Ker\tau = \left\{y + k' \mid k' \in Ker\tau, y + k' \in M_1\right\} \cup \left\{y\right\}$$

and the mapping τ is a nearness near-ring homomorphism,

$$\begin{array}{ll} & x+Ker\tau = y+Ker\tau \\ \Rightarrow & x \in y+Ker\tau \\ \Rightarrow & x \in \{y+k' \mid k' \in Ker\tau, y+k' \in M_1\} \text{ or } x \in \{y\} \\ \Rightarrow & x = y+k',\ k' \in Ker\tau,\ y+k' \in M_1 \text{ or } x=y \\ \Rightarrow & -y+x = (-y+y)+k',\ k' \in Ker\tau \text{ or } \tau(x)=\tau(y) \\ \Rightarrow & -y+x = k',\ k' \in Ker\tau \\ \Rightarrow & -y+x \in Ker\tau \\ \Rightarrow & \tau(-y+x) = 0_{\tau(M_1)} \\ \Rightarrow & \tau(-y)+\tau(x) = 0_{\tau(M_1)} \\ \Rightarrow & -\tau(y)+\tau(x) = 0_{\tau(M_1)} \\ \Rightarrow & \tau(x)=\tau(y) \\ \Rightarrow & \sigma_{M_1/\sim_\ell}(x+Ker\tau) = \sigma_{M_1/\sim_\ell}(y+Ker\tau). \end{array}$$

Therefore $\sigma_{M_1/\sim_\ell}$ is well defined.

For $A, B \in N_r(B)^*(M_1/_{\sim_\ell})$, we suppose that $A = B$. Since the mapping $\sigma_{M_1/\sim_\ell}$ is well defined,

$$\begin{aligned} \sigma(A) &= \begin{cases} \sigma_{M_1/\sim_\ell}(A) & , A \in \left(N_r(B)^* M_1\right)/_\sim \\ 0_{\tau(M_1)} & , A \notin \left(N_r(B)^* M_1\right)/_\sim \end{cases} \\ &= \begin{cases} \sigma_{M_1/\sim_\ell}(B) & , B \in \left(N_r(B)^* M_1\right)/_\sim \\ 0_{\tau(M_1)} & , B \notin \left(N_r(B)^* M_1\right)/_\sim \end{cases} \\ &= \sigma(B). \end{aligned}$$

Consequently, σ is well defined.

For all $x+Ker\tau, y+Ker\tau \in M_1/_{\sim_\ell} \subset N_r(B)^*(M_1/_{\sim_\ell})$,

$$\begin{array}{ll} & \sigma((x+Ker\tau) \oplus (y+Ker\tau)) \\ = & \sigma((x+y)+Ker\tau) \\ = & \sigma_{M_1/\sim_\ell}((x+y)+Ker\tau) \\ = & \tau(x+y) \\ = & \tau(x)+\tau(y) \\ = & \sigma_{M_1/\sim_\ell}(x+Ker\tau)+\sigma_{M_1/\sim_\ell}(y+Ker\tau) \\ = & \sigma(x+Ker\tau)+\sigma(y+Ker\tau) \end{array}$$

and

$$\begin{array}{ll} & \sigma((x+Ker\tau) \odot (y+Ker\tau)) \\ = & \sigma((x \cdot y)+Ker\tau) \\ = & \sigma_{M_1/\sim_\ell}((x \cdot y)+Ker\tau) \\ = & \tau(x \cdot y) \\ = & \tau(x) \cdot \tau(y) \\ = & \sigma_{M_1/\sim_\ell}(x+Ker\tau) \cdot \sigma_{M_1/\sim_\ell}(y+Ker\tau) \\ = & \sigma(x+Ker\tau) \cdot \sigma(y+Ker\tau). \end{array}$$

Therefore σ is a restricted nearness near-ring homomorphism by Definition 4.20. Consequently, $M_1/_{\sim_\ell} \simeq_{rn} \tau(M_1)$. □

5. Conclusion

To extend this work, one could study the properties of other algebraic structures on nearness approximation spaces. Hopefully this subject provides a basic framework for some theoretical sciences.

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