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Cyclic and constacyclic codes over the ring $\mathbb{Z}_4[u]/\langle u^3 - u^2 \rangle$ and their Gray images

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Abstract: In this article, the structure of generator polynomial of the cyclic codes with odd length is formed over the ring $\mathbb{Z}_4 + u\mathbb{Z}_4 + u^2\mathbb{Z}_4$ where $u^3 = u^2$. With the isomorphism we have defined, the generator polynomial of constacyclic codes with odd length over this ring is created from the generator of the cyclic codes. Additionally, necessary and sufficient conditions for a linear code in this ring to be a self dual code and a LCD code are mentioned. Furthermore, for all units over this ring, $\mathbb{Z}_4$-images of λ-constacyclic codes and also $\mathbb{Z}_4$-images of cyclic codes are examined by using related ones from defined three new Gray maps. Moreover, several new and optimal codes are constructed in terms of the Lee, Euclidean and Hamming weight in reference to the database.

Key words: Cyclic codes, constacyclic codes, codes over rings, $\mathbb{Z}_4$-linear codes

1. Introduction

In coding theory, linear and cyclic codes which have been studied on various rings for many years have a fabulous algebraic structure. For this reason they have been studied in a wide area with many diverse methods and diverse approaches, for instance in [1, 4–6, 10, 12–15, 18, 19, 21–23]. Cyclic codes have various remarkable generelizations. One of them is constacyclic codes which is an essential class of linear codes due to their rich algebraic structure and have effective applications in multitudinous departments. Berlekamp's work in [6] is one of the important studies that sheds light in coding theory. A lot of good articles have been made over fields and rings in this area. In [23] Wolfmann et al. searched all linear cyclic codes over of odd length whose Gray images are linear codes. Also they have shown that the Nechaev–Gray images of these codes are linear cyclic codes. Thus a new perspective was discovered. Based on this, it can easily be said that there is an extensive literature on constacyclic codes over $\mathbb{Z}_4$ and the extension rings of $\mathbb{Z}_4$ such as in [2, 8–11, 13, 15, 18, 19, 21]. The other generalization of the cyclic codes which is mentioned in our article is quasi-cyclic (QC) codes which was first studied in [2]. QC codes are cyclic codes with an index of 1. These codes are a considerable source of exploring new and good codes. For more information and search about the different opinions of $\mathbb{Z}_4$-codes, we redirect the reader to [2, 16, 20–22].

If we talk about some studies on the ring $\mathbb{Z}_4 + v\mathbb{Z}_4$ when $v^2 = v$; Dinh et al. [11] have identified a new Gray map in this ring and also they analysed the cyclic codes, constacyclic codes for units $1 + 2v$ and $3 + 2v$,

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negacyclic codes, the self dual codes of θ-constacyclic codes. They determined a generator polynomial for cyclic and constacyclic codes with odd length. They gave several examples and found new $\mathbb{Z}_4$-codes. Gao et al. [14] considered the linear codes over this ring and researched Euclidean self dual codes. By taking the Hermitian dual codes, the relationship between the unimodular complex lattices was pointed out. They also constituted generator polynomials by examining cyclic codes over the ring. And finally, by referring to quadratic codes, they found good and new $\mathbb{Z}_4$-linear codes. Kumar et al. [16] concentrated the DNA construction and binary images of DNA codes over the ring $\mathbb{Z}_4+v\mathbb{Z}_4$ when $v^2=v$. Besides all these studies, if we talked about the various rings with 64 elements; Özen et al. [19] investigated the structure of the ring $\mathbb{Z}_4+u\mathbb{Z}_4+u^2\mathbb{Z}_4$ where $u^3=0$ and Galois extensions of this ring. They also studied the ideal structure of these extensions and used their results to obtain for the cyclic codes over this ring. Furthermore, they obtained the minimal spanning set by constructing the generator polynomial of the ring with the help of kernel. They found several new codes according to Lee and Euclidean weight. Islam et al. ([15]) analysed the structure of cyclic and $(1+2u)$-constacyclic codes codes over the ring $\mathbb{Z}_4+u\mathbb{Z}_4+v\mathbb{Z}_4$ where $u^2=v^2=uv=vu=0$. They constituted a minimal spanning set of cyclic codes and also examined $\mathbb{Z}_4$-images of $(1+2u)$-constacyclic codes over the ring using the new maps they defined. The rings studied in these articles are finite commutative rings with unity as well as the ones having only maximal ideal. The most important feature that distinguishes the ring $\mathbb{Z}_4+u\mathbb{Z}_4+u^2\mathbb{Z}_4$ with $u^3=u^2$ we are working on from these rings is the arithmetic of the ring and the ring having more than one maximal ideal. In addition to the fact that the structure of the ring is different, the main purpose in our study of this ring is to discover the region that has not been scanned until now and to find some of the new codes there.

In this paper, we primarily focus on cyclic and λ-constacyclic codes over $\mathbb{Z}_4+u\mathbb{Z}_4+u^2\mathbb{Z}_4$ with $u^3=u^2$. This paper is organised as follows: In Section 2, we give the basic notions over the ring $\mathbb{Z}_4+u\mathbb{Z}_4+u^2\mathbb{Z}_4$ with $u^3=u^2$ and also identify three new Gray maps for this ring. In Section 3, using the decomposition method we create generator polynomials and idempotent polynomials for this ring with odd length. We also determine the necessary and sufficient conditions for linear code C_3 over this ring to be a self-dual code and a LCD code. Then utilizing these maps we observe that the $\mathbb{Z}_4$-images of λ-constacyclic codes of $\mathbb{Z}_4+u\mathbb{Z}_4+u^2\mathbb{Z}_4$ with $u^3=u^2$ are cyclic codes or quasicyclic codes with index 2 over $\mathbb{Z}_4$ for each unit of this ring. In Section 4, by using the Lee, Euclidean and Hamming weight we present many examples with tables via computational results. By accepting to the online database as a mainstay and using the $\mathbb{Z}_4$-images of cyclic codes over T_3, we find new codes which have 245 different generators with $(14,4^32^2,4_L),(14,4^32^2,8_E)$ and $(14,4^32^2,2_H)$ parameters; have 3 different generators with $(14,4^12^0,14_E)$ and $(14,4^12^0,14_H)$ parameters. Besides we obtain 342 different generators with $(14,4^32^4,4_L)$ and $(14,4^32^4,2_H)$ parameters; 1 generator with $(14,4^32^0,8_E)$ parameters; 22 different generators with $(14,4^12^6,4_L)$ and $(14,4^12^6,2_H)$ parameters. In reference to the online database, these parameters are new. Moreover we find optimal code via 1 different generator with the $(14,4^32^0,8_H)$ parameter; 2 different generators with $(14,4^12^3,12_L)$, $(14,4^12^3,14_E)$ and $(14,4^12^3,6_H)$ parameters; 2 different generators with $(14,4^02^4,24_E)$ and $(14,4^02^4,6_H)$ parameters; 6 different generators with $(14,4^32^3,8_L)$ and $(14,4^32^3,4_H)$ parameters; 8 different generators with $(14,4^02^6,8_L)$, $(14,4^02^6,16_E)$ and $(14,4^02^6,4_H)$ parameters; 352 different generators with $(14,4^32^1,8_L)$ and $(14,4^32^1,6_H)$ parameters; 7 different generators with $(14,4^02^7,4_L)$ and $(14,4^02^7,2_H)$ parameter. However, we do not write all of these examples in order to avoid density. Apart from this we attain 14 different new parameters by way of the $\mathbb{Z}_4$-images of $(1+2u^2)$-constacyclic codes over T_3. These are the followings: 3 different generators with $(14,4^42^3,4_L)$, $(14,4^42^3,6_E)$ and $(14,4^42^3,2_H)$ parameters; 1 generator with $(14,4^42^0,6_L)$ and $(14,4^42^0,6_E)$ parameter; 3 different generators with $(14,4^62^1,4_L)$, $(14,4^62^1,4_E)$ and

$(14, 4^6 2^1, 2_H)$ parameters; 3 different generators with $(14, 4^3 2^4, 4_L)$ and $(14, 4^3 2^4, 2_H)$ parameters; 1 generator with $(14, 4^3 2^1, 4_H)$ parameter; 3 generator with parameters $(14, 4^1 2^6, 4_L)$ and $(14, 4^1 2^6, 2_H)$. Again using the constacyclic codes, we find the optimal code with the parameter $(14, 4^4 2^0, 6_H)$ by means of 2 different generators; the parameters $(14, 4^3 2^3, 8_L)$ and $(14, 4^3 2^3, 4_H)$ via 1 generator; the parameters $(14, 4^0 2^7, 4_L)$ and $(14, 4^0 2^7, 2_H)$ through 2 different generators; the parameters $(14, 4^1 2^3, 12_L)$, $(14, 4^1 2^3, 14_E)$ and $(14, 4^1 2^3, 6_H)$ via 2 different generators; the parameters $(14, 4^3 2^1, 8_L)$ and $(14, 4^3 2^1, 6_H)$ through 5 different generators. The remaining codes are the best known $\mathbb{Z}_4$-linear codes determined in accordance with the online database*.

2. Preliminaries

In this paper we investigate the structure of the ring $\mathbb{Z}_4 + u\mathbb{Z}_4 + u^2\mathbb{Z}_4$ where $u^3 = u^2$. This ring is a finite commutative nonchain ring with identity and a nonlocal ring since it does not have a single maximal ideal. And also the set of units of the ring $\mathbb{Z}_4 + u\mathbb{Z}_4 + u^2\mathbb{Z}_4$ with $u^3 = u^2$ is $\{1, 3, 1+2u, 3+2u, 1+u+u^2, 3+u+u^2, 1+3u+u^2, 3+3u+u^2, 1+2u^2, 3+2u^2, 1+2u+2u^2, 3+2u+2u^2, 1+u+3u^2, 3+u+3u^2, 1+3u+3u^2, 3+3u+3u^2\}$. Throughout this paper, we will represent this ring by T_3. The ring T_3 is isomorphic to the quotient ring $\mathbb{Z}_4[u]/\langle u^3 - u^2 \rangle$.

Let z be any element of T_3 which can be expressed as $z = a_0 + ua_1 + u^2 a_2$ for each $a_i \in \mathbb{Z}_4$ and $i = 0, 1, 2$. A code of length m over T_3 is subset of T_3^m. C_3 is a linear iff C_3 is a T_3- submodule of T_3^m. The elements of the linear code are called codewords. Let σ and ρ_λ be maps from T_3^m to T_3^m given by $\sigma(z_0, z_1, \dots, z_{m-1}) = (z_{m-1}, z_0, z_1, \dots, z_{m-2})$ and $\rho_\lambda(z_0, z_1, \dots, z_{m-1}) = (\lambda z_{m-1}, z_0, z_1, \dots, z_{m-2})$, respectively. Let C_3 be a linear code of length m over T_3. Then C_3 is said to be cyclic if $\sigma(C_3) = C_3$, λ-constacyclic if $\rho_\lambda(C_3) = C_3$ where λ is unit over T_3. Each codeword $z = (z_0, z_1, \dots, z_{m-1})$ is symbolized via its polynomial form $z(x) = z_0 + z_1 x + \dots + z_{m-1}x^{m-1}$ for each $z_i = a_0^i + ua_1^i + u^2 a_2^i$ while $i = 0, 1, \dots, m-1$. We describe three new Gray maps for T_3 as follows: Our first Gray map is

$$\varphi_1 : T_3 \longrightarrow \mathbb{Z}_4^2$$

$$(a_0 + ua_1 + u^2 a_2) \to (a_0 + a_1 + 3a_2, 3a_0 + 3a_1 + a_2).$$

This map is extended component-wise to

$$\Phi_1 : T_3^m \mapsto \mathbb{Z}_4^{2m}$$

$$(z_0, z_1, \dots, z_{m-1}) \mapsto (a_0^0 + a_1^0 + 3a_2^0, \dots, a_0^{m-1} + a_1^{m-1} + 3a_2^{m-1}, 3a_0^0 + 3a_1^0 + a_2^0, \dots, 3a_0^{m-1} + 3a_1^{m-1} + a_2^{m-1})$$

where $z_i = a_0^i + ua_1^i + u^2 a_2^i$ for $i = 0, \dots, m-1$.

The second Gray map is as follows:

$$\varphi_2 : T_3 \longrightarrow \mathbb{Z}_4^2$$

$$(a_0 + ua_1 + u^2 a_2) \to (a_0 + a_1 + 3a_2, a_0 + 3a_1 + a_2)$$

*Asamov T (2007). Database of Z4 codes [online]. Website https://www.Z4Codes.info [accessed 22 October 2020]

This map is extended component-wise to

$$\Phi_2 : T_3^m \mapsto \mathbb{Z}_4^{2m}$$

$$(z_0, z_1, \ldots, z_{m-1}) \mapsto (a_0^0 + a_1^0 + 3a_2^0, \ldots, a_0^{m-1} + a_1^{m-1} + 3a_2^{m-1}, a_0^0 + 3a_1^0 + a_2^0, \ldots, a_0^{m-1} + 3a_1^{m-1} + a_2^{m-1})$$

where $z_i = a_0^i + ua_1^i + u^2 a_2^i$ for $i = 0, \ldots, m-1$.

And the last Gray map defined is as follows:

$$\varphi_3 : T_3 \longrightarrow \mathbb{Z}_4^2$$

$$(a_0 + ua_1 + u^2 a_2) \to (a_0 + a_1 + 3a_2, 3a_0 + a_1 + 3a_2)$$

This map is extended component-wise to

$$\Phi_3 : T_3^m \mapsto \mathbb{Z}_4^{2m}$$

$$(z_0, z_1, \ldots, z_{m-1}) \mapsto (a_0^0 + a_1^0 + 3a_2^0, \ldots, a_0^{m-1} + a_1^{m-1} + 3a_2^{m-1}, 3a_0^0 + a_1^0 + 3a_2^0, \ldots, 3a_0^{m-1} + a_1^{m-1} + 3a_2^{m-1})$$

where $z_i = a_0^i + ua_1^i + u^2 a_2^i$ for $i = 0, \ldots, m-1$.

Definition 2.1 *The Euclidean weight of $\beta \in \mathbb{Z}_4$ is defined as $w_E(\beta) = min\{\beta^2, (4-\beta)^2\}$ and the Lee weight is defined as $w_L(\beta) = min\{|\beta|, |4-\beta|\}$.*

Both the Euclidean, Lee and Hamming weight of any element of $z \in T_3^m$ is defined as the sum of the first and second components of both the Euclidean, Lee and Hamming weight of the $\Phi_i(z)$ with $i = 1, 2, 3$.

The Hamming weight of $\alpha \in T_3^m$ is $w_H(\alpha) = |\{i : \alpha_i \neq 0\}|$ and the Hamming distance is $d_H(\alpha, \kappa) = |\{i|\alpha_i \neq \kappa_i\}|$ such that $\alpha, \kappa \in T_3^m$. The Lee, Euclidean and Hamming distance are defined as $d_\epsilon(\alpha, \kappa) = w_\epsilon(\alpha - \kappa)$ which is between two codewords α and κ over T_3^m such that $\epsilon = L, E, H$.

Theorem 2.2 *The maps Φ_1, Φ_2 and Φ_3 are $\mathbb{Z}_4$-linear and distance-preserving from T_3^m to $\mathbb{Z}_4^{2m}$ according to Lee, Euclidean and Hamming distances.*

Proof Let $h = (h_0, h_1, \ldots, h_{m-1}), s = (s_0, s_1, \ldots, s_{m-1}) \in T_3^m$ such that $a_j^i, b_j^i \in \mathbb{Z}_4$, $h_i = a_0^i + ua_1^i + u^2 a_2^i$, $s_i = b_0^i + ub_1^i + u^2 b_2^i$ where $i = 0, 1, \ldots, m-1$ and $j = 0, 1, 2$.

For any $k_1, k_2 \in \mathbb{Z}_4$, $\Phi_1(k_1 h + k_2 s) = \Phi_1(k_1(a_0^0 + ua_1^0 + u^2 a_2^0, \ldots, a_0^{m-1} + ua_1^{m-1} + u^2 a_2^{m-1}) + k_2(b_0^0 + ub_1^0 +$

$u^2 b_2^0, \ldots, b_0^{m-1} + ub_1^{m-1} + u^2 b_2^{m-1}))$

$$= \Phi_1((k_1 a_0^0 + k_2 b_0^0) + u(k_1 a_1^0 + k_2 b_1^0) + u^2(k_1 a_2^0 + k_2 b_2^0), \ldots, (k_1 a_0^{m-1} +$$

$k_2b_0^{m-1}) + u(k_1a_1^{m-1} + k_2b_1^{m-1}) + u^2(k_1a_2^{m-1} + k_2b_2^{m-1}))$

$$= (k_1a_0^0 + k_2b_0^0 + k_1a_1^0 + k_2b_1^0 + 3k_1a_2^0 + 3k_2b_2^0, 3k_1a_0^0 + 3k_2b_0^0 + 3k_1a_1^0 +$$

$$3k_2b_1^0 + k_1a_2^0 + k_2b_2^0, \dots, k_1a_0^{m-1} + k_2b_0^{m-1} + k_1a_1^{m-1} + k_2b_1^{m-1} + 3k_1a_2^{m-1} + 3k_2b_2^{m-1}, 3k_1a_0^{m-1} + 3k_2b_0^{m-1} +$$

$$3k_1a_1^{m-1} + 3k_2b_1^{m-1} + k_1a_2^{m-1} + k_2b_2^{m-1})$$

$$= k_1\Phi_1(h) + k_2\Phi_1(s).$$

So, Φ_1 is a linear.

Since $h - s = (h_0 - s_0, \dots, h_{m-1} - s_{m-1})$ and Φ_1 is a linear, we get $\Phi_1(h - s) = \Phi_1(h) - \Phi_1(s)$. Using the definition of Lee distance, we obtain $d_L(h, s) = w_L(h - s) = \Sigma_{i=0}^{m-1}(h_i - s_i) = \Sigma_{i=0}^{m-1} w_L((a_0^i + ua_1^i + u^2a_2^i) - (b_0^i + ub_1^i + u^2b_2^i)$. If we edit this statement, then we get $d_L(h, s) = \Sigma_{i=0}^{m-1} w_L((a_0^i - b_0^i) + u(a_1^i - b_1^i) + u^2(a_2^i - b_2^i)$. This means that $d_L(h, s) = \Sigma_{i=0}^{m-1} w_L(a_0^i - b_0^i + a_1^i - b_1^i + 3a_2^i - 3b_2^i) + w_L(3a_0^i - 3b_0^i + 3a_1^i - 3b_1^i + a_2^i - b_2^i)$.

On the other hand, we have $w_L(\Phi_1(h - s)) = \Sigma_{i=0}^{m-1} w_L(h_i - s_i) = \Sigma_{i=0}^{m-1} w_L(\Phi_1(h_i) - \Phi_1(s_i))$. If we write $\Phi_1(h_i)$ and $\Phi_1(s_i)$ instead, we get $w_L(\Phi_1(h - s)) = \Sigma_{i=0}^{m-1} w_L((a_0^i + a_1^i + 3a_2^i, 3a_0^i + 3a_1^i + a_2^i) - (b_0^i + b_1^i + 3b_2^i, 3b_0^i + 3b_1^i + b_2^i))$. When we edit this equation, we obtain $w_L(\Phi_1(h - s)) = \Sigma_{i=0}^{m-1} w_L(a_0^i - b_0^i + a_1^i - b_1^i + 3a_2^i - 3b_2^i) + w_L(3a_0^i - 3b_0^i + 3a_1^i - 3b_1^i + a_2^i - b_2^i)$. This means that $w_L(\Phi_1(h - s)) = w_L(\Phi_1(h) - \Phi_1(s)) = d_L(\Phi_1(h), \Phi_1(s))$. Therefore, Φ_1 is a Lee distance preserving.

Linearity and for all distance preserving proofs all Gray maps can be demonstrated with the help of the above definition and proof for Φ_1. □

Definition 2.3 *Let $q = (q_0, \dots, q_{m-1})$ and $r = (r_0, \dots, r_{m-1})$ be an elements in T_3^m. The inner product is defined as $qr = q_0r_0 + \cdots + q_{m-1}r_{m-1}$. We can say that q and r are orthogonal if $qr = 0$. The dual of C is defined as $C^\perp = \{q | qr = 0, \forall r \in C\}$. If $C \subseteq C^\perp$ then a linear code C is said to be self-orthogonal, If $C = C^\perp$ then a linear code C is said to be self-dual.*

Definition 2.4 *The linear code that satisfies the condition $C \cap C^\perp = \{0\}$ is called a LCD code (linear code with complementary dual code).*

3. Cyclic and λ-constacyclic codes over T_3

If a shift of any codeword by one position is still a codeword then a linear code C_3 of length m over T_3 is called cyclic codes over the ring T_3. In this section we firstly search the structures of cyclic codes over T_3. Then we determine the relationship between cyclic and λ- constacyclic codes by setting a isomorphism between the $^{T_3[x]}/_{\langle x^m - 1\rangle}$ and $^{T_3[x]}/_{\langle x^m - \lambda\rangle}$, we also search the λ-constacyclic codes for each unit in this ring T_3 as a shift constant. Note that we can say that if a linear code C_3 of odd length m over T_3 is a cyclic code then $\sigma(z_0, z_1, \dots, z_{m-1}) = (z_{m-1}, z_0, \dots, z_{m-2})$ while σ is a cyclic shift operator. That is if $\sigma(C_3) = C_3$ then C_3 is cyclic. Also if a linear code of length m over T_3 is a λ-constacyclic code where λ is a unit then $\rho_\lambda(z_0, z_1, \dots, z_{m-1}) = (\lambda z_{m-1}, z_0, z_1, \dots, z_{m-2})$. That is if $\rho_\lambda(C_3) = C_3$ then C_3 is a λ-constacyclic code. It is known that cyclic codes over T_3 are the ideals determined with ideals in ${}^{T_3[x]}/_{\langle x^m-1\rangle}$ via the T_3-module isomorphism.

In this section our main target is first to create a generator polynomial of cyclic codes and λ-constacyclic codes of odd length m over T_3 and then to investigate their $\mathbb{Z}_4$-images via defined Gray maps. Throughout this paper we search for the structure of cyclic and constacyclic codes. And also we will represent the quotient ring ${}^{T_3[x]}/_{\langle x^m-1\rangle}$ with $T_{3,m}$ and the quotient ring ${}^{T_3[x]}/_{\langle x^m-\lambda\rangle}$ with T_{3,m_λ}.

3.1. The structure of cyclic codes over T_3

Theorem 3.1 *[22] Let R be finite commutative ring with identity. The following statements are equivalent:*

i. An idempotent family $(e_i)_{i=1}^n$ of R is exist for $i \neq j$ such that $e_ie_j = 0$, $\sum_{i=1}^{n} e_i = 1$ and $R_i = e_iR$.

ii. $R = R_1 + R_2 + \cdots + R_n$.

It is also known that any finite commutative ring decomposes uniquely as a direct sum of finite local commutative rings from [17]. Now we will form the generator polynomial of T_3 with the help of Chinese remainder theorem in this paper. In this case we need to a decomposition of T_3. We know that if $\Re$ and $\Im$ are two linear codes then $\oplus$ is defined by $\Re \oplus \Im = \{d + w | d \in \Re, w \in \Im\}$.

Motivated by all these explanations and the work [14], let us first consider generator polynomials of a cyclic code over the ring T_3. Using the u^2 and $1+3u^2$ idempotent elements of the ring, the ring is decomposed as $T_3 = u^2T_3 \oplus (1+3u^2)T_3$. Based on the Theorem 4.12 in [21], ${}^{T_3}/_{\langle u^2\rangle} \cong \mathbb{Z}_4 + u\mathbb{Z}_4$ with $u^2 = 0$ and ${}^{T_3}/_{\langle 1+3u^2\rangle} \cong \mathbb{Z}_4$ expressions are obtained since it is known that $T_3 = u^2T_3 \oplus (1 + 3u^2)T_3$. So $T_3 = u^2\mathbb{Z}_4 \oplus (1 + 3u^2)(\mathbb{Z}_4 + u\mathbb{Z}_4)$ is written.

Let $\Re$ and $\Im$ be linear codes of length m over $\mathbb{Z}_4$ and $\mathbb{Z}_4 + u\mathbb{Z}_4$, respectively, which are defined as $\Re = \{t + y + h \in \mathbb{Z}_4^m | t + uy + u^2h \in C_3\}$ and $\Im = \{t + uy \in (\mathbb{Z}_4 + u\mathbb{Z}_4)^m | t + uy + u^2h \in C_3$ for some $h \in \mathbb{Z}_4^m\}$. Thus we can say that T_3 is a Frobenius ring since it can be written as a direct sum of local rings with a single minimal ideal with the help of CRT.

Additionally, the linear code C_3 of odd length m over T_3 can be uniquely stated as $C_3 = u^2\Re \oplus (1+3u^2)\Im$. First of all we handle some basic results of cyclic codes over T_3.

Theorem 3.2 *A linear code $C_3 = u^2\Re \oplus (1 + 3u^2)\Im$ is cyclic if and only if $\Re$ and $\Im$ are both cyclic over $\mathbb{Z}_4$ and $\mathbb{Z}_4 + u\mathbb{Z}_4$, respectively.*

Proof Let $p_i = (p_0, p_1, \ldots, p_{m-1}) \in \Re$ and $v_i = (v_0, v_1, \ldots, v_{m-1}) \in \Im$. Assume that $z_i = u^2p_i \oplus (1 + 3u^2)v_i$ such that $v_i = t_i + uq_i$ for $i = 0, 1, \ldots, m-1$. Then the vectors reside in $z = (z_0, z_1, \ldots, z_{m-1}) \in C_3$. As C_3 is a cyclic code, we conclude that $\sigma(z) \in C_3$. Herefrom $\sigma(z) = u^2\sigma(p_i) + (1 + 3u^2)\sigma(v_i)$ is obtained. Thus $\sigma(p_i)$ be an element of $\Re$ and $\sigma(v_i)$ be an element of $\Im$ which indicate that $\Re$ and $\Im$ are both cyclic codes over $\mathbb{Z}_4$ and $\mathbb{Z}_4 + u\mathbb{Z}_4$, respectively. On the other hand, let $\Re$ and $\Im$ are cyclic codes over $\mathbb{Z}_4$ and $\mathbb{Z}_4 + u\mathbb{Z}_4$, respectively. Let $z = (z_0, z_1, \ldots, z_{m-1})$ be a codeword in C_3 and $z_i = u^2p_i + (1+3u^2)v_i$ such that $v_i = t_i + uq_i$ where $i = 0, 1, \ldots, m-1$. Since $\Re$ is cyclic, p_i is an element in $\Re$ and v_i is an element in $\Im$ it follows that $\sigma(p_i) \in \Re$ and also $\sigma(v_i) \in \Im$. By reason of the fact that $\sigma(z_i) \in C_3$, we obtain $\sigma(p_i) \in \Re$ and $\sigma(v_i) \in \Im$. So we get $\sigma(z_i) = u^2\sigma(p_i) + (1+3u^2)\sigma(v_i) \in u^2\Re \oplus (1+3u^2)\Im$. This means that $\sigma(z_i) = C_3$. In this way, C_3 is cyclic code over T_3. □

Considering all the above explanations and Theorem 3.2, the structure of the cyclic codes of odd length m over T_3 can be determined as follows:

Theorem 3.3 *Let $C_3 = u^2\Re \oplus (1+3u^2)\Im$ be a cyclic code of length m over T_3. Here $\Re$ is a code over $\mathbb{Z}_4$ and $\Im$ is a code over $\mathbb{Z}_4 + u\mathbb{Z}_4$ of odd length m. Then $C_3 = (u^2\langle f_1(x)(h_1(x)+2)\rangle) \oplus ((1+3u^2)\langle f_2(x)(h_2(x)+2) + uf_{1,2}(x)(h_{1,2}(x)+2), uf_3(x)(h_3(x)+2)\rangle)$ and $f_i(x), h_i(x), w_i(x)$ monic coprime pairwise polynomials over $\mathbb{Z}_4[x]$ such that $x^m - 1 = f_i(x)h_i(x)w_i(x)$ and $i = 1,2,3$.*

Proof Let $\widehat{C}_3 = (u^2\langle f_1(x)(h_1(x)+2)\rangle) \oplus ((1+3u^2)\langle f_2(x)(h_2(x)+2) + uf_{1,2}(x)(h_{1,2}(x)+2), uf_3(x)(h_3(x)+2)\rangle)$ such that $\Re = \langle f_1(x)(h_1(x)+2)\rangle$ and $\Im = \langle f_2(x)(h_2(x)+2) + uf_{1,2}(x)(h_{1,2}(x)+2), uf_3(x)(h_3(x)+2)\rangle$. It is clear that $\widehat{C}_3 \subseteq C_3$. For $u^2\Re$ and $(1+3u^2)\Im$, we have $u^2\Re = u^2C_3$ and $(1+3u^2)\Im = (1+3u^2)C_3$ since $u^3 = u^2$. Therefore, we attain $u^2\Re \oplus (1+3u^2)\Im \subseteq \widehat{C}_3$ that is $C_3 \subseteq \widehat{C}_3$. Since we show $\widehat{C}_3 \subseteq C_3$ and $C_3 \subseteq \widehat{C}_3$, we reach $C_3 = \widehat{C}_3$.

□

Theorem 3.4 *$C_3 = u^2\Re \oplus (1+3u^2)\Im$ be a cyclic code of length m over T_3 and suppose that $\varrho_1(x)$ and $\langle \varrho_2(x), \varrho_3(x)\rangle$ are generator polynomials of $\Re$ and $\Im$, respectively. Then $C_3 = \langle u^2\varrho_1(x), (1+3u^2)\langle \varrho_2(x), \varrho_3(x)\rangle\rangle$. If this generator polynomial of C_3 is organized it becomes $C_3 = \langle u^2\varrho_1(x), (1+3u^2)\varrho_2(x), (1+3u^2)\varrho_3(x)\rangle$.*

Proof To show $C_3 = \langle u^2\varrho_1(x), (1+3u^2)\langle \varrho_2(x), \varrho_3(x)\rangle\rangle$, we have to show that $C_3 \subseteq \langle u^2\varrho_1(x), (1+3u^{k-1})\langle \varrho_2(x), \varrho_3(x)\rangle\rangle$ and $\langle u^2\varrho_1(x), (1+3u^{k-1})\langle \varrho_2(x), \varrho_3(x)\rangle\rangle \subseteq C_3$. Due to $\Re = \langle \varrho_1(x)\rangle$, $\Im = \langle \varrho_2(x), \varrho_3(x)\rangle$ and $C_3 = u^2\Re \oplus (1+3u^2)\Im$, we obtain $C_3 = \{z(x) = u^2b_1(x)\varrho_1(x) + (1+3u^2)b_2(x)\langle \varrho_2(x), \tau_3(x)\rangle | b_1(x), b_2(x) \in \mathbb{Z}_4[x]\}$. Herefrom we get $C_3 \subseteq \langle u^2\varrho_1(x) + (1+3u^2)\langle \varrho_2(x), \varrho_3(x)\rangle\rangle \subseteq T_{3,m}$. Conversely let us take $u^2y_1(x)\varrho_1(x) + (1+3u^2)y_2(x)\langle \varrho_2(x), \varrho_3(x)\rangle \in \langle u^2\varrho_1(x), (1+3u^2)\langle \varrho_2(x), \varrho_3(x)\rangle\rangle$ such that $y_1(x), y_2(x) \in T_{3,m}$. Then we have $u^2y_1(x) = u^2b_1(x)$ and $(1+3u^2)y_2(x) = (1+3u^2)b_2(x)$ for some $b_1(x), b_2(x) \in \mathbb{Z}_4[x]$. Hence we reach $\langle u^2\varrho_1(x), (1+3u^2)\langle \varrho_2(x), \varrho_3(x)\rangle\rangle \subseteq C_3$. Therefore we demonstrate the equation $C_3 = \langle u^2\varrho_1(x), (1+3u^2)\langle \varrho_2(x), \varrho_3(x)\rangle\rangle$. □

Next we show that $e(x)$ which is called the generating idempotent of C_3 is unique.

Theorem 3.5 *Let C_3 be a cyclic codes length m. Then there exists a unique idempotent element $e(x) = u^2e_1(x) + (1+3u^2)e_2(x) \in T_3[x]$ such that $C_3 = \langle e(x)\rangle$.*

Proof There exist unique idempotent elements $e_1(x) \in \mathbb{Z}_4[x]$ and $e_2(x) \in (\mathbb{Z}_4 + u\mathbb{Z}_4)[x]$ such that $\Re = \langle e_1(x)\rangle$ and $\Im = \langle e_2(x)\rangle$ because m is odd. So we write $C_3 = \langle u^2e_1(x) + (1+3u^2)e_2(x)\rangle$. Let $e(x) = u^2e_1(x) + (1+3u^2)e_2(x)$. So $e^2(x) = u^2e_1(x) + (1+3u^2)e_2(x)$ which is equals to the idempotent element $e(x)$ of C_3. Well, $e(x)$ is unique? So if there is another $l(x) \in C_3$ such that $C_3 = \langle l(x)\rangle$ and also $l^2(x) = l(x)$. Due to $l(x) \in C_3 = \langle l(x)\rangle$ we have that $l(x) = k(x)e(x)$, for some $k(x) \in T_{3,m}$. From here we get $l(x)e(x) = k(x)e^2(x) = k(x)e(x) = l(x)$. Since $C_3 = \langle e(x)\rangle$, we can also find $l(x)e(x) = e(x)$. Therefore, we imply that $e(x)$ is an unique idempotent element. □

In accordance with the article of Dougherty et al. in ([13], Theorems 2.7 and 5.1), we can say $C_3^{\perp} = u^2\Re^{\perp} \oplus (1+3u^2)\Im^{\perp}$. Based on this, the following theorem can be written.

Theorem 3.6 *Let C_3 be a linear code over T_3 of length m. Then, $C_3 = u^2\Re \oplus (1+3u^2)\Im$ is a self-dual code over T_3 iff $\Re$ and $\Im$ are self-dual codes over $\mathbb{Z}_4$ and $\mathbb{Z}_4 + u\mathbb{Z}_4$, respectively.*

Proof Let C_3 be a linear code over T_3 of length m. If $\Re$ and $\Im$ are self-dual codes then we can say $\Re = \Re^{\perp}$ and $\Im = \Im^{\perp}$. Hence we obtain that $C_3 = u^2\Re \oplus (1+3u^2)\Im$ is equal to $C_3^{\perp} = u^2\Re^{\perp} \oplus (1+3u^2)\Im^{\perp}$. By this way we have $C_3 = C_3^{\perp}$. Conversely, assume that C_3 be a self-dual code over T_3. So we can write $C_3 = C_3^{\perp}$. Hence we get $C_3 = u^2\Re^{\perp} \oplus (1+3u^2)\Im^{\perp}$. This means that C_3 is equal to $C_3^{\perp}$. □

Corollary 3.7 *In addition to this theorem, it can be said as a result that $|T_3|^m = |C_3||C_3^{\perp}|$ since T_3 is a Frobenius ring.*

LCD codes, which have a wide application area such as cryptography, communications systems, data storage and consumer electronics, can be briefly mentioned with the following theorem.

Theorem 3.8 *Let C_3 be a linear code over T_3 of length m. Then $C_3 = u^2\Re \oplus (1+3u^2)\Im$ is a LCD code over T_3 iff $\Re$ and $\Im$ are LCD codes over $\mathbb{Z}_4$ and $\mathbb{Z}_4 + u\mathbb{Z}_4$, respectively.*

Proof Let C_3 be a linear code over T_3 of length m. If $\Re$ and $\Im$ are LCD codes then we have $\Re \cap \Re^{\perp} = \{0\}$ and $\Im \cap \Im^{\perp} = \{0\}$. Since $C_3^{\perp}$ is equal to $u^2\Re^{\perp} \oplus (1+3u^2)\Im^{\perp}$, we get $C_3 \cap C_3^{\perp} = u^2(\Re \cap \Re^{\perp}) \oplus (1+3u^2)(\Im \cap \Im^{\perp})$. Hence, we obtain $C_3 \cap C_3^{\perp} = \{0\}$. This means that C_3 is a LCD code over T_3. Conversely, assume that C_3 be a LCD code over T_3. Then we can write $C_3 \cap C_3^{\perp} = \{0\}$. Hence we get $C_3^{\perp} = u^2(\Re \cap \Re^{\perp}) \oplus (1+3u^2)(\Im \cap \Im^{\perp})$. Therefore, we attain $\Re \cap \Re^{\perp} = \{0\}$ and $\Im \cap \Im^{\perp} = \{0\}$. It is clear from here that $\Re$ and $\Im$ are LCD codes. □

3.2. The construction of λ-constacyclic codes over T_3 and their Gray images

The λ-constacyclic codes over T_3 are the ideals identified with ideals in the quotient ring ${}^{T_3[x]}/_{\langle x^m - \lambda \rangle}$ via the T_3-module isomorphism. The polynomial representation of λ-constacyclic codes:

$$\varepsilon : {T_3}^m \to T_{3,m_\lambda}$$

$$(z_0, z_1, \ldots, z_{m-1}) = z_0 + z_1 x + \cdots + z_{m-1}x^{m-1}(mod x^m - \lambda).$$

Now let us analyse the structure of λ-constacyclic codes over T_3. We identify a more spesific form for a generator of the $\mathbb{Z}_4$ image. In the consideration of $\lambda^{-1} = \lambda$, we have $\lambda^m = \lambda$ because m is odd. Note that if m is even then $\lambda^m = 1$.

Proposition 3.9 *Define $\xi : T_{3,m} \to T_{3,m_\lambda}$ by $z(x) = z(\lambda x)$. If m is odd then ξ is a ring isomorphism.*

Proof Consider that $\xi(x^m - 1) = \lambda^m x^m - 1 = \lambda x^m - 1 = \lambda(x^m - \lambda) = 0$. The remainder of the proof is obvious. □

Corollary 3.10 *There is a one-to-one correspondence between the ideals $T_{3,m}$ and the ideals of T_{3,m_λ} such that cyclic code C is equivalent to the constacyclic code $\xi(C)$.*

Now using the structure of cyclic code over $\mathbb{Z}_4$, Theorem 3.3 and the isomorphism ξ, we characterize the λ-constacyclic codes over T_3 as follows:

Theorem 3.11 *Let $C_3 = u^2\Re \oplus (1+3u^2)\Im$ be a constacyclic code of length m over T_3. Then C_3 is an ideal in $T_3[x]/\langle x^m - \lambda\rangle$ generated by $C_3 = (u^2\langle f_1(\check{x})(h_1(\check{x})+2)\rangle) \oplus ((1+3u^2)\langle f_2(\check{x})(h_2(\check{x})+2) + uf_{1,2}(\check{x})(h_{1,2}(\check{x})+2), uf_3(\check{x})(h_3(\check{x})+2)\rangle)$ where $\check{x} = \lambda x$ and $f_i(\check{x}), h_i(\check{x}), w_i(\check{x})$ are monic coprime pairwise polynomials in $\mathbb{Z}_4[x]$ such that $x^m - 1 = f_i(\check{x})h_i(\check{x})w_i(\check{x})$ and $i = 1,2,3$. Note that $\Re$ is a code over $\mathbb{Z}_4$ and $\Im$ is a code over $\mathbb{Z}_4 + u\mathbb{Z}_4$ of length m.*

We know that if we get a generator polynomial of λ-constacyclic code then we write λx instead of x in Theorem 3.2 and Theorem 3.3. Now we will examine the $\mathbb{Z}_4$-images of λ-constacyclic codes of the rings for each unit of this ring via the maps defined in Section 2.

Definition 3.12 *Let $z \in \mathbb{Z}_4^{2m}$ with $z = (z_0, z_1)$ where $z_i \in \mathbb{Z}_4$ for $i = 0,1$. Let υ_2 be a map from $\mathbb{Z}_4^{2m}$ to $\mathbb{Z}_4^{2m}$ defined by $\upsilon_2(z) = (\sigma(z_0)|\sigma(z_1))$ where σ is the cyclic shift from $\mathbb{Z}_4^{2m}$ to $\mathbb{Z}_4^{2m}$ given by $\sigma(z_i) = (z_i^{m-1}, z_i^1, \ldots, z_i^{m-2})$ for every $z_i = (z_i^0, \ldots, z_i^{m-1})$ where $z_i^j \in \mathbb{Z}_4$ and $j = 0,1,\ldots,m-1$. A code of length $2m$ over $\mathbb{Z}_4$ is called to be QC code of index 2 if $\upsilon_2(C) = C$.*

Recall that σ is a cyclic shift operator, ρ_λ is a constacyclic shift operator and υ_2 is a QC shift operator. Therefore, using these maps defined in preliminaries, let's analyze the $\mathbb{Z}_4$-images of all the λ-constacyclic codes over this ring.

Proposition 3.13 *For any $z \in T_3^m$,*

1. *If we use Φ_1 map and units $\lambda = 3, 1+2u, 1+2u^2, 3+2u+2u^2$, then we get $\Phi_1\rho_\lambda(z) = \sigma\Phi_1(z)$.*
2. *If we use Φ_2 map and units $\lambda = 1+u+u^2, 3+3u+u^2, 3+u+3u^2, 1+3u+3u^2$, then we have $\Phi_2\rho_\lambda(z) = \sigma\Phi_2(z)$.*
3. *If we use Φ_3 map and units $\lambda = 3+u+u^2, 1+3u+u^2, 1+u+3u^2, 3+3u+3u^2$, then we obtain $\Phi_3\rho_\lambda(z) = \sigma\Phi_3(z)$.*
4. *If we use units $\lambda = 3+2u^2, 1+2u+2u^2, 3+2u$, then we get $\Phi_i\rho_\lambda(z) = \upsilon_2\Phi_i(z)$ for each Gray map Φ_i and $i = 1,2,3$.*
5. *For each Gray map Φ_i, the equation $\Phi_i\sigma(z) = \upsilon_2\Phi_i(z)$ is obtained.*

Proof

1. Determine $z = (z_0, z_1, \ldots, z_{m-1})$ in T_3^m, where $z_j = a_0^i + ua_1^i + u^2a_2^i$ such that $0 \le j \le m-1$. We have $\Phi_1(z) = (a_0^0 + a_1^0 + 3a_2^0, \ldots, a_0^{m-1} + a_1^{m-1} + 3a_2^{m-1}, 3a_0^0 + 3a_1^0 + a_2^0, \ldots, 3a_0^{m-1} + 3a_1^{m-1} + a_2^{m-1})$ and $\sigma\Phi_1(z) = (3a_0^{m-1} + 3a_1^{m-1} + a_2^{m-1}, a_0^0 + a_1^0 + 3a_2^0, \ldots, a_0^{m-1} + a_1^{m-1} + 3a_2^{m-1}, 3a_0^0 + 3a_1^0 + a_2^0, \ldots, 3a_0^{m-2} + 3a_1^{m-2} + a_2^{m-2})$.

On the other hand,

$\rho_3(z) = (3z_{m-1}, z_0, z_1, \ldots, z_{m-2}) = (3a_0^{m-1} + 3ua_1^{m-1} + 3u^2a_2^{m-1}, a_0^0 + ua_1^0 + u^2a_2^0, \ldots a_0^{m-2} + ua_1^{m-2} + u^2a_2^{m-2})$,

$\rho_{1+2u}(z) = ((1+2u)z_{m-1}, z_0, z_1, \ldots, z_{m-2}) = (a_0^{m-1} + u(2a_0^{m-1} + a_1^{m-1}) + u^2(2a_1^{m-1} + 3a_2^{m-1}), a_0^0 + ua_1^0 + u^2a_2^0, \ldots a_0^{m-2} + ua_1^{m-2} + u^2a_2^{m-2})$,

$\rho_{1+2u^2}(z) = ((1+2u^2)z_{m-1}, z_0, z_1, \ldots, z_{m-2}) = (a_0^{m-1} + ua_1^{m-1} + u^2(2a_0^{m-1} + 2a_1^{m-1} + 3a_2^{m-1}), a_0^0 + ua_1^0 + u^2a_2^0, \ldots a_0^{m-2} + ua_1^{m-2} + u^2a_2^{m-2})$ and

$\rho_{3+2u+2u^2}(z) = ((3 + 2u + 2u^2)z_{m-1}, z_0, z_1, \ldots, z_{m-2}) = (3a_0^{m-1} + u(2a_0^{m-1} + 3a_1^{m-1}) + u^2(2a_0^{m-1} + 3a_2^{m-1}), a_0^0 + ua_1^0 + u^2a_2^0, \ldots a_0^{m-2} + ua_1^{m-2} + u^2a_2^{m-2})$.

The image of them under Φ_1;

$\Phi_1\rho_3(z) = (3a_0^{m-1} + 3a_1^{m-1} + a_2^{m-1}, a_0^0 + a_1^0 + 3a_2^0, \ldots, a_0^{m-1} + a_1^{m-1} + 3a_2^{m-1}, 3a_0^0 + 3a_1^0 + a_2^0, \ldots, 3a_0^{m-2} + 3a_1^{m-2} + a_2^{m-2})$,

$\Phi_1\rho_{1+2u}(z) = (3a_0^{m-1} + 3a_1^{m-1} + a_2^{m-1}, a_0^0 + a_1^0 + 3a_2^0, \ldots, a_0^{m-1} + a_1^{m-1} + 3a_2^{m-1}, 3a_0^0 + 3a_1^0 + a_2^0, \ldots, 3a_0^{m-2} + 3a_1^{m-2} + a_2^{m-2})$,

$\Phi_1\rho_{1+2u^2}(z) = (3a_0^{m-1} + 3a_1^{m-1} + a_2^{m-1}, a_0^0 + a_1^0 + 3a_2^0, \ldots, a_0^{m-1} + a_1^{m-1} + 3a_2^{m-1}, 3a_0^0 + 3a_1^0 + a_2^0, \ldots, 3a_0^{m-2} + 3a_1^{m-2} + a_2^{m-2})$ and

$\Phi_1\rho_{3+2u+2u^2}(z) = (3a_0^{m-1} + 3a_1^{m-1} + a_2^{m-1}, a_0^0 + a_1^0 + 3a_2^0, \ldots, a_0^{m-1} + a_1^{m-1} + 3a_2^{m-1}, 3a_0^0 + 3a_1^0 + a_2^0, \ldots, 3a_0^{m-2} + 3a_1^{m-2} + a_2^{m-2})$.

Therefore, we get $\Phi_1\rho_\lambda(z) = \sigma\Phi_1(z)$ such that $\lambda = 3, 1+2u, 1+2u^2, 3+2u+2u^2$.

The proof of others is done similarly. □

With the help of Proposition 3.13, the following theorem is obtained.

Theorem 3.14 *Let C_3 be a λ-constacyclic code of length m over T_3.*

1. *If λ is $3, 1+2u, 1+2u^2, 3+2u+2u^2$, then their $\mathbb{Z}_4$-images $\Phi_1(C_3)$ are cyclic codes over $\mathbb{Z}_4$.*
2. *If λ is $1+u+u^2, 3+3u+u^2, 3+u+3u^2, 1+3u+3u^2$, then their $\mathbb{Z}_4$-images $\Phi_2(C_3)$ are cyclic codes over $\mathbb{Z}_4$.*
3. *If λ is $3+u+u^2, 1+3u+u^2, 1+u+3u^2, 3+3u+3u^2$, then their $\mathbb{Z}_4$-images $\Phi_3(C_3)$ are cyclic codes over $\mathbb{Z}_4$.*
4. *If λ is $3+2u^2, 1+2u+2u^2, 3+2u$, then their $\mathbb{Z}_4$-images $\Phi_i(C_3)$ are QC codes of index 2 with length $2m$ over $\mathbb{Z}_4$ for $i = 1, 2, 3$.*
5. *If λ is 1, in other words C_3 is a cyclic code of length m over T_3, then $\Phi_i(C_3)$ are QC codes of index 2 with length $2m$ over $\mathbb{Z}_4$ for $i = 1, 2, 3$.*

Proof

1. Assume that C_3 is a λ-constacyclic code of length m over T_3 when $\lambda = 3, 1+2u, 1+2u^2, 3+2u+2u^2$. Then we obtain $\Phi_1\rho_\lambda(C_3) = \Phi_1(C_3)$. By the previous proposition, we reach $\sigma\Phi_1(C_3) = \Phi_1\rho_\lambda(C_3) = \Phi_1(C_3)$. This shows that $\Phi_1(C_3)$ is cyclic code of length $2m$ over $\mathbb{Z}_4$.

Others are done similarly to the proof of 1.

□

Definition 3.15 *π is a special permutation of $\mathbb{Z}_4^{2m}$ and it also be a called Nechaev permutation which is defined by $\pi(z_0, z_1, \ldots, z_{2m-1}) = (z_{\pi^\bullet_{(0)}}, z_{\pi^\bullet_{(1)}}, \ldots, z_{\pi^\bullet_{(2m-1)}})$ with the permutation $\pi^\bullet = (2i+1, m+2i+1)$ of $\{0, 1, \ldots, 2m-1\}$.*

Considering the definition of th Nechaev permutation, the following proposition can be given.

Proposition 3.16 *Let ρ_λ be the λ-constacyclic shift and υ_2 be the QC shift operator with index 2 over T_3 and Φ_i be the Gray map from T_3^m to $\mathbb{Z}_4^{2m}$ for $i = 1, 2, 3$.*

1. *For $z \in T_3^m$, we have $\Phi_1\rho_\lambda(C_3) = \pi\upsilon_2\Phi_1(C_3)$ such that $\lambda = 3, 1+2u, 1+2u^2, 3+2u+2u^2$.*

2. *For $z \in T_3^m$, we have $\Phi_2\rho_\lambda(z) = \pi\upsilon_2\Phi_2(z)$ such that $\lambda = 1+u+u^2, 3+3u+u^2, 3+u+3u^2, 1+3u+3u^2$.*

3. *For $z \in T_3^m$, we have $\Phi_3\rho_\lambda(z) = \pi\upsilon_2\Phi_3(z)$ such that $\lambda = 3+u+u^2, 1+3u+u^2, 1+u+3u^2, 3+3u+3u^2$.*

Proof

1. Let $z = (z_0, z_1, \ldots, z_{m-1})$ be in T_3^m, where $z_j = a_0^i + ua_1^i + u^2a_2^i$ such that $0 \leq j \leq m-1$. We know that

   $\Phi_1\rho_\lambda(z) = (3a_0^{m-1} + 3a_1^{m-1} + a_2^{m-1}, a_0^0 + a_1^0 + 3a_2^0, \ldots, a_0^{m-1} + a_1^{m-1} + 3a_2^{m-1}, 3a_0^0 + 3a_1^0 + a_2^0, \ldots, 3a_0^{m-2} + 3a_1^{m-2} + a_2^{m-2})$.

   Then $\upsilon_2\Phi_1(z) = (3a_0^{m-1}+3a_1^{m-1}+a_2^{m-1}, a_0^0+a_1^0+3a_2^0, \ldots, a_0^{m-1}+a_1^{m-1}+3a_2^{m-1}, 3a_0^0+3a_1^0+a_2^0, \ldots, 3a_0^{m-2}+3a_1^{m-2} + a_2^{m-2})$.

   Applying the permutation π to $\upsilon_2\Phi_1(z)$, we obtain $\Phi_1\rho_\lambda(C_3) = \pi\upsilon_2\Phi_1(C_3)$ for $\lambda = 3, 1+2u, 1+2u^2, 3+2u+2u^2$.

Proof of 2 and 3 is done in a similarly to 1.

□

From Proposition 3.16, the following theorem is written.

Theorem 3.17 *The $\mathbb{Z}_4$-image of a λ-constacyclic code over T_3 has a length m.*

1. *The permutation of $\Phi_1(C_3)$ is equivalent to a QC code of index 2 over $\mathbb{Z}_4$ of length $2m$ such that $\lambda = 3, 1+2u, 1+2u^2, 3+2u+2u^2$.*

2. *The permutation of $\Phi_2(C_3)$ is equivalent to a QC code of index 2 over $\mathbb{Z}_4$ of length $2m$ such that λ is $1+u+u^2, 3+3u+u^2, 3+u+3u^2$ and $1+3u+3u^2$.*

3. *The permutation of $\Phi_3(C_3)$ is equivalent to a QC code of index 2 over $\mathbb{Z}_4$ of length $2m$ such that λ is $3+u+u^2, 1+3u+u^2, 1+u+3u^2$ and $3+3u+3u^2$.*

Proof

1. Let C_3 be a λ-constacyclic codes of length m over T_3. Then $\rho_\lambda(C_3)=C_3$. If we apply the map Φ_1, then we get $\Phi_1\rho_\lambda(C_3)=\Phi_1(C_3)$. By the previous proposition, we attain $\Phi_1\rho_\lambda(C_3)=\pi\upsilon_2\Phi_1(C_3)=\rho_\lambda(C_3)$. Hence the permutation of $\Phi_1(C_3)$ is equivalent to a QC code of index 2 over $\mathbb{Z}_4$ with length $2m$.

The proof of others is made in a similar way. □

As a result of Proposition 3.9 and Corollary 3.10, we have the following result. And by taking advantage of this result, we can create Proposition 3.19, Theorem 3.20 and Corollary 3.21.

Corollary 3.18 *Define $\overline{\xi}(z)=(z_0,\lambda z_1,\ldots,\lambda^i z_i,\ldots,\lambda^{m-1}z_{m-1})$ to be a permutation which corresponds with the ring isomorphism ξ in the polynomial form over T_3^m. Then $C\subset T_3^m$ is a cyclic code iff $\overline{\xi}(C)$ is a λ-constacyclic code.*

Proof

Let C be a cyclic code. So $\sigma(z)\in C$ such that $z\in C$. Applying permutation $\overline{\xi}$, we get $\overline{\xi}(z)=(z_0,\lambda z_1,\ldots,z_{m-1})\in\overline{\xi}(C)$. On the other hand $\overline{\xi}(C)$ is a λ-constacyclic code iff $(\lambda z_{m-1},z_0,\lambda z_1,\ldots,\lambda z_{m-2})=\rho(\overline{\xi}(C))$. This statement can be written as $\lambda(z_{m-1},\lambda z_0,\ldots,z_{m-2})\in\overline{\xi}(C)$. Since $\overline{\xi}(C)$ is an ideal, we get $\rho(\overline{\xi}(C))\in\overline{\xi}(C)$. This means that $\overline{\xi}(C)$ is a λ-constacyclic code. □

Proposition 3.19 *For any $z\in T_3^m$, we have $\Phi_i\overline{\xi}(z)=\pi\Phi_i(z)$ such that $i=1,2,3$.*

Proof Let $z=(z_0,z_1,\ldots,z_{m-1})$ be in T_3^m, where $z_j=a_0^i+ua_1^i+u^2a_2^i$ such that $0\le j\le m-1$. Note that $\overline{\xi}(z)=(z_0,\lambda z_1,\ldots,\lambda^i z_i,\ldots,z_{m-1})$. Then $\overline{\xi}(z)=(a_0^0+ua_1^0+u^2a_2^0, a_0^1+ua_1^1+u^2(2a_0^1+2a_1^1+3a_2^1), a_0^2+ua_1^2+u^2a_2^2,\ldots,a_0^{m-1}+ua_1^{m-1}+u^2a_2^{m-1})$ such that $\lambda=1+2u^2$. Let us prove for the map Φ_1.

Define $\Gamma=(\Gamma_0,\Gamma_1,\ldots,\Gamma_{2m-1})\in\mathbb{Z}_4^{2m}$ by $\Gamma_s=a_0^s+a_1^s+3a_2^s$ and $\Gamma_{m+s}=3a_0^s+3a_1^s+a_2^s$ if m is even, $\Gamma_s=3a_0^s+3a_1^s+a_2^s$ and $\Gamma_{m+s}=a_0^s+a_1^s+3a_2^s$ if m is odd. Note that $s=0,1,\ldots,m-1$. Thus we get $\Phi_1\overline{\xi}(z)=(\Gamma_1,\Gamma_2,\ldots,\Gamma_{2m-1})$. That is, we have $\Phi_1\overline{\xi}(z)=(a_0^0+a_1^0+3a_2^0, 3a_0^0+3a_1^0+a_2^0, 3a_0^1+3a_1^1+a_2^1, a_0^1+a_1^1+3a_2^1,\ldots,a_0^{m-1}+a_1^{m-1}+3a_2^{m-1}, 3a_0^{m-1}+3a_1^{m-1}+a_2^{m-1})$.
We know that $\Phi_1(z)=(a_0^0+a_1^0+3a_2^0,\ldots,a_0^{m-1}+a_1^{m-1}+3a_2^{m-1}, 3a_0^0+3a_1^0+a_2^0,\ldots,3a_0^{m-1}+3a_1^{m-1}+a_2^{m-1})$. From this point of view, we attain $\pi\Phi_1(z)=(a_0^0+a_1^0+3a_2^0, 3a_0^0+3a_1^0+a_2^0, 3a_0^1+3a_1^1+a_2^1, a_0^1+a_1^1+3a_2^1,\ldots,a_0^{m-1}+a_1^{m-1}+3a_2^{m-1}, 3a_0^{m-1}+3a_1^{m-1}+a_2^{m-1})$.

Therefore, we see that $\Phi_1\overline{\xi}(z)=\pi\Phi_1(z)$. The rest of the proof is done in a similar way. □

Theorem 3.20 *Let δ_i be the $\mathbb{Z}_4$-images of cyclic codes over T_3 for $i=1,2,3$. Then $\pi(\delta_i)$ are cyclic codes.*

Proof Assume that C is a cyclic code over T_3, $\delta_i=\Phi_i(C)$ where $i=1,2,3$. In consideration of the previous two propositions, we get $\Phi_i\overline{\xi}(C)=\pi\Phi_i(C)=\pi\delta_i$. From Corollary 3.18, we see that $\overline{\xi}(C)$ is a λ-constacyclic code. Hence, we get $\Phi_i\overline{\xi}(C)=\Phi_i(C)$. By Theorem 3.14, we see that $\pi(\delta_i)$ are cyclic codes. □

Corollary 3.21 *$\mathbb{Z}_4$-images under the defined Gray maps of a cyclic code over T_3 are cyclic codes.*

4. Computational results

In this section, we investigate cyclic codes, $(1+2u^2)$-constacyclic codes and their $\mathbb{Z}_4$-images specifically via the Gray image Φ_1 for 7 length over $\mathbb{Z}_4+u\mathbb{Z}_4+u^2\mathbb{Z}_4$ when $u^3=u^2$. Predicated on Theorems 3.3, 3.4, and 3.11, we inform the results of a computer search on cyclic and $(1+2u^2)$-constacyclic codes over T_3. Note that we present each component of generator polynomial with $\tau_i(x)$ where $i=1,2,3$. In the circumstances generator polynomials over T_3 can be expressed as $\langle\tau_1(\lambda x),\tau_2(\lambda x),\tau_3(\lambda x)\rangle$. It is clear that the code is mentioned when $\lambda=1$ is cyclic codes and when $\lambda=1+2u^2$ is $1+2u^2$- constacyclic codes. Evaluations are realized using Magma software [7]. We have many cyclic codes over T_3 whose $\mathbb{Z}_4$-images produce new, optimal and best known linear codes over $\mathbb{Z}_4$ with reference to the online database, but we will write a few examples in terms of avoiding density.

In Table 1, we denominate elements of T_3 and we represent the elements of T_3 by Δ.

Table 1. Denominate of elements on ring $\mathbb{Z}_4+u\mathbb{Z}_4+u^2\mathbb{Z}_4$.

Elements of T_3	Δ	Elements of T_3	Δ	Elements of T_3	Δ
0	0	1	1	2	2
3	3	u	4	$2u$	5
$3u$	6	u^2	7	$2u^2$	8
$3u^2$	9	u^2+u	$1'$	u^2+2u	$2'$
u^2+3u	$3'$	$2u^2+u$	$4'$	$2u^2+2u$	$5'$
$2u^2+3u$	$6'$	$3u^2+u$	$7'$	$3u^2+2u$	$8'$
$3u^2+3u$	$9'$	$u+1$	A	$u+2$	B
$u+3$	D	$2u+1$	E	$2u+2$	F
$2u+3$	G	$3u+1$	H	$3u+2$	J
$3u+3$	K	u^2+1	L	u^2+2	M
u^2+3	N	$2u^2+1$	P	$2u^2+2$	R
$2u^2+3$	S	$3u^2+1$	U	$3u^2+2$	V
$3u^2+3$	Y	u^2+u+1	Z	u^2+u+2	b
u^2+u+3	c	$2u^2+u+1$	d	$2u^2+u+2$	e
$2u^2+u+3$	g	$3u^2+u+1$	l	$3u^2+u+2$	n
$3u^2+u+3$	p	u^2+2u+1	r	u^2+2u+2	s
u^2+2u+3	t	$2u^2+2u+1$	v	$2u^2+2u+2$	y
$2u^2+2u+3$	b'	$3u^2+2u+1$	c'	$3u^2+2u+2$	d'
$3u^2+2u+3$	e'	u^2+3u+1	g'	u^2+3u+2	l'
u^2+3u+3	n'	$2u^2+3u+1$	p'	$2u^2+3u+2$	r'
$2u^2+3u+3$	s'	$3u^2+3u+1$	t'	$3u^2+3u+2$	v'
$3u^2+3u+3$	y'				

In all the following Tables 2,3,4,5, we give information about new and optimal parameters over $\mathbb{Z}_4$ in recognition of Lee weight, Euclidean weight and Hamming weight attained from some cyclic codes and constacyclic codes over T_3 for length $n = 7$. The $\mathbb{Z}_4$-images of the codes of length is increased to 14 with the defined Gray map Φ_1. For ease of writing, we list the coefficients of polynomials in descending order starting from the highest degree of x.

Table 2. Some cyclic codes with $\mathbb{Z}_4$-images.

$\tau_1(x)$	$\tau_2(x)$	$\tau_3(x)$	$Type$	W_L	W_E	W_H
789^3	$U^3De'dD$	$5'5'05'$	4^32^{2*}	4^*	8^*	2^*
78999	$N^3De'D^2$	$7'7'3'03'$	4^32^4	4^*	8	2^*
7^3979^2	$(3')^3v'3'(v')^2$	$(5')^2$	4^12^0	14	14^*	14^*
7^2909	$dv's'Ns'$	$7'7'7'3'3'7'3'$	4^32^4	4^*	8	2^*
7^2909	$3'3'3'v'3'v'v'$	$3'5'3'7'$	4^32^1	$8{*}{*}$	8	$6{*}{*}$
789^3	$N^5s'D$	$7'5'3'7'3'$	4^32^{2*}	4^*	8^*	2^*
70779	N^5D^2	$3'5'3'7'$	4^32^4	4^*	8	2^*

For example, $(2u^2{+}2u{+}1)x^5{+}(u{+}1)x^3{+}3ux{+}1$ polynomial will be written as $v0A061$. If the coefficient Q is repeated n times consecutively, it is shortened to Q^n, e.g., the polynomial $(u{+}3)x^3{+}(u{+}3)x^2{+}(u{+}3)x{+}u{+}3$ is shown as D^4.

In these tables; Lee, Euclidean and Hamming weights are calculated for each of the generator polynomials. The codes that are found for the first time according to the online database and [3] are marked with '$*$'. The codes that have the best found parameters in length, size and minimum distances are called optimal code. And the optimal codes we find are marked with '$**$' in the table.

Columns not marked with '$*$' and '$**$' are best known linear codes over $\mathbb{Z}_4$. Here subscripts L, E and H denote the Lee minimum weight, the Euclidean minimum weight and the Hamming minimum weight, respectively.

5. Conclusion

This paper is dedicated to searching some algebric structures of cyclic and λ-constacyclic codes over the ring $\mathbb{Z}_4 + u\mathbb{Z}_4 + u^2\mathbb{Z}_4$ with $u^3 = u^2$. We explore the general form of generator polynomials via decomposition for cyclic and λ-constacyclic codes over this ring. We also analyze $\mathbb{Z}_4$-images, which are important observations of these codes. Besides we specifically investigate the cyclic and $(1 + 2u^2)$-constacylic codes of length 7 over the ring T_3 by way of Φ_1. According to online database, we present several new and optimal linear codes over $\mathbb{Z}_4$.

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Table 3. Some cyclic codes with $\mathbb{Z}_4$-images.

$\tau_1(x)$	$\tau_2(x)$	$\tau_3(x)$	$Type$	W_L	W_E	W_H
78999	$U^2de's'ds'$	$5'5'5'05'$	4^32^1	$8**$	8	$6**$
789^3	U^3N^2HD	$7'5'3'7'3'$	$4^32^{2^*}$	4^*	8^*	2^*
789^3	$U^3s'e'HD$	$5'0(5')^3$	$4^32^{2^*}$	4^*	8^*	2^*
70779	D^6s'	$3'5'3'7'$	4^32^4	4^*	8	2^*
70779	$N^5s'D$	$7'3'5'7'$	$4^32^{2^*}$	4^*	8^*	2^*
78999	$N^2s'e'Ds's'$	$7'5'3'7'3'$	4^32^1	$8**$	8	$6**$
70779	$N^5e'e'$	$3'5'3'7'$	4^32^4	4^*	8	2^*
79909	$7'7'dHs'bs'$	$7'5'7'7'$	4^32^1	$8**$	8	$6**$
88808	$3'3'3'v'3'v'v'$	$7'7'$	4^02^7	$4**$	8	$2**$
78999	$N^2De's'Ds'$	$7'7'3'5'3'$	4^32^4	4^*	8	2^*
70779	N^5D^2	$7'5'7'7'$	4^32^4	4^*	8	2^*
789^3	$b^2v'0v'$	$7'(3')^203'$	4^32^3	8^{**}	8	4^{**}
88808	$drs'3's'$	$5'5'05'$	4^02^6	8^{**}	16^{**}	4^{**}
7^39799	$3'3'3'v'3'v'v'$	$7'7'7'7'7'7'7'$	4^12^6	$4*$	8	$2*$
70779	$U^3s'e'HD$	$5'05'5'$	$4^32^{2^*}$	4^*	8^*	2^*
78999	$H^3s's'Hs'$	$3'7'$	4^32^4	4^*	8	2^*
789^3	N^5D^2	$5'05'5'$	$4^32^{2^*}$	4^*	8^*	2^*
70779	U^3N^2dD	$7'3'3'03'$	4^32^4	4^*	8	2^*
70779	$U^2de's'ds'$	$7'5'7'7'$	4^32^4	4^*	8	2^*
78999	$H^3s's'Hs'$	$7'5'7'7'$	4^32^4	4^*	8	2^*
79909	$7'7'dHs'bs'$	$(5')^7$	4^32^1	$8**$	8	$6**$
70779	$N^3De'D^2$	$5'05'5'5'$	$4^32^{2^*}$	4^*	8^*	2^*
70779	N^5D^2	$5'5'05'$	$4^32^{2^*}$	4^*	8^*	2^*
70779	$N^5s'D$	$(7')^63'$	$4^32^{2^*}$	4^*	8^*	2^*
88808	$drs'3's'$	$7'5'3'7'3'$	4^02^6	8^{**}	16^{**}	4^{**}
7^39^279	$dUDbs'$	$7'5'3'7'3'$	4^12^6	$4*$	8	$2*$
79909	$7'7'dHs'bs'$	$3'7'$	4^32^4	4^*	8	2^*
78999	$N^5s'D$	$7'7'3'5'3'$	4^32^4	4^*	8	2^*
88808	$drs'3's'$	$3'5'7'3'3'$	4^02^6	8^{**}	16^{**}	4^{**}
79^209	$r^2e'Re'$	$7'7'3'03'$	4^32^0	8	8^*	8^{**}
78999	D^6s'	$7'5'7'7'$	4^32^4	4^*	8	2^*
70779	$N^3s'e's'D$	$(5')^7$	$4^32^{2^*}$	4^*	8^*	2^*
78999	$N^2e'Ne'e'e'$	$3'3'7'5'3'$	4^32^4	4^*	8	2^*
70779	$N^2D^2s'e's'$	$3'5'7'3'3'$	$4^32^{2^*}$	4^*	8^*	2^*
70779	$(e')^7$	$3'5'3'7'$	4^32^4	4^*	8	2^*

Table 4. Some cyclic codes with $\mathbb{Z}_4$-images.

$\tau_1(x)$	$\tau_2(x)$	$\tau_3(x)$	$Type$	W_L	W_E	W_H
70779	$U^2de's'ds'$	$7'7'7'3'7'3'3'$	4^32^1	8$**$	8	6$**$
789^3	U^3N^2HD	$5'5'05'$	$4^32^{2^*}$	4^*	8^*	2^*
70779	$U^2de's'ds'$	$7'7'$	4^32^4	4^*	8	2^*
78999	$N^2s'e'Ds's'$	$3'5'3'7'$	4^32^4	4^*	8	2^*
70779	$r^3e'e're'$	$7'7'3'03'$	4^32^4	4^*	8	2^*
70779	$U^3De'dD$	$(5')^305'$	$4^32^{2^*}$	4^*	8^*	2^*
78999	$H^3s's'Hs'$	$5'5'5'05'$	4^32^1	8$**$	8	6$**$
78999	$N^3s'e's'D$	$7'7'$	4^32^4	4^*	8	2^*
789^3	$N^5s'D$	$3'7'5'7'$	$4^32^{2^*}$	4^*	8^*	2^*
70779	$N^2D^2s'e's'$	$5'5'5'05'$	$4^32^{2^*}$	4^*	8^*	2^*
88808	$drs'3's'$	$3'7'5'7'$	4^02^7	4$**$	8	2$**$
79909	$7'7'dHs'bs'$	$(3')^7$	4^32^1	8$**$	8	6$**$
70779	D^6s'	$3'3'7'5'3'$	4^32^4	4^*	8	2^*
70779	$r^3e'e're'$	$(7')^63'$	4^32^1	8	8	6$*$
70779	$U^3De'dD$	$7'7'$	4^32^4	4^*	8	2^*
70779	$r^3e'e're'$	$7'5'3'7'3'$	4^32^1	8$**$	8	6$**$
70779	$N^2D^2s'e's'$	$5'05'5'5'$	$4^32^{2^*}$	4^*	8^*	2^*
78999	$b^2v'0v'$	$7'7'7'3'7'3'3'$	4^32^4	4^*	8	2^*
70779	U^3N^2dD	$5'0(5')^3$	$4^32^{2^*}$	4^*	8^*	2^*
7^3979^2	$R0R^2$	$7'7'3'03'$	4^12^3	12^{**}	14^{**}	6^{**}
70779	$N^3s'e's'D$	$5'5'05'$	$4^32^{2^*}$	4^*	8^*	2^*
70779	$N^5s'D$	$5'05'5'$	$4^32^{2^*}$	4^*	8^*	2^*
70779	N^5D^2	$7'3'5'7'$	$4^32^{2^*}$	4^*	8^*	2^*
78999	$U^2de's'ds'$	$(3')^7$	4^32^1	8$**$	8	6$**$
70779	$U^2de's'ds'$	$7'7'3'03'$	4^32^4	4^*	8	2^*
7^3979^2	$(3')^3v'3'(v')^2$	$(7')^63'$	4^12^0	14	14^*	14^*
7^2909	$3'3'3'v'3'v'v'$	$7'5'3'7'3'$	4^32^4	4^*	8	2^*
78999	$N^2De's'Ds'$	$3'7'$	4^32^4	4^*	8	2^*
8^308	$(3')^3v'3'v'v'$	$(7')^63'$	4^02^4	12	24^{**}	6^{**}
789^3	$N^2D^2s'e's'$	$3'5'7'3'3'$	$4^32^{2^*}$	4^*	8^*	2^*
707^29	$b^2v'0v'$	$7'5'(3')^3$	4^32^3	8^{**}	8	4^{**}
78999	$U^2de's'ds'$	$7'7'3'03'$	4^32^4	4^*	8	2^*
789^3	U^3N^2HD	$5'05'5'5'$	$4^32^{2^*}$	4^*	8^*	2^*
789^3	$U^3s'e'HD$	$7'3'5'7'$	$4^32^{2^*}$	4^*	8^*	2^*
789^3	$N^3s'e's'D$	$7'07'7'3'$	$4^32^{2^*}$	4^*	8^*	2^*
78999	$U^2de's'ds'$	$7'3'3'03'$	4^32^4	4^*	8	2^*
70779	d^3D^2ds'	$7'5'7'7'$	4^32^4	4^*	8	2^*
70779	$U^3De'dD$	$7'5'3'7'3'$	$4^32^{2^*}$	4^*	8^*	2^*
79909	$7'7'dHs'bs'$	$7'5'3'7'3'$	4^32^4	4^*	8	2^*
70779	$N^3s'e's'D$	$3'7'$	4^32^4	4^*	8	2^*
7^39799	$drs'3's'$	$5'5'5'05'$	4^12^6	4$*$	8	2$*$

Table 4. (Continued).

$\tau_1(x)$	$\tau_2(x)$	$\tau_3(x)$	$Type$	W_L	W_E	W_H
$8^3 08$	$(3')^2v'3'(v')^2$	$(5')^2$	$4^0 2^4$	12	24^{**}	6^{**}
88808	$drs'3's'$	$3'5'7'3'3'$	$4^0 2^6$	8^{**}	16^{**}	4^{**}
70779	$H^3s's'Hs'$	$3'3'7'5'3'$	$4^3 2^4$	4^*	8	2^*
77909	$5'e'RNr$	$(5')^7$	$4^3 2^1$	8^{**}	8	6^{**}
789^3	N^5D^2	$5'05'5'5'$	$4^3 2^{2^*}$	4^*	8^*	2^*
70779	$H^3s's'Hs'$	$7'7'3'03'$	$4^3 2^4$	4^*	8	2^*
70779	$r^3e'e're'$	$3'5'3'7'$	$4^3 2^4$	4^*	8	2^*
78999	$r^3e'e're'$	$3'3'7'5'3'$	$4^3 2^4$	4^*	8	2^*
70779	$U^3s'e'HD$	$3'7'$	$4^3 2^4$	4^*	8	2^*
78999	$N^3Ds'e'D$	$5'5'5'05'$	$4^3 2^1$	$8{*}{*}$	8	$6{*}{*}$
70779	$r^3e'e're'$	$7'5'7'7'$	$4^3 2^4$	4^*	8	2^*

Table 5. Some $(1+2u^2)$-constacyclic codes with $\mathbb{Z}_4$-images.

$\tau_1(x)$	$\tau_2(x)$	$\tau_3(x)$	$Type$	W_L	W_E	W_H
79989	$D^3s'Ds'$	$(5')^7$	$4^3 2^1$	8^{**}	8	6^{**}
79	$3'3'7'e'H$	$7'7'7'3'3'7'3'$	$4^6 2^1$	4^*	4^*	2^*
78999	$7'5'3'v'v'$	$7'7'7'3'7'3'$	$4^3 2^1$	8^{**}	8	6^{**}
7789	$3'5'7'v'v'$	$(7')^6 3'$	$4^4 2^0$	6^*	6^*	6^{**}
79989	$H5'dHs'$	$7'3'5'7'$	$4^3 2^4$	4^*	8	2^*
88808	$drs'3's'$	$7'3'5'7'$	$4^0 2^7$	$4{*}{*}$	8	$2{*}{*}$
88808	$drs'3's'$	$3'5'3'7'$	$4^0 2^7$	$4{*}{*}$	8	$2{*}{*}$
97789	$5'5'rFe're'$	$5'5'5'05'$	$4^3 2^1$	8^{**}	8	4^*
99	$7'7'7'b7'bv'$	$7'7'7'3'7'3'3'$	$4^6 2^1$	4^*	4^*	2^*
97789	$N^2D^2s'Ns'$	$7'3'5'7'$	$4^3 2^4$	4^*	8	2^*
9989	$DFDd$	$7'7'7'7'7'7'3'$	$4^4 2^3$	4^*	6^*	2^*
9989	$U^3NUe'e'$	$7'5'3'7'3'$	$4^4 2^0$	8	8	6^{**}
9797979	$7'7'7'dDbH$	$7'7'$	$4^1 2^6$	4^*	8	2^*
79909	$5'5'e'e'rFe'$	$7'7'7'3'7'3'3'$	$4^3 2^1$	8^{**}	8	6^{**}
78979	$7'7'dv'DHs'$	$5'5'05'$	$4^3 2^1$	8^{**}	8	6^{**}
98779	$7'7'3'rH$	$5'05'5'$	$4^3 2^3$	8^{**}	8	4^{**}
77909	$5'e'RNr$	$(5')^7$	$4^3 2^1$	8^{**}	8	6^{**}
7979799	$UN^2 3'D$	$7'5'7'7'$	$4^1 2^6$	4^*	8	2^*
9989	$7'7'7'b7'bv'$	$7'5'7'7'$	$4^4 2^3$	4^*	6^*	2^*
7979799	$U3'drD$	$3'7'$	$4^1 2^6$	4^*	8	2^*
7977779	$R3'bFb$	$7'5'3'3'3'$	$4^1 2^3$	12^{**}	14^{**}	6^{**}
98779	$r^3e're'e'$	$7'5'7'7'$	$4^3 2^4$	4^*	8	2^*
79	$7'7'7's'bs'H$	$7'3'5'7'$	$4^6 2^1$	4^*	4^*	2^*
7979799	$U^3e'e'Ue'$	$7'07'7'3'$	$4^1 2^3$	12^{**}	14^{**}	6^{**}
7789	$7'7'dv's'ds'$	$7'7'$	$4^4 2^3$	4^*	6^*	2^*

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