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Maps on $\mathcal{S}(\mathcal{H})$ preserving the difference of noninvertible algebraic operators

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Abstract: The aim of this paper is to present the general structure of nonlinear surjective maps on $\mathcal{S}(\mathcal{H})$ preserving the operator pairs in which their difference is a noninvertible algebraic operator. $\mathcal{S}(\mathcal{H})$ represents the real Jordan algebra of bounded self-adjoint operators acting on an infinite dimensional Hilbert space $\mathcal{H}$.

Key words: Nonlinear preserver problem, algebraic operators, algebraic singularity

1. Introduction

Recently nonlinear preserver problems have been investigated by many authors, see for instance [1,2,3,6]. In [2] authors proved that if F is a map from the set of all complex $n \times n$ matrices into itself with $F(0) = 0$ such that $F(x) - F(y)$ and $x - y$ have at least one common eigenvalue then $F(x) = uxu^{-1}$ *or* $F(x) = ux^t u^{-1}$, for some invertible matrix u. Bourhim, Mashreghi and Stepanyan in 2014 [1] proved that a bicontinuous bijective map $\Phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ satisfies $c(\Phi(S) - \Phi(T)) = c(S - T)$ if and only if $\Phi(T) = UTV + R$ *or* $\Phi(T) = UT^*V + R$, for some bijective isometries U, V and $R \in \mathcal{B}(Y)$ where $c(.)$ stands either for minimum modulus or surjectivity modulus or the maximun modulus of T. Also in [4], Oudghiri and Souilah characterized all surjective maps of $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ that preserve operator pairs whose difference is a noninvertible algebraic operator. They proved that if $\Phi(I) = I + \Phi(0)$, then there exists an invertible either linear or conjugate linear operator $A : \mathcal{H} \to \mathcal{H}$ such that

$$\Phi(T) = ATA^{-1} + \Phi(0) \quad or \quad \Phi(T) = AT^*A^{-1} + \Phi(0), \qquad T \in \mathcal{B}(\mathcal{H}).$$

In this paper, we attempt to determine the general structure of Φ when it is restricted to the real Jordan algebra $\mathcal{S}(\mathcal{H})$.

Through out this paper $\mathcal{H}$ stands for an infinite dimensional separable complex Hilbert space. We denote $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$ and its self-adjoint part by $\mathcal{S}(\mathcal{H})$. The set of all finite rank operators in $\mathcal{S}(\mathcal{H})$ will be denoted by $\mathcal{F}(\mathcal{H})$. For $g, h \in \mathcal{H}$, $< g, h >$ stands for the inner product of g and h. For every $T \in \mathcal{B}(\mathcal{H})$, we use the notations $rank(T)$, $ker(T)$, $ran(T)$ and $\sigma(T)$ for the rank, kernel, range and the spectrum of T, respectively. A conjugate linear bijective operator U on $\mathcal{H}$ is called antiunitary, provided that $< Ux, Uy >=< y, x >$ for all $x, y \in \mathcal{H}$. The identity operator on $\mathcal{H}$ will be denoted by I. Two operators S, T in $\mathcal{S}(\mathcal{H})$ are called adjacent, provided that $S - T$ is a rank one operator. It is said that a

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surjective map $\psi : \mathcal{S}(\mathcal{H}) \longrightarrow \mathcal{S}(\mathcal{H})$ preserves adjacency of operators in both directions, if it preserves adjacent operators in both directions.

Definition 1.1 *The set of all nonzero polynomials of a single variable with real coefficients, will be denoted by* $P[\mathbb{R}]$. *An operator* $S \in \mathcal{S}(\mathcal{H})$ *is called algebraic if* $P(S) = 0$, *for some* $P \in P[\mathbb{R}]$.

We denote $\mathcal{A}(\mathcal{H})$, $\mathcal{NIA}(\mathcal{H})$ and $\mathcal{IA}(\mathcal{H})$, the set of all algebraic, noninvertible algebraic and invertible algebraic operators in $\mathcal{S}(\mathcal{H})$, respectively. A surjective map $\Lambda : \mathcal{S}(\mathcal{H}) \longrightarrow \mathcal{S}(\mathcal{H})$ is said to preserve operator pairs whose difference is a noninvertible algebraic operator, if for every $S, T \in \mathcal{S}(\mathcal{H})$

$$S - T \in \mathcal{NIA}(\mathcal{H}) \Longleftrightarrow \Lambda(S) - \Lambda(T) \in \mathcal{NIA}(\mathcal{H}).$$

2. Main results

Before we present the main result, we mention four auxiliary lemmas from [4], with necessary modifications for self-adjoint operator settings. The first two lemmas follow easily using almost same arguments as in [4]. However, in the second two lemmas some different phenomena take place, hence we prove them in details.

Lemma 2.1 [4, *Remark* 2.1] *Let* $T \in \mathcal{S}(\mathcal{H})$. *Then the following statements hold:*

(1) *Let* $h \in \mathcal{H}$ *be a unit vector,* $\lambda \in \mathbb{R}$ *and* T *is invertible. Then* $T - \lambda h \otimes h$ *is noninvertible if and only if* $< h, T^{-1}h >= \frac{1}{\lambda}$.

(2) $T \in \mathcal{A}(\mathcal{H})$, *if and only if* $T + F \in \mathcal{A}(\mathcal{H})$, *for every finite rank operator* $F \in \mathcal{S}(\mathcal{H})$.

(3) $T \in \mathcal{A}(\mathcal{H})$, *if and only if* $U^*TU \in \mathcal{A}(\mathcal{H})$, *for every unitary or antiunitary operator* $U \in \mathcal{S}(\mathcal{H})$.

(4) *If* $T \in \mathcal{A}(\mathcal{H})$, *then* $\sigma(T) \subset \mathbb{R}$ *is a finite set.*

Lemma 2.2 [4, *Lemma* 2.3] *Let* K *be a finite dimensional subspace of* $\mathcal{H}$ *and* $T \in \mathcal{S}(\mathcal{H})$ *be the operator represented by*

$$T = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix},$$

with respect to the decomposition of $\mathcal{H} = K \oplus K^\perp$. *Then* T *is algebraic if and only if* C *is algebraic. Furthermore, if* $B = 0$, *then* $\sigma(T) = \sigma(A) \bigcup \sigma(C)$.

Lemma 2.3 *Let* $A, B \in \mathcal{S}(\mathcal{H})$. *Then* A, B *are adjacent, if and only if there exists* $R \in \mathcal{S}(\mathcal{H}) \setminus \{A, B\}$ *such that* $R - B \in \mathcal{NIA}(\mathcal{H})$ *and for every* $T \in \mathcal{S}(\mathcal{H})$, $T - R, T - B \in \mathcal{NIA}(\mathcal{H})$ *imply* $T - A \in \mathcal{NIA}(\mathcal{H})$.

Proof Following the idea of [4, Proposition 2.2], we can restrict ourselves to the case where $B = 0$. If A is a rank one operator, then $A = \lambda h \otimes h$, for some unit vector $h \in \mathcal{H}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Set $R = -A$. Then $R \in \mathcal{NIA}(\mathcal{H}) \setminus \{A, 0\}$.
Assume $T \in \mathcal{NIA}(\mathcal{H})$ satisfies $T - R \in \mathcal{NIA}(\mathcal{H})$. We claim $T - A$ is noninvertible. Accordingly, there are two cases. If $ker(T) \cap \{h\}^\perp \neq \{0\}$, then $ker(T - A) \neq \{0\}$ and consequently $T - A$ is noninvertible. If

$ker(T) \cap \{h\}^{\perp} = \{0\}$, then $T+A$ is noninjective, as $\quad T-R = T+A \in \mathcal{NIA}(\mathcal{H})$.
Let $k \in ker(T+A)$ be a nonzero unit vector. Then $Tk = -\lambda < k,h > h$. Hence $k \notin \{h\}^{\perp}$. As $\mathcal{H} = \{h\}^{\perp} \oplus \mathbb{C}h$, it follows that $k = \mu h$, for some nonzero scalar $\mu \in \mathbb{C}$. Hence $Th = -\lambda h$. Consequently, as

$$T - A = T(I + h \otimes h),$$

by applying the facts that T is noninvertible and $I + h \otimes h$ is invertible, it follows that $T-A$ is noninvertible. Finally, as $T \in \mathcal{NIA}(\mathcal{H})$, from the second part of Lemma 2.1, it follows that $T - A \in \mathcal{NIA}(\mathcal{H})$.

For the inverse direction, it is assumed that $dim\ ran(A) \geq 2$. We claim that for every $R \in \mathcal{NIA}(\mathcal{H}) \setminus \{A, 0\}$, there exists $T \in \mathcal{NIA}(\mathcal{H})$ such that $T - R \in \mathcal{NIA}(\mathcal{H})$ and $T - A \notin \mathcal{NIA}(\mathcal{H})$. For this, let $R \in \mathcal{NIA}(\mathcal{H}) \setminus \{A, 0\}$ be fixed. There are two cases: if $A \notin \mathcal{NIA}(\mathcal{H})$, then it is enough to consider $T = 0$. If $A \in \mathcal{NIA}(\mathcal{H})$, then A is not injective and there exists some $h \in \mathcal{H}$ such that $(R-A)h \neq 0$, as $R \neq A$. Considering the fact that $dim\ ran(A) \geq 2$, it follows that there exist some $k \in \mathcal{H}$ such that the vectors $\{(R-A)h, Ak\}$ are linearly independent. By replacing k with $k + \theta$, for some $\theta \in ker(A)$ if it is necessary, we may assume $\{h, k\}$ are linearly independent. Let $K = span\{h, k, (R-A)h, Ak\}$. Then we can write

$$A = \begin{bmatrix} A_1 & A_2 \\ A_2^* & A_3 \end{bmatrix} \quad and \quad R = \begin{bmatrix} R_1 & R_2 \\ R_2^* & R_3 \end{bmatrix},$$

regarding to the decomposition of $\mathcal{H} = K \oplus K^{\perp}$. Set

$$T = \begin{bmatrix} S + A_1 & A_2 \\ A_2^* & cI \end{bmatrix},$$

where $c \in \mathbb{R} \setminus \sigma(A_3)$ and $S \in \mathcal{S}(K)$ is an invertible operator satisfying $Sh = (R_1 - A_1)h$ and $Sk = -A_1 k$. It follows from Lemma 2.2, that R, T and $T - R$ are algebraic operators. But as $Tk = (T-R)h = 0$, hence $T, T - R \in \mathcal{NIA}(\mathcal{H})$. On the other hand, since

$$T - A = \begin{bmatrix} S & 0 \\ 0 & cI - A_3 \end{bmatrix},$$

it follows that $T - A$ is invertible, thus $T - A \notin \mathcal{NIA}(\mathcal{H})$, which completes the proof. □

Lemma 2.4 *Let $S, T \in \mathcal{S}(\mathcal{H})$. Then $S = T$, under any of the following conditions.*

(i) *For every $N \in \mathcal{S}(\mathcal{H})$, $S - N \in \mathcal{NIA}(\mathcal{H})$ if and only if $T - N \in \mathcal{NIA}(\mathcal{H})$.*

(ii) *For every $N \in \mathcal{IA}(\mathcal{H})$, $S - N \in \mathcal{IA}(\mathcal{H})$ if and only if $T - N \in \mathcal{IA}(\mathcal{H})$.*

Proof (i) We follow the idea of [4, Proposition 2.4]. Since $\mathcal{NIA}(\mathcal{H})$ does not contain any invertible operator and $T - S \in \mathcal{NIA}(\mathcal{H})$, in order to prove $S = T$, it is enough to show that $T - S$ is a scalar operator. If this is not so, then there exists a unit vector $h \in \mathcal{H}$ such that h, $(T-S)h$ are linearly independent. Regarding to the decomposition of $\mathcal{H} = K \oplus K^{\perp}$, let $T - S$ be represented by

$$T - S = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix},$$

where $K = span\{h, (T-S)h\}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & z \end{bmatrix}$. Set

$$R = \begin{bmatrix} 0 & -B \\ -B^* & I-C \end{bmatrix}.$$

Then $R \in \mathcal{S}(\mathcal{H})$ and since $Rh = 0$, it follows that R is not invertible. Using the fact that $T - S \in \mathcal{NIA}(\mathcal{H})$, from Lemma 2.2, it follows that C and hence R are algebraic. Consequently, if we set $N = S - R$, then $S - N = R \in \mathcal{NIA}(\mathcal{H})$. But since

$$T - N = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$$

is invertible, we get a contradiction.
(ii)Following the idea of [4, Lemma 3.3] it follows that $\sigma(S) = \sigma(T)$. Hence it is enough to show that $S - T$ is a scalar operator. However, it is assumed that $S - T$ is not a scalar operator. Then, there exists $h \in \mathcal{H}$ such that the vectors h and $(S-T)h$ are linearly independent. There are two cases: either $\{h, Th\}$ or $\{h, Sh\}$ is a linearly independent set. It is enough to consider the first case. Let

$$\begin{bmatrix} S_1 & S_2 \\ S_2^* & S_3 \end{bmatrix}$$

be the representation of S regarding to the decomposition of $\mathcal{H} = K \bigoplus K^\perp$, where $K = span\{h, Th, Sh\}$. Let (s_{ij}) be the representation of S_1 regarding to the decomposition of K. Considering

$$\begin{bmatrix} 0 & I & o \\ 1 & s_{22} - I & s_{23} \\ 0 & s_{23}^* & s_{33} - I \end{bmatrix},$$

when $dim(K) = 3$ and $\begin{bmatrix} 0 & I \\ I & s_{22} \end{bmatrix}$, when $dim(K) = 2$, it follows that there exists an invertible operator, $A \in \mathcal{S}(K)$ such that $Ah = Th$ and $S_1 - A$ is invertible. Now consider

$$N = \begin{bmatrix} A & S_2 \\ S_2^* & \lambda I \end{bmatrix},$$

where $\lambda \in \mathbb{R} \setminus \sigma(S_3)$. Since A and $S_1 - A$ are invertible, it follows from Lemma 2.2 that N and $S - N$ are invertible algebraic operators. But since $(T-N)h = 0$, we conclude that $T - N \notin \mathcal{IA}(\mathcal{H})$, which is a contradiction. $\square$

The main idea for proving this theorem is taken from [4, Theorem B], however, a lot of new phenomena take place.

Theorem 2.5 *Let $\Lambda : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ be a surjective map satisfying $\Lambda(I) = I + \Lambda(0)$. Then Λ preserves operator pairs whose difference is a noninvertible algebraic operator if and only if there exists either a unitary or an antiunitary operator U on $\mathcal{H}$ such that $\Lambda(S) = USU^* + \Lambda(0)$ for every $S \in \mathcal{S}(\mathcal{H})$.*

Proof The "if" part is obvious. Conversely, assume Λ preserves operator pairs whose difference belongs to $\mathcal{NIA}(\mathcal{H})$. Through a few steps, we show that Λ has the desired structure.

Step 1. Λ is injective and preserves adjacency of operators in both directions.
Let $\Lambda(S) = \Lambda(T)$, for some $S, T \in \mathcal{S}(\mathcal{H})$. For every $N \in \mathcal{S}(\mathcal{H})$ by assumption,

$$T - N \in \mathcal{NIA}(\mathcal{H}) \iff \Lambda(T) - \Lambda(N) = \Lambda(S) - \Lambda(N) \in \mathcal{NIA}(\mathcal{H}),$$

which is equivalent to $S - N \in \mathcal{NIA}(\mathcal{H})$. Hence, from the first part of Lemma 2.4 it follows that $S = T$ and consequently Λ is injective. We consider that $A, B \in \mathcal{S}(\mathcal{H})$ such that $rank(A - B) = 1$. From Lemma 2.3, it follows that there exists $R \in \mathcal{S}(\mathcal{H})$ such that $R - B \in \mathcal{NIA}(\mathcal{H})$ and for every $T \in \mathcal{S}(\mathcal{H})$, $T - R, \quad T - B \in \mathcal{NIA}(\mathcal{H})$ which implies that $T - A \in \mathcal{NIA}(\mathcal{H})$. As Λ is injective, we get

$$\Lambda(R) \in \mathcal{S}(\mathcal{H}) \setminus \{\Lambda(A), \Lambda(B)\}.$$

By assumption $\Lambda(R) - \Lambda(B) \in \mathcal{NIA}(\mathcal{H})$. Let $S \in \mathcal{S}(\mathcal{H})$ be such that

$$S - \Lambda(R) \in \mathcal{NIA}(\mathcal{H}) \ \ and \ \ S - \Lambda(B) \in \mathcal{NIA}(\mathcal{H}).$$

Then, there exists $T \in \mathcal{S}(\mathcal{H})$ that $\Lambda(T) = S$, as Λ is surjective. Thus, $\Lambda(T) - \Lambda(R) \in \mathcal{NIA}(\mathcal{H})$ and $\Lambda(T) - \Lambda(B) \in \mathcal{NIA}(\mathcal{H})$, which implies $T - R \in \mathcal{NIA}(\mathcal{H})$ and $T - B \in \mathcal{NIA}(\mathcal{H})$.
Hence, we have $T - A \in \mathcal{NIA}(\mathcal{H})$ and consequently $S - \Lambda(A) \in \mathcal{NIA}(\mathcal{H})$. By applying Lemma 2.3, we get $rank(\Lambda(A) - \Lambda(B)) = 1$. Similarly, since Λ^{-1} has the same properties as Λ, the second assertion follows.

By replacing Λ with $\Lambda_1 = \Lambda - \Lambda(0)$, it follows that Λ_1 has the same properties as Λ. Furthermore, $\Lambda_1(0) = 0$ and $\Lambda_1(I) = I$.

Step 2. Λ_1 preserves rank one operators and maps $\mathcal{F}(\mathcal{H})$ into itself.

Consider a rank one operator $F \in \mathcal{S}(\mathcal{H})$. Then, F is adjacent to 0. It follows from step 1 that $\Lambda_1(F)$ and 0 are adjacent. Consequently, $rank(\Lambda_1(F)) = 1$. By using the same argument, it follows that $rank(\Lambda_1(E)) < \infty$, for every $E \in \mathcal{F}(\mathcal{H})$.

Step 3. Λ_1 preserves projections of rank one, and there exists either a unitary or antiunitary operator $U : \mathcal{H} \to \mathcal{H}$ such that $\Lambda_1(T) = UTU^*$, for every $T \in \mathcal{F}(\mathcal{H})$.

Since $\Lambda_1 : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H})$ preserves adjacency and satisfies $\Lambda_1(0) = 0$, it follows from [7, Theorem 2.1] that either

- there exists a rank one operator $R \in \mathcal{S}(\mathcal{H})$ such that the range of Λ_1 is contained in the linear span of R; or
- there exists an injective linear or conjugate linear operator $U : \mathcal{H} \to \mathcal{H}$ such that $\Lambda_1(\sum_{j=1}^{k} t_j x_j \otimes x_j) = \sum_{j=1}^{k} t_j U(x_j \otimes x_j)U^*$, for every $\sum_{j=1}^{k} t_j x_j \otimes x_j \in \mathcal{F}(\mathcal{H})$; or
- there exists an injective linear or conjugate linear operator $U : \mathcal{H} \to \mathcal{H}$ such that $\Lambda_1(\sum_{j=1}^{k} t_j x_j \otimes x_j) = -\sum_{j=1}^{k} t_j U(x_j \otimes x_j)U^*$ for every $\sum_{j=1}^{k} t_j x_j \otimes x_j \in \mathcal{F}(\mathcal{H})$.

As Λ_1 is bijective, the first case is not happening. Since both Λ_1 and Λ_1^{-1} have the same properties, from above discussion it follows that there exists either an invertible linear or conjugate linear operator $U : \mathcal{H} \to \mathcal{H}$ and $\lambda \in \{-1, 1\}$ such that

$$\Lambda_1(T) = \lambda UTU^* \ \ , \ \forall \ T \in \mathcal{F}(\mathcal{H}).$$

Note that for an arbitrary unit vector $f \in \mathcal{H}$, $I - f \otimes f \in \mathcal{NIA}(\mathcal{H})$. Hence, by assumption we should have

$$\Lambda_1(I) - \Lambda_1(f \otimes f) = I - \lambda Uf \otimes Uf \in \mathcal{NIA}(\mathcal{H}).$$

But this happens precisely when $\lambda = 1$. Now, consider an arbitrary vector $e \in \mathcal{H}$. Then

$$< e, e >= 1 \Longleftrightarrow I - e \otimes e \in \mathcal{NIA}(\mathcal{H}) \Longleftrightarrow I - Ue \otimes eU^* \in \mathcal{NIA}(\mathcal{H}) \Longleftrightarrow < Ue, Ue >= 1.$$

Consequently, Λ_1 preserves projections of rank one. Furthermore, as for every unit vector $e \in \mathcal{H}$, $||Ue|| = \sqrt{< Ue, Ue >} = 1$, it follows that U is either a unitary or an antiunitary operator on $\mathcal{H}$.

By replacing Λ_1 with $\Lambda_2 = U^*\Lambda_1 U$, in the sequel we may assume $\Lambda_2(F) = F$, for every $F \in \mathcal{F}(\mathcal{H})$.

Step 4. Λ_2 preserves the difference of $\mathcal{IA}(\mathcal{H})$ in both directions, that is, for every $S, T \in \mathcal{S}(\mathcal{H})$ we have

$$S - T \in \mathcal{IA}(\mathcal{H}) \Longleftrightarrow \Lambda_2(S) - \Lambda_2(T) \in \mathcal{IA}(\mathcal{H}).$$

Let $S, T \in \mathcal{S}(\mathcal{H})$ be such that $T - S \in \mathcal{IA}(\mathcal{H})$. Then for some unit vectors $e \in \mathcal{H}$, $< e, (T-S)^{-1}e >= 1$. Set $F = e \otimes e$. It follows from the first part of Lemma 2.1 that $T - (S + F)$ is not invertible. Hence $T - (S + F) \in \mathcal{NIA}(\mathcal{H})$, which implies

$$\Lambda_2(T) - \Lambda_2(S + F) \in \mathcal{NIA}(\mathcal{H}).$$

On the other hand, since $(S + F) - S$ is rank one then so is $\Lambda_2(S + F) - \Lambda_2(S)$. Therefore, since

$$\Lambda_2(T) - \Lambda_2(S) = \Lambda_2(T) - \Lambda_2(S + F) + (\Lambda_2(S + F) - \Lambda_2(S)),$$

it follows that $\Lambda_2(T) - \Lambda_2(S) \in \mathcal{A}(\mathcal{H})$. But since by assumption $T - S$ is invertible, $T - S \notin \mathcal{NIA}(\mathcal{H})$, which implies

$$\Lambda_2(T) - \Lambda_2(S) \notin \mathcal{NIA}(\mathcal{H}).$$

Hence

$$\Lambda_2(T) - \Lambda_2(S) \in \mathcal{IA}(\mathcal{H}).$$

Similarly, since Λ_2^{-1} satisfies the same properties as Λ_2, we conclude that Λ_2 preserves the difference of $\mathcal{IA}(\mathcal{H})$ in both directions.

Step 5. $\Lambda_2(T) = T$ for every $T \in \mathcal{IA}(\mathcal{H}) \bigcup \mathcal{NIA}(\mathcal{H})$.

First let assume $T \in \mathcal{IA}(\mathcal{H})$ and because of $T - 0 \in \mathcal{IA}(\mathcal{H})$, it follows from step 4 that $\Lambda_2(T) = \Lambda_2(T) - \Lambda_2(0) \in \mathcal{IA}(\mathcal{H})$. If $\Lambda_2(T) \neq T$, then there exists a unit vector $e \in \mathcal{H}$ such that $T^{-1}e \neq \Lambda_2(T)^{-1}e$, $< e, T^{-1}e >= 1$ while $< e, \Lambda_2(T)^{-1}e > \neq 1$. By considering the first part of Lemma 2.1 that $T - e \otimes e \notin \mathcal{IA}(\mathcal{H})$ but

$$\Lambda_2(T) - e \otimes e = \Lambda_2(T) - \Lambda_2(e \otimes e) \in \mathcal{IA}(\mathcal{H}),$$

there appears a contradiction. This contradiction shows that $\Lambda_2(T) = T$. Now by considering $T \in \mathcal{NIA}(\mathcal{H})$. Then

$$\Lambda_2(T) = \Lambda_2(T) - \Lambda_2(0) \in \mathcal{NIA}(\mathcal{H}).$$

For every $N \in \mathcal{IA}(\mathcal{H})$, from the first part we have $\Lambda_2(N) = N$ and $T - N \in \mathcal{IA}(\mathcal{H})$ if and only if $\Lambda_2(T) - N \in \mathcal{IA}(\mathcal{H})$. Hence, from the second part of Lemma 2.4, it follows that $\Lambda_2(T) = T$.

Step 6. $\Lambda_2(T) = T$ for every $T \in \mathcal{S}(\mathcal{H})$.

Temporarily, we denote $\mathcal{LNIA}(\mathcal{H})$ the real linear span of $\mathcal{NIA}(\mathcal{H})$. It follows from [5, Theorem 3] that the elements of $\mathcal{S}(\mathcal{H})$ can be represented by a real linear combination of at most eight projections. Hence by considering suitable polynomials and applying the fact that every nontrivial projection is noninvertible algebraic, it follows that $\mathcal{LNIA}(\mathcal{H}) = \mathcal{S}(\mathcal{H})$. Consequently, if we show that $\Lambda_2 |_{\mathcal{NIA}(\mathcal{H})}$ is additive, then the desired result follows from step 5. This is, let $T_1, T_2 \in \mathcal{NIA}(\mathcal{H})$ be fixed and consider the map $\Phi : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ that for every $T \in \mathcal{S}(\mathcal{H})$ is defined by

$$\Phi(T) := \Lambda_2(T - T_2) - T_2.$$

It follows from previous steps that Φ is bijective. It preserves the difference of $\mathcal{NIA}(\mathcal{H})$ in both directions, $\Phi(I) = I$ and $\Phi(F) = F$ for every $F \in \mathcal{F}(\mathcal{H})$. Hence, for every $T \in \mathcal{NIA}(\mathcal{H})$, $\Phi(T) = T$. In particular, we get

$$T_1 = \Phi(T_1) = \Lambda_2(T_1 + T_2) - T_2,$$

which implies

$$\Lambda_2(T_1 + T_2) = T_1 + T_2.$$

Hence $\Lambda_2 |_{\mathcal{NIA}(\mathcal{H})}$ is additive.

Finally, it follows from step 6, that for every $T \in \mathcal{S}(\mathcal{H})$, $\Lambda_2(T) = T$. From this we get

$$T = \Lambda_2(T) = U^*\Lambda_1(T)U = U^*(\Lambda(T) - \Lambda(0))U$$

Hence

$$\Lambda(T) = UTU^* + \Lambda(0)$$

for every $T \in \mathcal{S}(\mathcal{H})$ which is the desired result and it completes the proof. □

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