TÜBİTAK

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 566 – 578
© TÜBİTAK
doi:10.3906/mat-2008-26

A generalization of parabolic potentials associated to Laplace–Bessel differential operator and its behavior in the weighted Lebesgue spaces

Çağla SEKİN*
Department of Mathematics, Faculty of Science, Akdeniz University, Antalya, TURKEY

Received: 10.08.2020 • **Accepted/Published Online:** 23.12.2020 • **Final Version:** 21.01.2021

Abstract: In this work we introduce some generalizations of the singular parabolic Riesz and parabolic Bessel potentials. Namely, Δ_ν being the Laplace–Bessel singular differential operator, we define the families of operators

$$H^{\alpha}_{\beta,\nu} = \left(\frac{\partial}{\partial t} + (-\Delta_\nu)^{\beta/2}\right)^{-\alpha/\beta} \text{ and } \mathcal{H}^{\alpha}_{\beta,\nu} = \left(I + \frac{\partial}{\partial t} + (-\Delta_\nu)^{\beta/2}\right)^{-\alpha/\beta}, \ (\alpha, \beta > 0),$$

and investigate their properties in the special weighted $L_{p,\nu}$-spaces.

Key words: Laplace–Bessel differential operator, Fourier–Bessel transform, singular parabolic potentials, generalized translation operator, Hardy–Littlewood–Sobolev type inequality

1. Introduction

Singular parabolic Riesz and parabolic Bessel potentials are defined in terms of the Fourier–Bessel transform by

$$(H^{\alpha}_{\nu} f)^{\wedge}(x,t) = \left(|x|^2 + it\right)^{-\alpha/2} f^{\wedge}(x,t) \tag{1.1}$$

and

$$(\mathcal{H}^{\alpha}_{\nu} f)^{\wedge}(x,t) = \left(1 + |x|^2 + it\right)^{-\alpha/2} f^{\wedge}(x,t), \tag{1.2}$$

where $x \in \mathbb{R}^n_+ = \{\xi \mid \xi = (\xi_1, ..., \xi_{n-1}, \xi_n); \xi_n > 0\}$, $|x|^2 = x_1^2 + ... + x_n^2$, $t \in (-\infty, \infty)$.

These potentials are interpretated as negative fractional powers of the singular heat operators $\left(\frac{\partial}{\partial t} - \Delta_\nu\right)$ and $\left(I + \frac{\partial}{\partial t} - \Delta_\nu\right)$, respectively. Here I is the identity operator and $\Delta_\nu = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_k^2} + \frac{2\nu}{x_n}\frac{\partial}{\partial x_n}$ is the Laplace–Bessel singular differential operator.

The singular parabolic potentials $H^{\alpha}_{\nu} f$ and $\mathcal{H}^{\alpha}_{\nu} f$, initially defined by (1.1) and (1.2), can be represented as integral operators

$$\left(H^{\alpha}_{\nu} f\right)(x,t) = \frac{1}{\Gamma(\alpha/2)} \int\limits_{\mathbb{R}^n_+} \int_0^{\infty} \tau^{\frac{\alpha}{2}-1} W_\nu(y,\tau) T^{y,\tau} f(x,t) y_n^{2\nu} dy d\tau \tag{1.3}$$

*Correspondence: caglasekin@akdeniz.edu.tr
2010 *AMS Mathematics Subject Classification:* 47G40, 47G10, 26A33

and

$$\left(\mathcal{H}_\nu^\alpha f\right)(x,t)=\frac{1}{\Gamma(\alpha/2)}\int\limits_{\mathbb{R}_+^n}\int_0^\infty \tau^{\frac{\alpha}{2}-1}e^{-\tau}W_\nu(y,\tau)T^{y,\tau}f(x,t)y_n^{2\nu}dyd\tau, \tag{1.4}$$

where,

$$W_\nu(y,\tau)=\sqrt{c(n,\nu)}(2\tau)^{-\frac{n+2\nu}{2}}\exp(-|y|^2/4\tau),\ y\in\mathbb{R}_+^n,\tau>0 \tag{1.5}$$

is the generalized Gauss–Weierstrass kernel, the operator $T^{y,\tau}$ is the generalized translation associated to the Laplace–Bessel differential operator and

$$c(n,\nu)=\left[(2\pi)^{n-1}2^{2\nu-1}\Gamma^2(\nu+\frac{1}{2})\right]^{-1},\ (\text{see [1, 3, 5, 22]}). \tag{1.6}$$

In the present work we introduce the operators

$$H_{\beta,\nu}^\alpha=\left(\frac{\partial}{\partial t}+(-\Delta_\nu)^{\beta/2}\right)^{-\alpha/\beta} \tag{1.7}$$

and

$$\mathcal{H}_{\beta,\nu}^\alpha=\left(I+\frac{\partial}{\partial t}+(-\Delta_\nu)^{\beta/2}\right)^{-\alpha/\beta},\ (\alpha,\beta>0), \tag{1.8}$$

which have the following integral representations:

$$\left(H_{\beta,\nu}^\alpha f\right)(x,t)=\frac{1}{\Gamma(\alpha/\beta)}\int\limits_{\mathbb{R}_+^n}\int_0^\infty \tau^{\frac{\alpha}{\beta}-1}W_\nu^{(\beta)}(y,\tau)T^{y,\tau}f(x,t)y_n^{2\nu}dyd\tau \tag{1.9}$$

and

$$\left(\mathcal{H}_{\beta,\nu}^\alpha f\right)(x,t)=\frac{1}{\Gamma(\alpha/\beta)}\int\limits_{\mathbb{R}_+^n}\int_0^\infty \tau^{\frac{\alpha}{\beta}-1}e^{-\tau}W_\nu^{(\beta)}(y,\tau)T^{y,\tau}f(x,t)y_n^{2\nu}dyd\tau. \tag{1.10}$$

Here, the kernel function $W_\nu^{(\beta)}(y,\tau)$ is defined as the inverse Fourier–Bessel transform of the function $\exp(-\tau|y|^\beta)$ with respect to $y\in\mathbb{R}_+^n$-variable, i.e.

$$F_\nu\left(W_\nu^{(\beta)}(.,\tau)\right)(y)=e^{-\tau|y|^\beta},\ (y\in\mathbb{R}_+^n,\ \tau>0,\ \beta>0). \tag{1.11}$$

It is clear that in case of $\beta=2$ the integral operators (1.9) and (1.10) coincide with the singular parabolic Riesz and parabolic Bessel potentials (1.3) and (1.4), respectively.

In this work we investigate some properties of the operators (1.9) and (1.10) within the framework of special weighted L_p-spaces, defined as

$$L_{p,\nu}=\left\{f:|\ \|f\|_{p,\nu}\equiv\left(\int\limits_{\mathbb{R}_+^n}\int_{-\infty}^\infty |f(x,t)|^p x_n^{2\nu}dxdt\right)^{1/p}<\infty\right\}. \tag{1.12}$$

Remark 1.1 *The classical parabolic potentials, generated by the ordinary (nonsingular) heat operators* $(-\Delta + \frac{\partial}{\partial t})$ *and* $(I - \Delta + \frac{\partial}{\partial t})$ *were introduced by Sampson [21] and Jones [13].*

Various properties of these potentials and the suitable anisotropic Sobolev-type spaces were studied by many authors: Gopala Rao [11, 12], Chanillo [9], Bagby [8], Sampson [21], Nogin and Rubin [15–17]. Extensive information on this subject can be found in the books [18, 19], see also [4, 7, 20, 24] and references therein. Singular parabolic potentials $H_\nu^\alpha f$ *and* $\mathcal{H}_\nu^\alpha f$ *associated to the singular heat operators* $(-\Delta_\nu + \frac{\partial}{\partial t})$ *and* $(I - \Delta_\nu + \frac{\partial}{\partial t})$ *were introduced by Aliev [1]. The wavelet approach to singular parabolic potentials were introduced by Aliev and Rubin [3] (see, also [5, 22]).*

2. Preliminaries and main results

The Fourier–Bessel and inverse Fourier–Bessel transforms of a function $g(x,t)$, $((x,t) \in \mathbb{R}_+^n \times \mathbb{R}^1)$ are defined by

$$g^\wedge(y,\tau) = \int_{\mathbb{R}_+^n \times \mathbb{R}^1} g(x,t) e^{-i(x'\cdot y' + t\tau)} j_{\nu-\frac{1}{2}}(x_n y_n) d\mu(x) dt, \tag{2.1}$$

$$g^\vee(y,\tau) = c(n,\nu) g^\wedge(-y_1, \cdots, -y_{n-1}, y_n, -\tau), \tag{2.2}$$

where $x' \cdot y' = x_1 y_1 \cdots + x_{n-1} y_{n-1}$; $d\mu(x) = x_n^{2\nu} dx \equiv x_n^{2\nu} dx_1 \cdots dx_n$; $\nu > 0$ is a fixed parameter; $j_\lambda(z) = 2^\lambda \Gamma(\lambda+1) z^{-\lambda} J_\lambda(z)$ is the normalized Bessel function such that $j_\lambda(0) = 1$, $j_\lambda'(0) = 0$ and the normalized coefficient $c(n,\nu) = \left[(2\pi)^{n-1} 2^{2\nu-1} \Gamma^2\left(\nu + \frac{1}{2}\right)\right]^{-1}$ (see, e.g., [1, 3, 14, 22]). Here we actually deal with the ordinary Fourier transform in $x' = (x_1, \cdots, x_{n-1})$ and t variables and the Bessel transform in $x_n > 0$ variable.

The generalized translation operator of $g : \mathbb{R}_+^n \times \mathbb{R} \to \mathbb{C}$ is defined as

$$T^{y,\tau} g(x,t) = \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma(\nu)\Gamma\left(\frac{1}{2}\right)} \int_0^\pi g(x' - y', \sqrt{x_n^2 - 2x_n y_n \cos\theta + y_n^2}; t - \tau) \sin^{2\nu-1}\theta d\theta. \tag{2.3}$$

In fact, the operator $T^{y\cdot\tau}$ is the ordinary (Euclidean) translation in x' and t variable and the Bessel translation in x_n variable.

The relevant convolution is defined by

$$(h_1 \circledast h_2)(x,t) = \int_{\mathbb{R}_+^n \times \mathbb{R}^1} h_1(y,\tau) T^{y,\tau} h_2(x,t) d\mu(y) d\tau. \tag{2.4}$$

It is well known that

$$(h_1 \circledast h_2)^\wedge = h_1^\wedge \cdot h_2^\wedge$$

and

$$\|h_1 \circledast h_2\|_{q,\nu} \le \|h_1\|_{s,\nu} \|h_2\|_{p,\nu}, \ 1 \le p,q,s \le \infty, \ \frac{1}{q} = \frac{1}{p} + \frac{1}{s} - 1. \tag{2.5}$$

The Gauss–Weierstrass kernel associated to the Fourier–Bessel transform (2.1) is defined by

$$W_\nu(y,\tau) = \sqrt{c(n,\nu)} (2\tau)^{-\frac{n+2\nu}{2}} \exp(-|y|^2/4\tau), \ (y \in \mathbb{R}_+^n, \ \tau > 0). \tag{2.6}$$

Here, $c(n,\nu)$ is defined as in (1.6) (see [23] for $n=1$ and [1, 3] for any $n>1$). Note that the kernel function $W_\nu(y,\tau)$ is the inverse Fourier–Bessel transform of the function $e^{-s|x|^2}$ with respect to $x\in\mathbb{R}^n_+$ variable, i.e.

$$\int_{\mathbb{R}^n_+} W_\nu(y,\tau)e^{-ix'\cdot y'}j_{\nu-\frac{1}{2}}(x_n y_n)d\mu(y)=e^{-\tau|x|^2}. \tag{2.7}$$

The generalization of the kernel $W_\nu(y,\tau)$ has been introduced in [6] as the inverse Fourier–Bessel transform of $\exp(-t\,|x|^\beta)$, $\beta>0$. Namely, for a given $\beta>0$ denote

$$W_\nu^{(\beta)}(y,\tau)=(\exp(-\tau\,|\cdot|^\beta))^\vee(y)\equiv c(n,\nu)\int_{\mathbb{R}^n_+} e^{-\tau|x|^\beta}e^{ix'\cdot y'}j_{\nu-\frac{1}{2}}(x_n y_n)d\mu(x). \tag{2.8}$$

We give here some properties of the kernel $W_\nu^{(\beta)}(y,\tau)$ that we will need below.

Lemma 2.1 *(cf. [6]) Let $\beta>0$, $\tau>0$ and $y\in\mathbb{R}^n_+$. Then*

(a) *$W_\nu^{(\beta)}(y,\tau)$ is radial with respect to the variable $y\in\mathbb{R}^n_+$ and has the following anisotropic homogeneity property:*

$$W_\nu^{(\beta)}(\lambda^{1/\beta}y,\lambda\tau)=\lambda^{-(n+2\nu)/\beta}W_\nu^{(\beta)}(y,\tau),\ \lambda>0. \tag{2.9}$$

In particular, for $\lambda=1/\tau$ we have

$$\tau^{-(n+2\nu)/\beta}W_\nu^{(\beta)}(\tau^{-1/\beta}y,1)=W_\nu^{(\beta)}(y,\tau). \tag{2.10}$$

(b) *For $0<\beta\le 2$, the kernel function $W_\nu^{(\beta)}(y,\tau)$ is positive.*

(c) *If $\beta=2k$, $(k\in\mathbb{N})$ then $W_\nu^{(\beta)}(y,\tau)$ is rapidly decreasing as $|y|\to\infty$ and infinitely smooth with respect to y-variable.*

(d) *For any $\tau>0$,*

$$\int_{\mathbb{R}^n_+} W_\nu^{(\beta)}(y,\tau)d\mu(y)=1, \tag{2.11}$$

provided that $0<\beta\le 2$ or $\beta=2k$, $(k\in\mathbb{N})$.

Remark 2.2 *In particular cases of $\beta=1$ and $\beta=2$ the kernel $W_\nu^{(\beta)}(y,\tau)$ can be computed explicitly (see, [2, 10]), namely,*

$$W_\nu^{(1)}(y,\tau)=\frac{2\Gamma((n+2\nu+1)/2)}{\pi^{n/2}\Gamma(\nu+1/2)}\frac{\tau}{(|y|^2+\tau^2)^{(n+2\nu+1)/2}}; \tag{2.12}$$

$$\begin{aligned} W_\nu^{(2)}(y,\tau) &= \sqrt{c(n,\nu)}(2\tau)^{-(n+2\nu)/2}\exp(-|y|^2/4\tau) \\ &= \frac{2\pi^{\nu+1/2}}{\Gamma(\nu+1/2)}(4\pi\tau)^{-(n+2\nu)/2}e^{-|y|^2/4\tau}. \end{aligned} \tag{2.13}$$

(The functions $W_\nu^{(1)}(y,\tau)$ and $W_\nu^{(2)}(y,\tau)$ are named as modified Poisson and Gauss kernels, associated to Laplace–Bessel differential operator Δ_ν).

Remark 2.3 *From now on it will be assumed that,* $W_{\nu}^{(\beta)}(y,\tau)=0$*, for* $\tau\leq 0$*, i.e.*

$$W_{\nu}^{(\beta)}(y,\tau)=\left\{\begin{array}{ll}\left(\exp(-\tau\,|\cdot|^{\beta})\right)^{\vee}(y), & \text{if } \tau>0\\ 0, & \text{if } \tau\leq 0\end{array}\right\}.$$

By taking into account Remark 2.3 and setting

$$\tau_{+}^{\frac{\alpha}{\beta}-1}=\left\{\begin{array}{ll}\tau^{\frac{\alpha}{\beta}-1}, & \text{if } \tau>0\\ 0, & \text{if } \tau\leq 0\end{array}\right\},$$

the formulas (1.9) and (1.10) can be written as generalized convolution:

$$\left(H_{\beta,\nu}^{\alpha}f\right)(x,t)=\frac{1}{\Gamma(\alpha/\beta)}\int\limits_{\mathbb{R}_{+}^{n}\times\mathbb{R}^{1}}\tau_{+}^{\frac{\alpha}{\beta}-1}W_{\nu}^{(\beta)}(y,\tau)T^{y,\tau}f(x,t)d\mu(y)d\tau=(p_{\alpha}\circledast f)(x,t) \tag{2.14}$$

and

$$\left(\mathcal{H}_{\beta,\nu}^{\alpha}f\right)(x,t)=\frac{1}{\Gamma(\alpha/\beta)}\int\limits_{\mathbb{R}_{+}^{n}\times\mathbb{R}^{1}}\tau_{+}^{\frac{\alpha}{\beta}-1}e^{-\tau}W_{\nu}^{(\beta)}(y,\tau)T^{y,\tau}f(x,t)d\mu(y)d\tau=(q_{\alpha}\circledast f)(x,t). \tag{2.15}$$

Here,

$$p_{\alpha}(y,\tau)=\frac{1}{\Gamma(\alpha/\beta)}\tau_{+}^{\frac{\alpha}{\beta}-1}W_{\nu}^{(\beta)}(y,\tau) \tag{2.16}$$

and

$$q_{\alpha}(y,\tau)=\frac{1}{\Gamma(\alpha/\beta)}\tau_{+}^{\frac{\alpha}{\beta}-1}e^{-\tau}W_{\nu}^{(\beta)}(y,\tau). \tag{2.17}$$

If f is a Schwarz test function, we have

$$\left(H_{\beta,\nu}^{\alpha}f\right)^{\wedge}=p_{\alpha}^{\wedge}\cdot f^{\wedge} \text{ and } \left(\mathcal{H}_{\beta,\nu}^{\alpha}f\right)^{\wedge}=q_{\alpha}^{\wedge}\cdot f^{\wedge}.$$

Further,

$$\begin{aligned} p_{\alpha}^{\wedge}(x,t) &= \frac{1}{\Gamma(\alpha/\beta)}\int_{-\infty}^{\infty}\tau_{+}^{\frac{\alpha}{\beta}-1}e^{-it\tau}\left(\int_{\mathbb{R}_{+}^{n}}W_{\nu}^{(\beta)}(y,\tau)e^{-ix'\cdot y'}j_{\nu-\frac{1}{2}}(x_{n}y_{n})d\mu(y)\right)d\tau\\ &= \frac{1}{\Gamma(\alpha/\beta)}\int_{0}^{\infty}\tau^{\frac{\alpha}{\beta}-1}e^{-it\tau}e^{-\tau|x|^{\beta}}d\tau\\ &= \frac{1}{\Gamma(\alpha/\beta)}\int_{0}^{\infty}\tau^{\frac{\alpha}{\beta}-1}e^{-\tau(|x|^{\beta}+it)}d\tau\\ &= (|x|^{\beta}+it)^{-\alpha/\beta}. \end{aligned} \tag{2.18}$$

Similarly, we have

$$q_{\alpha}^{\wedge}(x,t)=(1+|x|^{\beta}+it)^{-\alpha/\beta}. \tag{2.19}$$

(2.18) and (2.19) show that the operators $H_{\beta,\nu}^{\alpha}f$ and $\mathcal{H}_{\beta,\nu}^{\alpha}f$ can be interpreted as fractional $(-\alpha/\beta)th$ powers of the fractional differential operators $((-\Delta_{\nu})^{\beta/2}+\partial/\partial t)$ and $(I+(-\Delta_{\nu})^{\beta/2}+\partial/\partial t)$.

Remark 2.4 *From now on, regarding to the parameter* β *it will be assumed that* $0<\beta\leq 2$ *or* $\beta=2k$, $k\in\mathbb{N}$.

The main results of this study are as follows.

Theorem 2.5 *Let the operators* $\mathcal{H}^{\alpha}_{\beta,\nu}$, $(\alpha,\beta,\nu>0)$ *be defined as in (1.10). Then*

(a) *These operators* $L_{p,\nu}\to L_{p,\nu}$ *are bounded, i.e.*

$$\left\|\mathcal{H}^{\alpha}_{\beta,\nu}f\right\|_{p,\nu}\leq c_{\beta}\left\|f\right\|_{p,\nu},\ (1\leq p\leq\infty);$$

(b) *If* $1\leq p\leq q\leq\infty$ *and* $\alpha>(\beta+n+2\nu)(\frac{1}{p}-\frac{1}{q})$, *then*

$$\left\|\mathcal{H}^{\alpha}_{\beta,\nu}f\right\|_{q,\nu}\leq c_{\beta}(p,q)\left\|f\right\|_{p,\nu};$$

(c) *If* $\alpha>(\beta+n+2\nu)\frac{1}{p}$, *then*

$$\operatorname*{ess\,sup}_{(x,t)\in\mathbb{R}^n_+\times\mathbb{R}^1}\left|\left(\mathcal{H}^{\alpha}_{\beta,\nu}f\right)(x,t)\right|\leq c_{\beta}(p)\left\|f\right\|_{p,\nu}.$$

The next theorem is a generalization of the Hardy–Littlewood–Sobolev theorem for parabolic Riesz potentials, associated to the Laplace–Bessel differential operator.

Theorem 2.6 *Let* $f\in L_{p,\nu}$, $1<p<\infty$ *and* $0<\alpha<(\beta+n+2\nu)\frac{1}{p}$. *Then*

(a) *The integrals* $\left(H^{\alpha}_{\beta,\nu}f\right)(x,t)$ *converge absolutely a.e. in* $\mathbb{R}^n_+\times\mathbb{R}^1$;

(b) *For* $1\leq p<q<\infty$, *the operators* $H^{\alpha}_{\beta,\nu}$ *are of the weak* (p,q)*-type, i.e.*

$$m\left\{(x,t)\in\mathbb{R}^n_+\times\mathbb{R}^1:\left|\left(H^{\alpha}_{\beta,\nu}f\right)(x,t)\right|>\lambda\right\}\leq\left(\frac{c\left\|f\right\|_{p,\nu}}{\lambda}\right)^q,$$

where $\alpha=(\beta+n+2\nu)(\frac{1}{p}-\frac{1}{q})$ *and* $c=c(p,q,n,\nu)>0$. *Here, the measure of a measurable subset* $E\subset\mathbb{R}^n_+\times\mathbb{R}^1$ *is defined by*

$$m\{E\}=\int\limits_E x_n^{2\nu}dxdt.$$

(c) *For* $1<p<q<\infty$, *the operators* $H^{\alpha}_{\beta,\nu}$ *are bounded from* $L_{p,\nu}$ *to* $L_{q,\nu}$ *if and only if*

$$\alpha=(\beta+n+2\nu)(\frac{1}{p}-\frac{1}{q}).$$

Remark 2.7 *In the case of* $\beta=2$, *Theorem 2.6 has been proven in the paper [3] by Aliev and Rubin. For the ordinary (nonsingular) parabolic-type potentials, the analogues of the Theorems 2.5-2.6 were studied by Aliev and Sekin in [7]. Note also that, in the forthcoming studies we plan to apply these results to the characterization of the functional spaces associated to these potential type operators.*

3. Proofs of main results

Proof [of Theorem 2.5] **(a)** Applying the Minkowski inequality to the formula (2.15) and using the inequality

$$\|T^{y,\tau}f\|_{p,\nu} \leq \|f\|_{p,\nu}\,,\ (\forall (y,\tau)\in \mathbb{R}^n_+\times\mathbb{R}^1) \tag{3.1}$$

we have

$$\begin{aligned}\left\|\mathcal{H}^{\alpha}_{\beta,\nu}f\right\|_{p,\nu} &\leq \left(\frac{1}{\Gamma(\alpha/\beta)}\int_{\mathbb{R}^n_+\times\mathbb{R}^1}\tau_+^{\frac{\alpha}{\beta}-1}e^{-\tau}\left|W^{(\beta)}_{\nu}(y,\tau)\right|d\mu(y)d\tau\right)\|f\|_{p,\nu}\\ &= \frac{1}{\Gamma(\alpha/\beta)}\|f\|_{p,\nu}\int_0^\infty \tau^{\frac{\alpha}{\beta}-1}e^{-\tau}\left(\int_{\mathbb{R}^n_+}\left|W^{(\beta)}_{\nu}(y,\tau)\right|d\mu(y)\right)d\tau.\end{aligned}$$

The anisotropic homogeneity property (2.10) gives

$$\begin{aligned}\int_{\mathbb{R}^n_+}\left|W^{(\beta)}_{\nu}(y,\tau)\right|d\mu(y) &= \int_{\mathbb{R}^n_+}\tau^{-(n+2\nu)/\beta}\left|W^{(\beta)}_{\nu}(\tau^{-1/\beta}y,1)\right|y_n^{2\nu}dy\\ (\text{set } y &= \tau^{1/\beta}z)\\ &= \int_{\mathbb{R}^n_+}\left|W^{(\beta)}_{\nu}(z,1)\right|z_n^{2\nu}dz\equiv c_\beta<\infty.\end{aligned}$$

As a result,

$$\left\|\mathcal{H}^{\alpha}_{\beta,\nu}f\right\|_{p,\nu}\leq c_\beta\|f\|_{p,\nu}\,.$$

(Note that, if $0<\beta\leq 2$, then $W^{(\beta)}_{\nu}(y,\tau)$ is positive and therefore, by virtue of (2.11) we have $c_\beta=1$.)

(b) By making use of the generalized Young inequality (2.5) we have from (2.15) that

$$\left\|\mathcal{H}^{\alpha}_{\beta,\nu}f\right\|_{q,\nu}\leq\|q\|_{s,\nu}\|f\|_{p,\nu}\,,\ \left(\frac{1}{q}=\frac{1}{p}+\frac{1}{s}-1\right).$$

Here,

$$\begin{aligned}\|q\|_{s,\nu} &= \frac{1}{\Gamma(\alpha/\beta)}\left(\int_{\mathbb{R}^n_+}\int_0^\infty\left|\tau^{\frac{\alpha}{\beta}-1}e^{-\tau}W^{(\beta)}_{\nu}(y,\tau)\right|^s d\mu(y)d\tau\right)^{\frac{1}{s}}\\ &= \frac{1}{\Gamma(\alpha/\beta)}\left(\int_0^\infty\tau^{s\left(\frac{\alpha}{\beta}-1\right)}e^{-\tau s}\left(\int_{\mathbb{R}^n_+}\left|W^{(\beta)}_{\nu}(y,\tau)\right|^s y_n^{2\nu}dy\right)d\tau\right)^{1/s}.\end{aligned}$$

Using the anisotropic homogeneity property and changing variables as in previous section (a), we have

$$\begin{aligned}\int_{\mathbb{R}^n_+}\left|W^{(\beta)}_{\nu}(y,\tau)\right|^s y_n^{2\nu}dy &= \int_{\mathbb{R}^n_+}\tau^{-s(n+2\nu)/\beta}\left|W^{(\beta)}_{\nu}(\tau^{-1/\beta}y,1)\right|^s y_n^{2\nu}dy\\ &= \int_{\mathbb{R}^n_+}\tau^{-s(n+2\nu)/\beta}\tau^{\frac{n+2\nu}{\beta}}\left|W^{(\beta)}_{\nu}(z,1)\right|^s z_n^{2\nu}dz.\end{aligned}$$

Therefore,

$$\|q\|_{s,\nu} = \frac{1}{\Gamma(\alpha/\beta)} \left(\int_0^\infty \tau^{s\left(\frac{\alpha}{\beta}-1\right)-\frac{(n+2\nu)(s-1)}{\beta}} e^{-s\tau} d\tau \right)^{\frac{1}{s}} \left(\int_{\mathbb{R}_+^n} \left| W_\nu^{(\beta)}(z,1) \right|^s z_n^{2\nu} dz \right)^{\frac{1}{s}}.$$

The last integral on $(0,\infty)$ is finite if and only if

$$\begin{aligned} s\left(\frac{\alpha}{\beta}-1\right) - \frac{1}{\beta}(n+2\nu)(s-1) > -1 \quad &\Leftrightarrow \quad \frac{\alpha}{\beta} - \frac{1}{\beta}(n+2\nu)\left(1-\frac{1}{s}\right) > 1-\frac{1}{s} \\ &\Leftrightarrow \quad \frac{\alpha}{\beta} > \left(1-\frac{1}{s}\right)\left(1+\frac{n+2\nu}{\beta}\right) = \left(\frac{1}{p}-\frac{1}{q}\right)\left(1+\frac{n+2\nu}{\beta}\right) \\ &\Leftrightarrow \quad \alpha > (\beta+n+2\nu)\left(\frac{1}{p}-\frac{1}{q}\right). \end{aligned}$$

This completes the proof of part (b). The part (c) follows from (b) by putting $q=\infty$. □

Proof [of Theorem 2.6] **(a)** We have

$$\left(H_{\beta,\nu}^\alpha f\right)(x,t) = i_1(x,t) + i_2(x,t),$$

where

$$i_1(x,t) = \frac{1}{\Gamma(\alpha/\beta)} \int_{\mathbb{R}_+^n} \int_0^1 \tau^{\frac{\alpha}{\beta}-1} W_\nu^{(\beta)}(y,\tau) T^{y,\tau} f(x,t) d\mu(y) d\tau$$

and

$$i_2(x,t) = \frac{1}{\Gamma(\alpha/\beta)} \int_{\mathbb{R}_+^n} \int_1^\infty \tau^{\frac{\alpha}{\beta}-1} W_\nu^{(\beta)}(y,\tau) T^{y,\tau} f(x,t) d\mu(y) d\tau.$$

By making use of the Minkowski inequality and the anisotropic homogeneity property of $W_\nu^{(\beta)}$ we have

$$\begin{aligned} \|i_1\|_{p,\nu} &\le \frac{1}{\Gamma(\alpha/\beta)} \int_{\mathbb{R}_+^n} \int_0^1 \tau^{\frac{\alpha}{\beta}-1} \left| W_\nu^{(\beta)}(y,\tau) \right| \left\| \left(T^{y,\tau} f\right)(.,.) \right\|_{p,\nu} d\mu(y) d\tau \\ &\overset{(3.1)}{\le} \frac{1}{\Gamma(\alpha/\beta)} \|f\|_{p,\nu} \int_{\mathbb{R}_+^n} \int_0^1 \tau^{\frac{\alpha}{\beta}-1} \left| W_\nu^{(\beta)}(y,\tau) \right| d\mu(y) d\tau \\ &= \frac{1}{\Gamma(\alpha/\beta)} \|f\|_{p,\nu} \int_{\mathbb{R}_+^n} \left| W_\nu^{(\beta)}(z,1) \right| z_n^{2\nu} dz \cdot \int_0^1 \tau^{\frac{\alpha}{\beta}-1} d\tau \\ &= \frac{1}{\Gamma(\frac{\alpha}{\beta}+1)} c_\beta \|f\|_{p,\nu} < \infty, \end{aligned}$$

where $c_\beta = \int_{\mathbb{R}_+^n} \left| W_\nu^{(\beta)}(z,1) \right| z_n^{2\nu} dz < \infty$. Therefore, $i_1(x,t)$ is finite for almost all $(x,t) \in \mathbb{R}_+^n \times (0,\infty)$.

The application of the Hölder inequality yields

$$|i_2(x,t)| \le \frac{1}{\Gamma(\alpha/\beta)} \|f\|_{p,\nu} \left(\int_{\mathbb{R}_+^n} \int_1^\infty \tau^{\left(\frac{\alpha}{\beta}-1\right)q} \left| W_\nu^{(\beta)}(y,\tau) \right|^q d\mu(y) d\tau \right)^{1/q}$$

$$= \frac{1}{\Gamma(\alpha/\beta)} \|f\|_{p,\nu} \left(\int_{\mathbb{R}^n_+} \int_1^\infty \tau^{\left(\frac{\alpha}{\beta}-1\right)q} \tau^{-q\frac{n+2\nu}{\beta}} \left| W_\nu^{(\beta)}(\tau^{-1/\beta},1) \right|^q d\mu(y) d\tau \right)^{1/q}$$

$$= \frac{1}{\Gamma(\alpha/\beta)} \|f\|_{p,\nu} c_{\beta,q} \left(\int_1^\infty \tau^{q\left(\frac{\alpha}{\beta}-1-\frac{n+2\nu}{\beta}\right)+\frac{n+2\nu}{\beta}} d\tau \right)^{1/q}, \tag{3.2}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $c_{\beta,q} = \left(\int_{\mathbb{R}^n_+} \left| W_\nu^{(\beta)}(y,1) \right|^q y_n^{2\nu} dy \right)^{1/q} < \infty$.

The last integral in (3.2) is convergent if and only if

$$q\left(\frac{\alpha}{\beta} - 1 - \frac{n+2\nu}{\beta}\right) + \frac{n+2\nu}{\beta} < -1$$

which is equivalant to the inequality $\alpha < (\beta+n+2\nu)\frac{1}{p}$. As a result, the integrals $\left(H^\alpha_{\beta,\nu} f\right)(x,t)$ converge absolutely for almost all $(x,t) \in \mathbb{R}^n_+ \times (0,\infty)$. The case of $p=1$ is proved by a slight modification:

(b) Let now $1 < p < q < \infty$, $f \in L_{p,\nu}$ and $\alpha = (\beta+n+2\nu)\left(\frac{1}{p}-\frac{1}{q}\right)$. We have to show that $\left\| H^\alpha_{\beta,\nu} f \right\|_{q,\nu} \le c \|f\|_{p,\nu}$, where c does not depend on f. We will use some of the techniques from our paper [7]. Taking into account the expression of the operator $H^\alpha_{\beta,\nu} f$ in formula (1.9), we introduce the function K_ν, K_ν^1 and K_ν^∞ as follows

$$K_\nu \equiv K_\nu(y,\tau) = \frac{1}{\Gamma(\alpha/\beta)} \tau_+^{\frac{\alpha}{\beta}-1} W_\nu^{(\beta)}(y,\tau),\ (y \in \mathbb{R}^n_+, \tau \in \mathbb{R}^1);$$

$$K_\nu^1 = \left\{ \begin{array}{ll} K_\nu, & \tau \le \mu \\ 0, & \tau > \mu \end{array} \right\} \text{ and } K_\nu^\infty = \left\{ \begin{array}{ll} 0, & \tau < \mu \\ K_\nu, & \tau \ge \mu \end{array} \right\},$$

that is $K_\nu = K_\nu^1 + K_\nu^\infty$ (we will choose the parameter μ later).

Everywhere below, the notation $m\{E\}$ will denote the following measure of the set $E = \{(y,\tau) : y \in \mathbb{R}^n_+, \tau \in \mathbb{R}^1\}$:

$$m\{E\} = \int_E y_n^{2\nu} dy d\tau.$$

Let $\lambda > 0$. Then

$$\begin{aligned} m_0(\lambda) &\equiv m\{|K_\nu \circledast f| > 2\lambda\} \equiv m\{(y,\tau) \in \mathbb{R}^n_+ \times \mathbb{R}^1 : |(K_\nu \circledast f)(y,\tau)| > 2\lambda\} \\ &\le m\{\left|K_\nu^1 \circledast f\right| > \lambda\} + m\{|K_\nu^\infty \circledast f| > \lambda\} \equiv m_1(\lambda) + m_\infty(\lambda). \end{aligned} \tag{3.3}$$

The Chebyshev inequality and the Young inequality (2.5) yield

$$\begin{aligned} m_1(\lambda) &\equiv m\{\left|K_\nu^1 \circledast f\right| > \lambda\} = m\{\left|K_\nu^1 \circledast f\right|^p > \lambda^p\} \\ &\le \lambda^{-p} \left\|K_\nu^1 \circledast f\right\|^p_{p,\nu} \le \lambda^{-p} \left\|K_\nu^1\right\|^p_{1,\nu} \|f\|^p_{p,\nu}. \end{aligned} \tag{3.4}$$

Let us calculate $\left\|K_\nu^1\right\|_{1,\nu}$. By making use of the anisotropic homogeneity property (2.10) and then setting $y = \tau^{\frac{1}{\beta}} z$, ($z \in \mathbb{R}^n_+$) we have

$$\left\|K_\nu^1\right\|_{1,\nu} = \frac{1}{\Gamma(\alpha/\beta)} \int_{\mathbb{R}^n_+} \int_0^\mu \tau^{\frac{\alpha}{\beta}-1} \tau^{-(n+2\nu)/\beta} \left| W_\nu^{(\beta)}(\tau^{-1/\beta},1) \right| y_n^{2\nu} dy d\tau$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha/\beta)} \left(\int_{\mathbb{R}^n_+} \left| W_\nu^{(\beta)}(z,1) \right| z_n^{2\nu} dz \right) \int_0^\mu \tau^{\frac{\alpha}{\beta}-1} d\tau \\
&= \frac{1}{\Gamma\left(\frac{\alpha}{\beta}+1\right)} \mu^{\frac{\alpha}{\beta}} c_\beta, \qquad (3.5)
\end{aligned}
$$

where

$$
c_\beta = \int_{\mathbb{R}^n_+} \left| W_\nu^{(\beta)}(z,1) \right| z_n^{2\nu} dz < \infty.
$$

From (3.4) and (3.5) we have

$$
m_1(\lambda) \le A \lambda^{-p} \mu^{\frac{\alpha}{\beta} p}, \qquad (3.6)
$$

where

$$
A = \left(\frac{c_\beta}{\Gamma\left(\frac{\alpha}{\beta}+1\right)} \|f\|_{p,\nu} \right)^p .
$$

Further, the application of Hölder's inequality gives

$$
\|K_\nu^\infty \circledast f\|_\infty \equiv ess\sup \left| K_\nu^\infty \circledast f \right| (y,\tau) \le \|K_\nu^\infty\|_{p',\nu} \|f\|_{p,\nu} , \qquad (3.7)
$$

where $\frac{1}{p'} + \frac{1}{p} = 1$.

Furthermore, using the anisotropic homogeneity property of $W_\nu^{(\beta)}$ and changing variables as in (3.5) we have

$$
\begin{aligned}
\|K_\nu^\infty\|_{p',\nu} &= \frac{1}{\Gamma(\alpha/\beta)} \left(\int_{\mathbb{R}^n_+} \int_\mu^\infty \left(\tau^{\frac{\alpha}{\beta}-1} \left| W_\nu^{(\beta)}(y,\tau) \right| \right)^{p'} y_n^{2\nu} dy d\tau \right)^{1/p'} \\
&= \frac{1}{\Gamma(\alpha/\beta)} \left(\int_\mu^\infty \tau^{\left(\frac{\alpha}{\beta}-1-\frac{n+2\nu}{\beta}\right)p' + \frac{n+2\nu}{\beta}} d\tau \right)^{1/p'} \left(\int_{\mathbb{R}^n_+} \left| W_\nu^{(\beta)}(z,1) \right|^{p'} z_n^{2\nu} dz \right)^{1/p'} .
\end{aligned}
$$

Since $\alpha = (\beta + n + 2\nu)\left(\frac{1}{p} - \frac{1}{q}\right)$ and $\frac{1}{p'} + \frac{1}{p} = 1$, we have

$$
\begin{aligned}
\left(\frac{\alpha}{\beta} - 1 - \frac{n+2\nu}{\beta} \right) p' + \frac{n+2\nu}{\beta} + 1 &= \frac{1}{\beta} \left[(\alpha - \beta - n - 2\nu) p' + \beta + n + 2\nu \right] \\
&= \frac{1}{\beta} \left[\alpha p' - (\beta + n + 2\nu)(p'-1) \right] \\
&= \frac{p'-1}{\beta} \left[\alpha\beta - (\beta + n + 2\nu) \right] \\
&= -\frac{p'-1}{\beta} \frac{p}{q} (\beta + n + 2\nu) \\
&= -\frac{p'}{\beta q} (\beta + n + 2\nu).
\end{aligned}
$$

Then

$$\left(\int_{\mu}^{\infty} \tau^{\left(\frac{\alpha}{\beta}-1-\frac{n+2\nu}{\beta}\right)p'+\frac{n+2\nu}{\beta}} d\tau\right)^{1/p'} = \left(\frac{\beta q}{(\beta+n+2\nu)p'}\right)^{1/p'} \mu^{-\frac{\beta+n+2\nu}{\beta q}},$$

and therefore,

$$\|K_\nu^\infty\|_{p',\nu} \leq \frac{1}{\Gamma(\alpha/\beta)}\left(\frac{\beta q}{(\beta+n+2\nu)p'}\right)^{1/p'} \left\|W_\nu^{(\beta)}(\cdot,1)\right\|_{p',\nu} \mu^{-\frac{\beta+n+2\nu}{\beta q}}.$$

By taking this into account in (3.7) we get

$$\|K_\nu^\infty\|_\infty \leq B\mu^{-\frac{\beta+n+2\nu}{\beta q}}, \tag{3.8}$$

where

$$B = \frac{1}{\Gamma(\alpha/\beta)}\left(\frac{\beta q}{(\beta+n+2\nu)p'}\right)^{1/p'} \left\|W_\nu^{(\beta)}(\cdot,1)\right\|_{p',\nu} \|f\|_{p,\nu}.$$

Now let us choose the parameter μ so that

$$B\mu^{-\frac{\beta+n+2\nu}{\beta q}} = \lambda, \text{ i.e. } \mu = \left(\frac{\lambda}{B}\right)^{-\frac{\beta q}{\beta+n+2\nu}}. \tag{3.9}$$

Then we obtain from (3.8) that

$$\|K_\nu^\infty \circledast f\|_\infty \leq \lambda$$

and therefore, $m_\infty(\lambda) \equiv m\{|K_\nu^\infty \circledast f| > \lambda\} = 0$; see (3.3).

Now, from (3.3) we have

$$\begin{aligned} m_0(\lambda) &\leq m_1(\lambda) \overset{(3.6)}{\leq} A\lambda^{-p}\mu^{\frac{\alpha}{\beta}p} \overset{(3.9)}{=} A\lambda^{-p}\left(\frac{\lambda}{B}\right)^{-\frac{\beta q}{\beta+n+q}\frac{\alpha}{\beta}p} \\ &= AB^{\frac{\alpha pq}{\beta+n+2\nu}}\lambda^{-p-\frac{\alpha pq}{\beta+n+2\nu}}. \end{aligned} \tag{3.10}$$

Setting $\alpha = (\beta+n+2\nu)\left(\frac{1}{p}-\frac{1}{q}\right)$, we have

$$-p - \frac{\alpha pq}{\beta+n+q} = -p - pq\left(\frac{1}{p}-\frac{1}{q}\right) = -q$$

and

$$\begin{aligned} AB^{\frac{\alpha pq}{\beta(\beta+n+2\nu)}} &= \left(\frac{c_\beta}{\Gamma\left(\frac{\alpha}{\beta}+1\right)}\|f\|_{p,\nu}\right)^p \left(\frac{1}{\Gamma(\alpha/\beta)}\frac{\beta q}{(\beta+n+2\nu)p'}\left\|W_\nu^{(\beta)}(\cdot,1)\right\|_{p',\nu}\|f\|_{p,\nu}\right)^{q-p} \\ &= C\|f\|_{p,\nu}^q, \text{where } C \text{ does not depend on } f. \end{aligned}$$

As a result,

$$m\{|K_\nu \circledast f| > 2\lambda\} \leq C\left(\frac{\|f\|_{p,\nu}}{\lambda}\right)^q,$$

and therefore the operator $H^{\alpha}_{\beta,\nu}$ is of the weak (p,q)-type. The case of $p=1$ is proved by a slight modification. From the Marcinkiewicz interpolation theorem it follows that $H^{\alpha}_{\beta,\nu}f$ is of strong (p,q)-type, where $1<p<q<\infty$ and $\alpha=(\beta+n+2\nu)\left(\frac{1}{p}-\frac{1}{q}\right)$.

(c) The "necessity part" of proposition (c) follows from the homogeneity property of the kernel $W^{(\beta)}_{\nu}(y,\tau)$. For completeness, we present a sketch of the proof.

Let $\alpha>0$, $1<p<q<\infty$ and there exist $c>0$ such that

$$\left\|H^{\alpha}_{\beta,\nu}f\right\|_{q,\nu}\le c\left\|f\right\|_{p,\nu},\ \forall f\in L_{p,\nu}. \tag{3.11}$$

Then the inequality

$$\left\|H^{\alpha}_{\beta,\nu}g\right\|_{q,\nu}\le c\left\|g\right\|_{p,\nu} \tag{3.12}$$

must hold for $g(y,\tau)=f(\lambda y,\lambda^{\beta}\tau)$, $\forall\lambda\in(0,\infty)$, as well.

Further,

$$\left\|H^{\alpha}_{\beta,\nu}g\right\|_{q,\nu}=\left(\int_{\mathbb{R}^n_+\times\mathbb{R}^1}\left|\int_{\mathbb{R}^n_+}\int_0^{\infty}\tau^{\frac{\alpha}{\beta}-1}W^{(\beta)}_{\nu}(y,\tau)f(\lambda x-\lambda y,\lambda^{\beta}t-\lambda^{\beta}\tau)y_n^{2\nu}dyd\tau\right|^q x_n^{2\nu}dxdt\right)^{1/q}.$$

Changing variables as $y\to\lambda^{-1}y$, $\tau\to\lambda^{-\beta}\tau$, $x\to\lambda^{-1}x$, $t\to\lambda^{-\beta}t$ and using anisotropic homogeneity property $W^{(\beta)}_{\nu}(\lambda^{-1}y,\lambda^{-\beta}\tau)=\lambda^{n+2\nu}W^{(\beta)}_{\nu}(y,\tau)$ we have

$$\left\|H^{\alpha}_{\beta,\nu}g\right\|_{q,\nu}=\lambda^{-\alpha-\frac{\beta+n+2\nu}{q}}\left\|H^{\alpha}_{\beta,\nu}f\right\|_{q,\nu}. \tag{3.13}$$

On the other hand

$$\left\|g\right\|_{p,\nu}=\left(\int_{\mathbb{R}^n_+\times\mathbb{R}^1}\left|f(\lambda y,\lambda^{\beta}\tau)\right|^p y_n^{2\nu}dyd\tau\right)^{1/p}=\lambda^{-\frac{\beta+n+2\nu}{p}}\left\|f\right\|_{p,\nu}. \tag{3.14}$$

Since $\lambda>0$ is arbitrary, we have from (3.11), (3.12), (3.13) and (3.14) that it must be

$$-\alpha-\frac{\beta+n+2\nu}{q}=-\frac{\beta+n+2\nu}{p},\ \text{i.e. }\ \alpha=(\beta+n+2\nu)\left(\frac{1}{p}-\frac{1}{q}\right).$$

The theorem is completely proved. □

References

[1] Aliev IA. The properties and inversions of B-parabolic potentials. In: Special Problems of Mathematics and Mechanics. Baku, Azerbaijan: Bilik, 1992, pp. 56-75 (in Russian).

[2] Aliev IA, Bayrakci S. On inversion of B-elliptic potentials by the method of Balakrishman-Rubin. Fractional Calculus and Applied Analysis 1998; 1(4): 365-384.

[3] Aliev IA, Rubin B. Parabolic potentials and wavelet transforms with the generalized translation. Studia Mathematica 2001; 145: 1-16.

[4] Aliev IA, Rubin B. Parabolic wavelet transforms and Lebesque spaces of parabolic potentials. Rocky Mountain Journal of Mathematics 2002; 32 (2): 391-408.

[5] Aliev IA, Rubin B, Sezer S, Uyhan SB. Composite wavelet transforms: applications and perspectives. AMS Contemporary Mathematics 2008; 464: 1-25.

[6] Aliev IA, Sağlık E. Generalized Riesz potentials spaces and their characterization via wavelet-type transforms. Filomat 2016; 30 (10): 2809-2823.

[7] Aliev IA , Sekin Ç. A generalization of parabolic Riesz and parabolic Bessel potentials. Rocky Mountain Journal of Mathematics 2020; 50 (3): 815-824.

[8] Bagby R. Lebesgue spaces of parabolic potentials. Illinois Journal of Mathematics 1971; 15 (4): 610-634.

[9] Chanillo S. Hypersingular integrals and parabolic potentials. Transactions of the American Mathematical Society 1981; 267: 531-547.

[10] Gadjiev AD, Aliev IA. Riesz and Bessel potentials generated by a generalized translation and their inverses. In: Proc. IV All-Union Winter Conference, Theory of functions and approximation; Saratov, Russia; 1988. pp. 7-53.

[11] Gopala VR. A characterization of parabolic function spaces. American Journal of Mathematics 1977; 99: 985-993.

[12] Gopala VR. Parabolic function spaces with mixed norm. Transactions of the American Mathematical Society 1978; 246: 451-461.

[13] Jones BF. Lipschitz spaces and heat equation. Journal of Applied Mathematics and Mechanics 1968; 18: 379-410.

[14] Kipriyanov IA. Singular Eliptic Boundary Value Problems. Moscow, Russia: Nauka Fizmatlit, 1997 (in Russian).

[15] Nogin VA, Rubin BS. Inversion and description of parabolic potentials with L_p-densities. Proceedings of the USSR Academy of Sciences 1985; 284: 535-538.

[16] Nogin VA, Rubin BS. Inversion of parabolic potentials with L_p-densities. Matematicheskie Zametki 1986; 38: 831-840.

[17] Nogin VA, Rubin BS. The spaces $L^{\alpha}_{p,r}(\mathbb{R}^{n+1})$ of parabolic potentials. Analysis Mathematica 1987; 13: 321-338 (in Russian).

[18] Samko SG, Kilbas AA, Marichev OI. Fractional Integrals and Derivatives: Theory and applications. New York, NY, USA: Cordon and Breach Science Publishers Inc., 1993.

[19] Samko SG. Hypersingular Integrals and Their Applications. Oxfordshire, UK: Taylor& Francis, 2002.

[20] Samko SG. Fractional powers of operators via hypersingular integrals. In: Balakrishnan AV (editor). Semigroups of Operators: Theory and Applications. Progress in Nonlinear Differential Equations and Their Applications, Vol. 42. Basel, Switzerland: Birkhäuser Verlag AG, 2000, pp. 259-273.

[21] Sampson CH. A characterization of parabolic Lebesgue spaces. PhD, Rice University, Houston, TX, USA, 1968.

[22] Sezer S, Aliev IA. On space of parabolic potentials associated with the singular heat operator. Turkish Journal of Mathematics 2005; 29: 299-319.

[23] Stempak K. La theorie de Littlewood-Paley pour la transformation de Fourier-Bessel. Comptes rendus de l'Académie des Sciences Series I 1986; 303: 15-18 (in French).

[24] Uyhan S, Gadjiev AD, Aliev IA. On approximation properties of the parabolic potentials. Bulletin of the Australian Mathematical Society 2006; 74 (3): 449-460.