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An extension of maximum principle with some applications

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Abstract: Let $U \subseteq R^n$ (*res.* $D \subset R^n$) be an open (*res.* a compact) subset, and let L be an elliptic operator defined on $C^2(U,R)$ (*res.* $C^2(D,R)$). In the present paper, we are going to extend the maximum principle for the function $f \in C^2(U,R)$ (*res.* $f \in C^2(D,R)$) satisfying the equation $Lf = \varepsilon$, where ε is a real everywhere nonzero continuous function on U (*res.* D). Finally, we obtain some applications in mathematics and physics.

Key words: Boundary behavior, elliptic operator, maximum principle, positive definite matrix

1. Preliminary notes

Let $U \subseteq R^n$ be an open set. A function $f : U \to R$ with two continuous partial derivatives is called a C^2 function. The set of all real C^2 functions defined on U, is denoted by $C^2(U,R)$. Let $f : U \to R$ be an element of $C^2(U,R)$, and let ∇^2 denotes the Laplace operator. The function f is said to be harmonic if $\nabla^2 f = 0$. Suppose that $\varepsilon : U \to R$ is an everywhere positive (*res.* negative) continuous function. The function f is said to be $\varepsilon-$strictly subharmonic (*res.* $\varepsilon-$strictly superharmonic) if $\nabla^2 f = \varepsilon$. For a nonempty set $E \subseteq R^n$, the function $f : E \to R$ is said to be harmonic on E if there exists an open set U containing E and a function $g \in C^2(U,R)$ such that, $g_{|E} = f$ and $\nabla^2 g = 0$ on E. Similarly, for a nonempty set $E \subseteq R^n$ and an everywhere positive (*res.* negative) continuous function $\varepsilon : E \to R$, the function $f : E \to R$ is said to be $\varepsilon-$strictly subharmonic (*res.* $\varepsilon-$strictly superharmonic) on E, if there exists an open set U containing E and a function $g \in C^2(U,R)$ such that, $g_{|E} = f$ and $\nabla^2 g = \varepsilon$ on E. The sets of all harmonic, $\varepsilon-$strictly subharmonic, and $\varepsilon-$strictly superharmonic functions on E are denoted by $H(E)$, $SBH(\varepsilon,E)$ and $SPH(\varepsilon,E)$, respectively.

Let $A = [a_{ij}]$ be an $n \times n$ positive definite symmetric matrix and let $L = (\frac{\partial}{\partial X})A(\frac{\partial}{\partial X})^t$. Then L is said to be an elliptic operator, and the C^2 function $f : U \to R$ is said to be $L-$harmonic on U if $L(f) = 0$. For an everywhere positive (*res.* negative) continuous function $\varepsilon : U \to R$, the function $f \in C^2(U,R)$ is said to be $\varepsilon L-$strictly subharmonic (*res.* $\varepsilon L-$strictly superharmonic) on U if $L(f) = \varepsilon$. The sets of all $L-$harmonic, $\varepsilon L-$strictly subharmonic and $\varepsilon L-$strictly superharmonic functions on U, are denoted by $H(L,U)$, $SBH(\varepsilon,L,U)$ and $SPH(\varepsilon,L,U)$ respectively. For an arbitrary nonempty set $E \subseteq R^n$, the sets $H(L,E)$, $SBH(\varepsilon,L,E)$ and $SPH(\varepsilon,L,E)$ are defined as $H(E)$, $SBH(\varepsilon,E)$ and $SPH(\varepsilon,E)$, respectively.

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2. Introduction

The maximum principle refers to a collection of results and techniques in the study of partial differential equations. It is also a very valuable tool for most results concerning existence, uniqueness and qualitative properties of solutions to quasilinear, linear, elliptic and parabolic types of partial differential equations. There are numerous references, in which the authors have provided some of the aspects of this principle.

Some early results, related to the maximum principle for harmonic functions of two variables defined on a domain of the Euclidean plane, is interpreted in [8, 25, 29]. In these discussions, the harmonic functions are proposed as the real part of some analytic functions.

Some of the generalizations of maximum principle in the last three dedicates are given in Banach spaces. For instance, the maximum principle for "Banach space valued harmonic functions" is established by Dowling in [17]. The author showed that for an open connected subset D of the complex field C and the real Banach space X, the maximum principle holds for every function $f: D \to X$, if and only if X is a strictly convex Banach space. The result is a real analogue of Thorp and Whitley on the strong maximum modulus theorem for "Banach space valued analytic functions" [36]. Some extensions of maximum principle for vector-valued analytic and harmonic functions are also considered by the same author in [16].

One of the research topics in this theory, is the study of the analogies of the maximum principles known for different types of partial differential equations, as well as their applications to analysis of their solutions.

For instance, in [13], De Figueiredo and Mitidieri discussed the elliptic system

$$-\triangle u = f(x, u) - v, -\triangle v = \delta u - \gamma v, in\ \Omega \tag{2.1}$$

subject to Dirichlet boundary conditions $u = v = 0$ on the boundary of Ω, $\partial\Omega$. Here $\triangle$ is the Laplacian, δ, γ are positive constants, and $\Omega \subset R^n (n \geq 2)$ is a bounded smooth domain. They used a maximum principle for a linear elliptic system associated with (2.1), as the key ingredient for the question of existence of positive solutions.

In [15], the same authors investigated the weakly coupled elliptic system

$$\begin{cases} L(D)U = A(x)U + F & in\ \Omega \subset R^n, \\ U = 0 & on\ \partial\Omega \end{cases}$$

concerning maximum principle, where $\Omega \subset R^n (n \geq 1)$ is a bounded smooth domain, $L(D)$ is a diagonal matrix of second order elliptic operators, $A(x)$ is an $n \times n$ coefficient matrix and F is a given $n-$vector function defined in Ω. Here, the maximum principle means that "$U \geq 0$ in Ω when the given function $F \geq 0$ in Ω".

In [14], the authors established maximum principles for weakly coupled elliptic systems of the form

$$\begin{cases} L_k D = \Sigma a_{kj} u_j + f_k & in\ \Omega,\ k = 1, \ldots, n \\ u_k = 0 & on\ \partial\Omega \end{cases}$$

where the set

$$L_k(D) = -\Sigma_{ij} b_{ij} D_i D_j + \Sigma_i b_i^k D_i, k = 1, \ldots, n$$

of second order elliptic operators with real cofficients, defined in some bounded domain $\Omega \subset R^n$.

In [19], Fleckinger et al. considered another cooperative elliptic system on a bounded smooth domain $\Omega \subset R^d$. They studied the problem

$$\begin{cases} -\triangle_p u_i = \Sigma a_{ij}(x) |u_j|^{p-2} u_j + f_i & in\ \Omega, \\ u_i = 0 & on\ \partial\Omega \end{cases}$$

where the coefficients $a_{ij}(1 \leq i, j \leq n)$ are constant, $a_{ij} \geq 0$ for $i \neq j$ (cooperative systems), and the operator $\triangle_p$ defined by $\triangle_p u = div(|\nabla u|^{p-2}\nabla u), p > 1$, is called p-Laplacian. They are concerned with existence of positive solutions and with the following form of maximum principle, called inverse positivity: The hypothesis $f_i > 0$ on Ω implies $u_i > 0$ on Ω for any solution $U = (u_i)$.

In [12], Corrêa and Souto investigated the maximum principle for the linear system

$$\begin{cases} L(U) = BU + F(x) & in\ \Omega, \\ \beta(U) = 0 & on\ \partial\Omega \end{cases} \tag{2.2}$$

where Ω is a bounded smooth domain in R^n, $L = [L_1, \ldots, L_p]$ is a diagonal-matrix with second order elliptic operators $L_k u = -a_{ij}^k D_{ij} u + a_i^k(x) D_i u, 1 \leq k \leq p$, $B(x) = (b_{ij}(x))$ is a $p \times p$ cooperative matrix (i.e. $b_{ij} \geq 0$ in Ω, if $i \neq j$), $F(X) = (f_1(x), \ldots, f_p(x))^t$ is a given p-vector function defined in Ω, $u = (u_1, \ldots, u_p)^t$ is the p-vector solution and $\beta(U)$ is some boundary condition. By a maximum principle they meant the statement: if $F \geq 0$ in Ω, then $U \geq 0$ in Ω whenever U is a solution of (2.2). As usual $U \geq 0$ means $u_i \geq 0$ for all $i = 1, \ldots, p$.

In [31], Serag and El-Zahrani studied the problem

$$\begin{cases} -\triangle_p u = a(x)|u|^{p-2}u + b(x)|u|^{\alpha}|v|^{\beta}v + f & x \in R^n, \\ -\triangle_q v = c(x)|u|^{\alpha}|v|^{\beta}u + d(x)|v|^{q-2}v + g & x \in R^n, \\ lim_{|x|\to\infty} u(x) = lim_{|x|\to\infty} v(x) = 0, u, v > 0 & in\ R^n \end{cases}$$

where $\alpha, \beta > 0, 1 < p, q < n$, the degenerated $p-$Laplacian defined as $\triangle_p u = div|\nabla u|^{p-2}\nabla u]$ with $p > 1$, $p \neq 2$, and f, g are given functions. Moreover, the cofficients $a(x), b(x), c(x)$ and $d(x)$ are given positive smooth functions satisfied some inequlities as the authors stated in [31]. They obtained some necessary and sufficient conditions for having a maximum principle.

In [33], Serag and Qamlo obtained some necessary and sufficient conditions for having the maximum principle and existence of positive solutions for some cooperative systems involving Schrödinger operators defined on unbounded domains. Then, they deduced the existence of solutions for semilinear systems. Finally they discussed the generalized maximum principle for non cooperative systems.

In [22], Khafagy considered the problem

$$\begin{cases} -\triangle_{P,p} u = -div[P(x)|\nabla u|^{p-2}\nabla u] = a(x)|u|^{p-2}u + b(x)|u|^{\alpha}|\beta|^{\beta}v + f & x \in R^n, \\ -\triangle_{Q,q} v = -div[Q(x)|\nabla v|^{p-2}\nabla v] = c(x)|u|^{\alpha}|v|^{\beta}u + d(x)|v|^{q-2}|v + g & x \in R^n, \\ lim_{|x|\to\infty} u(x) = lim_{|x|\to\infty} v(x) = 0, u, v > 0 & in\ R^n \end{cases}$$

where $P(x)$ is a weight function and $\triangle_{P,p}$ with $1 < p < n$, $p \neq 2$ denotes the degenerate p-Laplacian defined by $\triangle_{P,p} u = div[P(x)|\nabla u|^{p-2}\nabla u]$. They gave necessary and sufficient conditions to have a maximum principle for this system and proved the existence of weak solutions for the same system by using an approximation method.

In [32], Serag and Khafagy studied the problem

$$\begin{cases} -\triangle_{P,p} u_i = \Sigma_{j=1}^{j=n} a_{ij}(x)|u_j|^{p-2}u_j + f_i(x, u_1, \ldots, u_n) & in\ \Omega, \\ u_i = 0, i = 1, \ldots, n & on\ \partial\Omega, \end{cases}$$

where $P(x)$ is a weight function and $\triangle_{P,p}$ with $p>1$, $p\neq 2$ defined as above. The author gave some conditions for having the maximum principle for this system and then proved the existence of positive weak solutions for the quasilinear system.

The study of Enache [18] is concerned with a class of boundary value problems for fully nonlinear elliptic PDEs involving the $p-$Hessian operator. Here, Enache derived a maximum principle for a suitable function involving the solution u and its gradient of the following type of boundary value problems

$$\begin{cases} S_p(D^2(u)) = g(u)h(|\nabla u|^2) & in\ \Omega, \\ u = 0 & on\ \partial\Omega \end{cases}$$

where $n\geq 2, \Omega\subset R^n$ is a bounded domain containing the origin, $p\in\{1,\ldots,n\}$, and g, h are positive C^1 functions assumed to satisfy the following condition

$$h^{-1}\frac{g^{'}}{g}+2\left(\begin{array}{c} n \\ p \end{array}\right)^{\frac{-1}{p}} g^{\frac{1}{p}}\frac{h^{'}}{h}\geq 0.$$

This maximum principle is then applied to obtain some sharp estimates for the solution and the magnitude of its gradient.

Some of the recent works in this area, are related to the fractional derivative of functions. Proving the maximum principle, uniqueness of solutions to the initial boundary value problems for the time-fractional partial differential equations are some of the results in these works.

In [2], the initial boundary value problems for linear and nonlinear multiterm fractional diffusion equations with the Riemann–Liouville time-fractional derivatives are considered. To guarantee the uniqueness of solutions, Al-Refai and Luchko employed the weak and the strong maximum principles for the equations under consideration that were formulated and proved. An essential element of their proof of the maximum principles is an estimation for the value of the Riemann–Liouville fractional derivative of a function at its maximum point that is established for a wider space of functions compared to those used in their previous works.

In [37], Ye et al. considered a multiterm time-space Riesz–Caputo fractional differential equation over an open bounded domain and proved a maximum principle for the equation. They also derived the uniqueness and continuous dependence of the solution.

In [3], Al-Refai and Luchko analyzed the initial boundary value problems for the one-dimensional linear and nonlinear fractional diffusion equations with the Riemann–Liouville time-fractional derivative. First, they derived a weak and a strong maximum principles for solutions of the linear problems. Then they employed these principles to show uniqueness of solutions of the initial boundary value problems for the nonlinear fractional diffusion equations under some standard assumptions posed on the nonlinear part of the equations.

In [4], Alsaedi et al. presented an inequality for the fractional derivatives related to the Leibniz rule to obtain a modern proof of the maximum principle for the fractional differential equations.

In [1], Al-Refai and Luchko formulated and proved the weak and strong maximum principles for a general parabolic-type fractional differential operator with the Riemann–Liouville time-fractional derivative of distributed order. The proofs of the maximum principles are based on an estimate of the Riemann–Liouville fractional derivative at its maximum point that was recently derived by the authors.

In [27], Luchko and Yamamoto after introducing the general fractional derivatives of the types of Caputo and Riemann–Liouville, derived some important estimates for the general time-fractional derivatives of the mentioned types of a function at its maximum point. Then these estimates are applied to prove a weak maximum principle for the general time-fractional diffusion equation.

In [38], Zhenhai et al. dealt with maximum principles for "multiterm space-time variable-order Riesz–Caputo fractional differential equations". They firstly derived several important inequalities for variable-order fractional derivatives at extreme points. Then, based on these inequalities, they obtained the maximum principles. Finally, these principles were employed to show the uniqueness of solutions of the "multiterm space-time variable-order Riesz–Caputo fractional differential equations" and continuous dependance of solutions on initial boundary value conditions.

In [10], Chan and Li established a weak maximum principle for a fractional diffusion equation involving the Riemann–Liouville fractional derivative. Then they used it to prove the uniqueness and the continuous dependence of a solution on the initial data.

In [24], Kochubei and Luchko devoted an in-depth discussion of the maximum principle for the time-fractional partial differential equations, with applications including uniqueness of solutions to the initial boundary value problems for the time-fractional partial differential equations.

In [28], Luchko and Yamamoto dealt with the following initial boundary value problem for the single-term time-fractional diffusion equation

$$\begin{cases} \partial_t^{\alpha}(x,t) = \Sigma_{i,j=1}^{n}\partial_i(a_{ij}(x)\partial_j u(x,t)) + c(x)u(x,t) + F(x,t) & x \in \Omega, t > 0, \\ u(x,t) = 0 & x \in \partial\Omega, t > 0, \\ u(x,0) = a(x) & x \in \Omega \end{cases}$$

where $0 < \alpha < 1$, Ω is a bounded domain with smooth boudary $\partial\Omega$, $c \in C^1(\Omega)$, $a_{ij} = a_{ji} \in C(\Omega)$ for $1 \le i,j \le n$, and there exists a constant $\mu_0 > 0$ such that $\Sigma_{i,j=1}a_{ij}(x)\xi_i\xi_j \ge \mu_0\Sigma_{i=1}^{n}\xi_i^2$ for all $x \in \Omega$ and $\xi_1,\dots,\xi_n \in R$. They introduced a key lemma, that is a basis for the proofs of all other results. Then the key lemma and the fixed point theorem, are employed to prove the maximum and comparison principles and some of their corollaries.

In [9], Cao et al. proposed maximum and minimum principles for time-fractional Caputo–Katugampola diffusion operators. They proved several inequalities at extreme points, and considered uniqueness and continuous dependence of solutions for fractional diffusion equations of initial boundary value problems.

In [11], Chen and Li considered nonlinear equations involving the following fractional p-Laplacian

$$(-\triangle)_p^s u(x) \equiv C_{n,s,p} PV \int_{R^n} \frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)]}{|x-z|^{n+ps}} dy = f(x,u).$$

They proved a maximum principle for antisymmetric functions and obtained other key ingredients for carrying on the method of moving planes, such as a variant of the Hopf lemma. Then they established radial symmetry and monotonicity for positive solutions to semilinear equations involving the fractional p-Laplacian in a unit ball and in the whole space.

In [5], Bahaa investigated the optimal control problem for fractional order cooperative system governed by Schrödinger operator, and discussed the maximum principle for the fractional order cooperative system.

In [7], Borikhanov et al. formulated and proved a maximum principle for the one-dimensional subdiffusion equation

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \frac{\partial^2}{\partial x^2} D_{*t}^{1-\alpha} u(x,t) + F(x,t,u) & in \ (0,a)\times(0,T] = \Omega, \\ u(x,t) = \varphi(x) & x \in [0,a], \\ u(0,t) = \lambda(t), u(a,t) = \mu(t) & t \in [0,T] \end{cases} \tag{2.3}$$

with Atangana–Baleanu fractional derivative. Their proof of the maximum principle is based on an extremum principle for the Atangana–Baleanu fractional derivative that is given in the paper. Then they applied the maximum principle and showed that the initial boundary value problem for the linear and nonlinear time-fractional diffusion equations possesses at most one classical solution and this solution continuously depends on the initial and boundary conditions.

In [23], Kirane and Torebek obtained some new estimates of the Hadamard fractional derivatives of a function at its extreme points. They applied the extremum principle to show that the initial boundary value problem for linear and nonlinear time-fractional diffusion equations possesses at most one classical solution and this solution depends continuously on the initial and boundary conditions. Moreover, they proved the extremum principle for an elliptic equation with a Hadamard fractional derivative.

In [6], Boccardo and Orsina proved that the strong maximum principle holds for solutions of some quasilinear elliptic equations having lower order terms with quadratic growth with respect to the gradient of the solution.

In [34], Shengda et al. obtained two significant inequalities for generalized time fractional derivatives at extreme points. They applied the inequalities to establish the maximum principles for multiterm time-space fractional variable-order operators. Finally, they employed the principles to investigate two kinds of diffusion equations involving generalized time-fractional Caputo derivatives and space-fractional Riesz–Caputo derivatives.

In the present paper, we obtain the maximum principle for the elements of $H(\varepsilon, E)$, $SBH(\varepsilon, L, E)$, and $SPH(\varepsilon, L, E)$ where E is an open or compact subset of R^n, $\varepsilon : E \to R$ is an everywhere nonzero continuous function, and $L = (\frac{\partial}{\partial X})A(\frac{\partial}{\partial X})^t$ is an elliptic operator on $C^2(E, R)$. Finally, we employ the results to deduce some applications in the theory of elliptic boundary value problems, existence of optimum LP solution in operational research, greatest distance in Euclidian geometry, smallest force, and smallest light intensity in physics. Among other things, we provide a new theorem about extremum values of quadratic forms on the unit disc in the general inner product spaces.

3. Equivalence

At first we indicate that two arbitrary sets X and Y are said to be equivalent if there exists a 1-1 mapping of X onto Y. To capture this idea in set theoretic terms, we write $X \sim Y$ [26]. Now, we would intend to show that the following theorems hold.

Theorem 3.1 *For a nonzero constant function $\varepsilon > 0\,(res.and \varepsilon < 0)$, and for an arbitrary set $E \subseteq R^n$, we have $H(E) \sim SBH(\varepsilon, E)$, (res. $H(E) \sim SPH(\varepsilon, E)$).*

Proof Let $\varphi : H(E) \to SBH(\varepsilon, E)$ ($res.$ $\varphi : H(E) \to SPH(\varepsilon, E)$) defined by

$$\varphi(f) = f + h, \tag{3.1}$$

where $h : R^n \to R$ is the function $h(x_1 \dots, x_n) = \frac{\varepsilon}{2n} \sum_{i=1}^{n} x_i^2$. Then φ is well defined, one to one and surjective. $\square$

Suppose that $A = [a_{ij}]$ is an $n \times n$ positive definite symmetric matrix, and $L = (\frac{\partial}{\partial X})A(\frac{\partial}{\partial X})^t$ is an elliptic operator. Theorem 3.1 can be generalized as follows:

Theorem 3.2 *For a nonzero constant function $\varepsilon > 0\,(res.and\varepsilon < 0)$, and for an arbitrary set $E \subseteq R^n$, we have ($H(L, E) \sim SBH(\varepsilon, L, E)$), (res.$H(L, E) \sim SPH(\varepsilon, L, E)$).*

Proof Let $\varphi : H(L, E) \to SBH(\varepsilon, L, E)$ (*res.* $\varphi : H(L, E) \to SPH(\varepsilon, L, E)$) be defined as in (3.1) for $M = 2\Sigma_i a_{ii} + 4\Sigma_{i<j} a_{ij}$ for $(1 \le i, j \le n)$ and

$$h(x_1, \dots, x_n) = \frac{\varepsilon}{M}(\sum_i x_i^2 + 2\sum_{i<j} x_i x_j).$$

A simple computation shows that $M \neq 0$, and φ is well defined, one to one and surjective. $\square$

4. Existence and uniqueness

In this section, we are going to obtain some results on the existence and uniqueness of extremum points of the elements of $H(D)$, $H(L, D)$, $SBH(\varepsilon, L, D)$ and $SPH(\varepsilon, L, D)$, for a compact subset $D \subset R^n$. We divide the problem into two steps.

4.1. Step one

We begin with the elements of $H(D)$, $SBH(\varepsilon, D)$ and $SPH(\varepsilon, D)$.

Theorem 4.1 *Let $\emptyset \neq U \subseteq R^n$ be an open set, and $\varepsilon > 0$ (res. $\varepsilon < 0$) on U. Then every $f \in SBH(\varepsilon, U)$ (res. $f \in SPH(\varepsilon, U)$) does not have a local maximum (res. minimum) point on U.*

Proof Let $x_0 \in U$ be the local maximum point of $f \in SBH(\varepsilon, U)$ for $\varepsilon > 0$. Then $\frac{\partial f}{\partial x_i}(x_0) = 0$ and $\frac{\partial^2 f}{\partial x_i^2}(x_0) \le 0$ for all $i = 1, \dots, n$. Therefore $\nabla^2 f = \varepsilon \le 0$, contradics the hypothesis. The proof of the elements of $SPH(\varepsilon, U)$ is similar. $\square$

As a consequence of the properties of continuous functions (see [30]) and Theorem 4.1, we have the following corollary.

Corollary 4.2 *Let $\emptyset \neq U \subset R^n$ be an open set, ∂U be its boundary in R^n, $D = U \cup \partial U$ be a bounded set and $\varepsilon > 0$ (res. $\varepsilon < 0$). Then every $f \in SBH(\varepsilon, D)$ (res. $f \in SPH(\varepsilon, D)$) has an absolute maximum (res. minimum) point on ∂U.*

Theorem 4.3 *Let $\emptyset \neq U \subset R^n$ be an open set, and let $D = U \cup \partial U$ be a bounded set. If $f \in H(D)$ has an absolute maximum (res. minimum) point on U, then it has an absolute maximum (res. minimum) point on ∂U with the same value.*

Proof Let $x_0 \in U$ be the absolute maximum point of f. Let $w \in H(\varepsilon, D)$ for some $\varepsilon > 0$, and define the sequence of functions $\{f_k\}_{k\in N}$ by $f_k(x) = f(x)+\frac{1}{k}w(x)$. Therefore $\nabla^2 f_k(x) = \frac{\varepsilon}{k} > 0$ and so, $f_k \in SBH(\frac{\varepsilon}{k}, D)$. According to Corollary 4.2, let $x_k \in \partial U$ be the absolute maximum point of f_k, then $f_k(x_k) \geq f_k(x_0)$ and

$$f(x_0) \geq f(x_k) \geq f(x_0) + \frac{1}{k}(w(x_0) - w(x_k)),$$

therefore $w(x_0) \leq w(x_k)$. Let $lim_{k\to+\infty} x_k = x_\infty \in \partial U$, then

$$f(x_0) \geq lim_{k\to+\infty} f(x_k) \geq f(x_0) + lim_{k\to+\infty}\frac{1}{k}(w(x_0) - w(x_k))$$

and so $f(x_0) \geq f(x_\infty) \geq f(x_0)$, therefore $f(x_0) = f(x_\infty)$. The proof of the absolute minimum point is similar. □

A celebrated theorem in complex analysis asserts that a holomorphic function defined on a disc, is completely determined by its values on the boundary of the disc [29]. The following important corollaries are some similar versions for the stated theorem.

Corollary 4.4 *Let $\emptyset \neq U \subset R^n$ be an open set, and let $D = U \cup \partial U$ be a bounded set. Let $f : D \to R$ be an element of $H(D)$ such that $f_{|\partial U} = c$ for some constant c. Then $f_{|D} = c$.*

Corollary 4.5 *Let $\emptyset \neq U \subset R^n$ be an open set, $D = U \cup \partial U$ be a bounded set, and $f, g \in H(D)$ (res. $\nabla^2 f = \nabla^2 g = \epsilon$ for an arbitrary continuous function ϵ defined on D) such that $f_{|\partial U} = g_{|\partial U} = \varphi$ for some continuous function $\varphi : \partial U \to R$. Then $f_{|D} = g_{|D}$. In other words, φ has at most one extension on D.*

The following example shows that the extenstion of φ on D in Corollary 4.5, in general, does not exist. It also shows that an elliptic boundary value problem does not have essentially a zero.

Example 4.6 *Let $U = (1, 2) \cup (3, 4)$, $\varphi_1(x) = x$, $\varphi_2(x) = 0$ and $\varepsilon(x) = 6x$. Then $D = [1, 2] \cup [3, 4]$ and the elliptic boundary value problem*

$$\begin{cases} \frac{d^2 f}{dx^2} = \varepsilon & if\ x \in D, \\ f(x) = \varphi_i(x) & if\ x \in \partial D \end{cases}$$

does not have any solution for $i = 1, 2$.

The following example shows that the function φ in Corollary 4.5, may have infinitely many extensions on a set properly containing D.

Example 4.7 *Let $m \in N$, and the functions $\alpha, \beta_m, \gamma_m : R \to R$ are defined by*

$$\alpha(x) = \begin{cases} exp(\frac{-1}{x}) & if\ x > 0, \\ 0 & if\ x \leq 0 \end{cases}$$

and

$$\beta_m(x) = \alpha(x - m)\alpha(m + 1 - x), \quad \gamma_m(x) = \begin{cases} \frac{\int_x^{m+1} \beta_m(t)dt}{\int_m^{m+1} \beta_m(t)dt} & if\ x \leq m + 1, \\ 0 & if\ x > m + 1. \end{cases}$$

Let $U = \{(x_1, \dots, x_n) | x_1^2 + \cdots + x_n^2 < \frac{1}{2}\}$, *and* $\varphi : \partial U \to R$ *is defined by* $\varphi(x_1, \dots, x_n) = 1$, *then a computation shows that* $f_m : R^n \to R$ *given by* $f_m(x_1, \dots, x_n) = \gamma_m(\sqrt{x_1^2 + \cdots + x_n^2})$ *is an element of* $C^2(D, R) \cap H(D)$, *and also an extension of* φ *on* R^n *for all* $m \in N$.

The following example shows that an element of $H(U)$, in general, does not have an extension in $H(D)$.

Example 4.8 *Let* $n \in N$, $U = \{(x_1, \dots, x_{4n}) | 0 < \Sigma_{i=1}^{i=4n} x_i^2 < 1\}$ *and*

$$f(x_1, \dots, x_{4n}) = (\Sigma_{i=1}^{i=4n} x_i^2)^{-1},$$

then $\partial U = (x_1, \dots, x_{4n}) | \Sigma_{i=1}^{i=4n} x_i^2 = 1\} \cup \{0\}$, $f \in H(U)$, *and an argument using continuity shows that* f *does not have an extension* $\hat{f} \in H(D)$.

4.2. Step two

Here we will generalize the previous theorems for the elements of $H(L, E)$, $SBH(\varepsilon, L, E)$ and $SPH(\varepsilon, L, E)$, where E is an open or compact subset of R^n, ε is an everywhere nonzero continuous function on E, and L is an elliptic operator on $C^2(E, R)$.

Any $n \times n$ symmetric positive definite matrix is orthogonally similar to the diagonal matrix $\Lambda = diag(\lambda_1, \dots, \lambda_n)$ with real entries of its eigenvalues. Therefore there exists an invertible matrix C such that $\Lambda = CAC^{-1}$ and $C^{-1} = C^t$ [20]. Let $\frac{\partial}{\partial x}$ be the $1 \times n$ matrix $\frac{\partial}{\partial x} = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$, x be the $1 \times n$ matrix $x = (x_1, \dots, x_n)$, and $\frac{\partial}{\partial x_i} \cdot \frac{\partial}{\partial x_j} = \frac{\partial^2}{\partial x_i \partial x_j}$ symbolically. Define the new matrix $y = (y_1, \dots, y_n)$ by $y^t = Cx^t$. A simple computation shows that $\lambda_i > 0$ for all $1 \le i \le n$, $A = C^t \Lambda C$ and $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} C$. Therefore

$$L = \frac{\partial}{\partial x} A (\frac{\partial}{\partial x})^t = (\frac{\partial}{\partial y} C) A C^t \left(\frac{\partial}{\partial y}\right)^t = \frac{\partial}{\partial y} \Lambda \left(\frac{\partial}{\partial y}\right)^t = \Sigma_{i=1}^n \lambda_i \frac{\partial^2}{\partial y_i^2}.$$

The following results are immediate consequences of subsection 4.1 and the former discussion.

Theorem 4.9 *Let* $\emptyset \ne U \subseteq R^n$ *be an open set,* L *be an elliptic operator and* $\varepsilon > 0$ *(res.* $\varepsilon < 0$*) on* U. *Then every* $f \in SBH(\varepsilon, L, U)$, *(res.* $SPH(\varepsilon, L, U)$*), does not have a local maximum (res. minimum) point on* U.

Corollary 4.10 *Let* $\emptyset \ne U \subset R^n$ *be an open set,* $D = U \cup \partial U$ *be a bounded set,* L *be an elliptic operator on* $C^2(D, R)$ *and* $\varepsilon > 0$ *(res.* $\varepsilon < 0$*) on* D. *Then every* $f \in SBH(\varepsilon, L, D)$, *(res.* $SPH(\varepsilon, L, D)$*) has an absolute maximum (res. minimum) point on* ∂U.

Theorem 4.11 *Let* $\emptyset \ne U \subset R^n$ *be an open set,* $D = U \cup \partial U$ *be a bounded set, and let* L *be an elliptic operator on* $C^2(D, R)$. *If* $f \in H(L, D)$ *has an absolute maximum (res. minimum) point on* U, *then it has an absolute maximum (res. minimum) point on* ∂U *with the same value.*

Corollary 4.12 *Let* $\emptyset \ne U \subset R^n$ *be an open set,* $D = U \cup \partial U$ *be a bounded set, and let* L *be an elliptic operator on* $C^2(D, R)$. *If* $f : D \to R$ *be an element of* $H(L, D)$ *with* $f = c$ *on* ∂U *for some constant* c, *then* $f = c$ *on* D.

Corollary 4.13 *Let $\emptyset \neq U \subset R^n$ be an open set, $D = U \cup \partial U$ be a bounded set, $\varphi : \partial U \to R$ be a continuous function, L be an elliptic operator on $C^2(D,R)$, and $f,g \in H(L,D)$ (res. $Lf = Lg = \epsilon$ for an arbitrary continuous function ϵ defined on D) such that $f_{|\partial U} = g_{|\partial U} = \varphi$. Then $f_{|D} = g_{|D}$.*

Remark 4.14 *Suppose that $\emptyset \neq U \subset R^n$ is an open subset, $D = U \cup \partial U$, and $\varepsilon : D \to R$, $\varphi : \partial D \to R$ are continuous functions. Let L be an elliptic operator on $C^2(D,R)$ with matrix A, $f : D \to R$ be a C^2 function such that $Lf = \varepsilon$ on D, and $f = \varphi$ on ∂D. Let C be an orthogonal matrix such that $CAC^{-1} = \Lambda = diag(\lambda_1, \ldots, \lambda_n)$, $\lambda_i > 0$ for $i = 1, \ldots, n$. Let $y = (y_1, \ldots, y_n), x = (x_1, \ldots, x_n)$ and define the linear isomorphism $l : R^n \to R^n$ by $l(x) = xC^t = y$. If $V = l(U) = UC^t$ and $\tilde{D} = V \cup \partial V$, then $\partial\tilde{D} = l(\partial D) = (\partial D)C^t$. Define the functions $g : \tilde{D} \to R$, $\psi : \partial\tilde{D} \to R$ and $\epsilon : \tilde{D} \to R$ by $g(y) = f(x)$, $\psi(y) = \varphi(x)$ and $\epsilon(y) = \varepsilon(x)$ respectively, so $g(\partial\tilde{D}) = f(\partial D) = \varphi(\partial D) = \psi(\partial\tilde{D})$. Since f is a C^2 function, then so is g. Moreover, $\frac{\partial g}{\partial y} = \frac{\partial f}{\partial x}C^t$ and*

$$\begin{aligned} \frac{\partial g}{\partial y}\Lambda(\frac{\partial g}{\partial y})^t &= (\frac{\partial f}{\partial x}C^t)\Lambda(\frac{\partial f}{\partial x}C^t)^t \\ &= \frac{\partial f}{\partial x}(C^t\Lambda C)(\frac{\partial f}{\partial x})^t = \frac{\partial f}{\partial x}A(\frac{\partial f}{\partial x})^t = \varepsilon(x) = \epsilon(y). \end{aligned}$$

Therefore the following two boundary value problems are equivalent, in the sense that one of them has a solution if and only if the other has a solution, i.e.

$$\begin{cases} L(f)(x) = 0 & if\ x \in D, \\ f(x) = \varphi(x) & if\ x \in \partial D. \end{cases} \equiv \begin{cases} \Lambda(g)(y) = 0 & if\ y \in \tilde{D}, \\ g(y) = \psi(y) & if\ y \in \partial\tilde{D}. \end{cases}$$

Similarly,

$$\begin{cases} L(f)(x) = \varepsilon & if\ x \in D, \\ f(x) = \varphi(x) & if\ x \in \partial D. \end{cases} \equiv \begin{cases} \Lambda(g)(y) = \epsilon & if\ y \in \tilde{D}, \\ g(y) = \psi(y) & if\ y \in \partial\tilde{D}. \end{cases}$$

5. Applications

In this section, some applications of the preceding results would be provided.

5.1. Uniqueness of solution in the elliptic boundary value problems

The following uniqueness theorem, is based on the Corollary 4.13, that we have just obtained.

Theorem 5.1 *Let $\emptyset \neq U \subset R^n$ be an open set, $D = U \cup \partial U$ be a bounded set, L be an elliptic operator on $C^2(D,R)$ and $\varphi : \partial D \to R$ be a continuous function. Then the boundary value problem*

$$\begin{cases} L(f)(x) = 0 & if\ x \in D, \\ f(x) = \varphi(x) & if\ x \in \partial D \end{cases}$$

has at most one solution on D.

5.2. Property of the extremum values for the quadratic forms

In the following, a theorem on the eigenvalues of the matrix of a quadratic form would be proved. This theorem provides a notable result on the eigenvalues of the self adjoint linear transformations.

Let $m \in N$, σ_S be the inner product on R^{m+1} with symmetric positive definite $(m+1)\times(m+1)$ matrix S, and let A be an $(m+1)\times(m+1)$ symmetric matrix such that $AS = SA$. Let $U = \{x \in R^{m+1} | \sigma_S(x,x) < 1\}$, and $f(x) = \sigma_S(xA, x)$ for $x \in R^{m+1}$. Then $\partial U = \{x \in R^{m+1} | \sigma_S(x,x) = 1\}$ and $D = U \cup \partial U$ is a compact subset of R^{m+1}. A simple computation shows that f is an element of $H(\varepsilon, D) \cup H(D)$ for $\varepsilon = 2trAS$. Corollary 4.2 and Theorem 4.3 imply that $f_{|D}$ takes its extremum on ∂U. Let $\varepsilon \geq 0$, and f takes its maximum at a point $u \in \partial U$. Then $x \in \partial U$ implies that $f(x) \leq f(u)$. Let $\lambda = f(u)$, so $\sigma_S(x,x) = 1$ implies that $f(u) = \lambda\sigma_S(x,x) = \sigma_S(\lambda x, x)$, or $\sigma_S(xA, x) \leq \sigma_S(\lambda x, x)$. The last inequality is valid for all $z \in R^{m+1}$, because let $a > 0$ and $\sigma_S(z,z) = a^2$, then $z = ay$, where $\sigma_S(y,y) = 1$, hence $\sigma_S(zA, z) = \sigma_S((ay)A, ay) = a^2\sigma_S(yA, y)$ and $\sigma_S(\lambda z, z) = a^2\sigma_S(\lambda y, y)$. But $\sigma_S(yA, y) \leq \sigma_S(\lambda y, y)$ since $\sigma_S(y,y) = 1$. Therefore

$$\sigma_S(zA, z) = a^2\sigma_S(yA, y) \leq a^2\sigma_S(\lambda y, y) = \sigma_S(\lambda z, z) \tag{5.1}$$

and the inequality is proved.

If μ is an eigenvalue of A, with corresponding eigenvector $\omega \in R^{m+1}$, then (5.1) implies that

$$\mu\sigma_S(\omega,\omega) = \sigma_S(\mu\omega,\omega) = \sigma_S(\omega A,\omega) \leq \sigma_S(\lambda\omega,\omega) = \lambda\sigma_S(\omega,\omega),$$

and so $\mu \leq \lambda$, i.e. λ is the largest eigenvalue of A.

Let $B = A - \lambda I$, so (5.1) asserts that $\sigma_S(zB, z) \leq 0$ for all $z \in R^{m+1}$. Let $g(z) = \sigma_S(zB, z)$, then $g(z) \leq 0$ for all $z \in R^{m+1}$ and

$$\begin{aligned} g(u) = \sigma_S(uB, u) &= \sigma_S(uA - \lambda u, u) \\ &= \sigma_S(uA, u) - \sigma_S(\lambda u, u) = f(u) - \lambda\sigma_S(u,u) = 0. \end{aligned}$$

Let $v \in R^{m+1} - \{0\}$ be arbitrary, and consider the function $h(t) = g(u + tv)$ for $t \in R$. A simple computation shows that $h(t) = 2t\sigma_S(uB, v) + t^2 g(v)$. Therefore $h(0) = 0$ and the quadratic polynomial h takes its maximum at $t = 0$, so $h'(0) = 2\sigma_S(uB, v) = 0$. Since v is arbitrary, we have $uB = 0$, or $uA = \lambda u$.

The following theorem is a consequence of what we have just proved.

Theorem 5.2 *For $m \in N$ and any inner product space (R^{m+1}, σ_S) with symmetric positive definite matrix $S_{(m+1)\times(m+1)}$, any symmetric matrix $A_{(m+1)\times(m+1)}$ satisfying $AS = SA$ and $trAS \geq 0$ (res. $trAS \leq 0$), the maximum (res. minimum) point of $f(x) = \sigma_S(xA, x)$ defined on $D = \{x \in R^{m+1} | \sigma_S(x,x) \leq 1\}$ lies in ∂D, with the value equal to the largest (res. smallest) eigenvalue of A.*

Note that if S be the identity matrix, then a special case of Theorem 5.2, would be obtained.

Theorem 5.3 *Let $A = [a_{ij}]$ be a symmetric $(m+1)\times(m+1)$ matrix of real numbers. Then at least one of the maximum or minimum points of the function $f(x_1, ..., x_{m+1}) = \Sigma_{i,j=1}^{m+1} a_{ij}x_ix_j$ defined on the unit ball $B = \{(x_1, ..., x_{m+1}) \in R^{m+1} | x_1^2 + \cdots + x_{m+1}^2 \leq 1\}$ lies in ∂B, with the value equal to the largest or smallest eigenvalue of the matrix A, respectively. If $trA = 0$, then both of the maximum and minimum points of the function f on B, lies in ∂B.*

The following corollary, is also a special case of Theorem 5.2.

Corollary 5.4 *Let (R^{m+1}, σ_S) be any inner product space with symmetric positive definite matrix $S_{(m+1)\times(m+1)}$. Then for any symmetric matrix $A_{(m+1)\times(m+1)}$, and any polynomial Γ with real coefficients, satistying $AS = SA$ and $tr\Gamma(A)S \geq 0$ (res. $tr\Gamma(A)S \leq 0$), the maximum (res. minimum) point of $f(x) = \sigma_S(x\Gamma(A), x)$ defined on $D = \{x \in R^{m+1} | \sigma_S(x, x) \leq 1\}$ lies in ∂D, with the value equal to the largest (res. smallest) eigenvalue of $\Gamma(A)$.*

If M is a $p \times q$ matrix of real entries, the norm of $M = [m_{ij}]$ denoted by $\|M\|$, is defined to be the nonnegative number given by the formula $\|M\| = \Sigma_{i=1}^{i=p}\Sigma_{j=1}^{j=q}|m_{ij}|$. For an infinite sequence of $p \times q$ matrices $T_k (k = 1, 2, \dots)$ whose entries are real numbers, denote the $ij-$entry of T_k by t_{ij}^k. If for a sequence $\{a_k\}_{k\in N}$ of real numbers, all pq series

$$\Sigma_{k=1} a_k t_{ij}^k \quad (i = 1, \dots, p; j = 1, \dots, q) \tag{5.2}$$

are convergent, then we say that the series of matrices $\Sigma_{k=1} a_k T_k$ is convergent, and its sum, denoted by $\Sigma_{k=1}^{\infty} a_k T_k$ is defined to be the $p \times q$ matrix whose $ij-$entry is the sum of the series in (5.2). If $\Sigma_{k=1}|a_k|\|T_k\|$ converges, then Weierstrass test implies that the matrix series $\Sigma_{k=1} a_k T_k$ is convergent [30]. In the case that $p = q$, each T_k be a symmetric matrix, and $\Sigma_{k=1} a_k T_k$ be a convergent matrix series, then $\Sigma_{k=1}^{\infty} a_k T_k$ is also a symmetric matrix.

The following theorem is a consequence of the above discussion and Corollary 5.4.

Theorem 5.5 *For any inner product space (R^{m+1}, σ_S), any power series $\Sigma_{k=1} a_k x^k$ with radius of convergence $R > \|S\|$, $A = \Sigma_{k=1}^{\infty} a_k S^k$ satisfying $trAS \geq 0$ (res. $trAS \leq 0$), the maximum (res. minimum) point of $f(x) = \sigma_S(xA, x)$ on $D = \{x \in R^{m+1} | \sigma_S(x, x) \leq 1\}$ lies in ∂D, with the value equal to the largest (res. smallest) eigenvalue of $\Sigma_{k=1}^{\infty} a_k S^k$.*

5.3. Existence of optimum solution in operational research

In operational research, the feasible solution space of a two variables LP problem represents the compact set D, in the first quadrant in which all the constraints are satisfied simultaneously. An important characteristic of the optimum LP solution is that, it is always associated with a corner point in ∂D of the solution space where two lines intersect. Since an affine function is a harmonic function, the following result, generalizes the theorem of the existence of a solution to an n variables problem [35].

Theorem 5.6 *Let $\emptyset \neq U \subset R^n$ be an open set, and let $D = U \cup \partial U$ be a bounded set. Then every affine function defined on D, takes its extremums on ∂U.*

5.4. Greatest distance between a point and a compact set

Let $\emptyset \neq D \subset R^n$ be a compact set, $(p_1, \dots, p_n) \in R^n$ and $f : R^n \to R$ be defined by the equation $f(x_1, \dots, x_n) = \Sigma_{i=1}^{i=n}(x_i - p_i)^2$, then $\nabla^2 f = 2n > 0$ and so, Corollary 4.2 implies the following theorem.

Theorem 5.7 *The farthest point of a compact set $\emptyset \neq D \subset R^n$ from a point $p \in R^n$, lies on ∂D.*

5.5. Inverse-square law

The inverse-square law in physics, is any physical law stating that a specified physical quantity or intensity, is inversely proportional to the square of the distance from the source of that physical quantity. The law of light intensity, Newton's law for universal gravitation, and Coulomb's law for the electrostatic force of interaction between two point charges, follow an inverse-square behavior [21].

As the consequences of Theorem 5.7, we have the following results.

Corollary 5.8 *The smallest light intensity emitted from a spherical source at a point p of a thin plate, is occurred necessarily when p belongs to the boundary of the plate.*

Corollary 5.9 *The smallest gravitational (res. electrostatic) force, between a point mass (res. charge) p and a point mass (res. charge) q of a body, is created necessarily when q belongs to the outer shell of the body.*

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