Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 110 – 124
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doi:10.3906/mat-2007-67

On the Generalization of κ-Fractional Hilfer-Katugampola Derivative with Cauchy Problem

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Received: 17.07.2020 • **Accepted/Published Online:** 07.10.2020 • **Final Version:** 21.01.2021

Abstract: We generalize the κ-fractional Hilfer–Katugampola derivative and set some properties of the generalized operator resulting from this. As an application, we demonstrate that the Cauchy problem with this new definition is equivalent to a second kind of Volterra integral equation. We discuss some specific cases for this problem.

Key words: κ-gamma function, κ-Mittag–Leffler function, κ-Riemann–Liouville fractional integral, generalized κ-fractional derivative

1. Description

There are a few issues where noninteger order derivatives play a pivotal role [1, 3, 20, 22, 26]. It ought to be stressed that fractional derivative are introduced from distinct ways, specifically, describing three unmistakable ways, which we will make reference to so as to build up the work in one of them.

Normally, every typical fractional derivative is defined in respect of a specific integral. Among the most notable definitions of noninteger order derivatives we may make reference to the Caputo, Grunwald–Letnikov, Hadamard and Riemann–Liouville derivatives [2, 6], whose details include integrals with singular kernels and which are utilized to contemplate, for instance, issues including the memory impact [19]. Then again, various plans of fractional derivatives have appeared in the literature [15], in the year 2010. From a few points of view, these new formulations vary from the old style ones. For example, classical fractional derivatives are described so that one recovers the old style derivatives in the sense of Newton and Leibniz in the cutoff, where there is a whole number in the request for the derivative. A new fractional derivative has also been recently proposed in [5, 18]. Another fractional derivative with a comparing integral whose kernel can be a nonsingular function, such as a Mittag–Leffler function [29], was additionally proposed as of late. Integer order derivatives are also managed to recover in such cases by considering adequate limits for the values of their parameters.

In comparison, there are numerous approaches to acquire a speculation of classical fractional derivatives. A few authors describe the parameters in classical definitions or in some specific functions [8, 10, 14, 20, 27]. Likewise, in an ongoing paper [13], the authors present a boundary and discuss on two specific spaces, which they call the generalization of fractional derivative and further propose a Caputo modification of this generalization.

In this paper, we are interested in the κ-fractional Hilfer–Katugampola derivative, which generalize the Hilfer–Katugampola derivative. Particularly, to study the existence and uniqueness of its solutions and their

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2010 *AMS Mathematics Subject Classification:* 26A33

dependence on initial conditions. We propose a new generalization of κ-fractional derivatives and discuss a general Cauchy problem. We retrieve a broad class of fractional derivatives as a product.

Gauhar et al. [23] presented the extended Caputo fractional derivative operator and by using Mittag–Leffler function as kernel. They generate the relations for the hypergeometric functions. Martin et al. [4] establish the extended Riemann–Liouville fractional derivative operator by using the extended beta function and use the Bessel function as the kernel. They presented some new results like Mellin transform, hypergeometric function and relation Appell's function of generating functions. Fully extended beta function, extended hypergeometric function and an extended confluent hypergeometric function were presented by Mubeen et al. [24] in 2018. Recently, Subashini et al. [28] presented the nonlocal functional integro differential equations by using the Hilfer fractional derivative. They use the Mönch fixed point theorem and presented some theoretical results in 2020. Gauhar et al. [21] in 2018 defined a fractional operator and use it to find the Mellin transform and some fascinating results. Gauhar et al. [25] presented the generalized κ-fractional with the help of κ-Mittag–Leffler function and wright hypergeometric functions.

In this paper, the contents are sorted out in various sections. In Section 2, the definition of the κ-fractional integrals in the senses of Riemann–Liouville and Hadamard, the spaces and κ-Mittag–Leffler functions which mainly focus on our work are presented. Some properties of κ-fractional Hilfer–Katugampola derivative are presented in Section 3. In Section 4, we introduce a generalized κ-fractional Hilfer–Katugampola derivative and demonstrate that a wide list of definitions of fractional derivatives can be recovered using appropriate parameters. As an application, introduced using theorems in the previous section, we approach linear fractional differential equations via study the Cauchy problem, the existence and uniqueness of its solution and its dependence on the initial conditions in the Sections 5, 6, and 7. In Section 8, concluding remarks are given.

2. Prelude

To present the κ-Mittag–Leffler function, let us present the idea of gamma function, beta function or the Pochhammer symbol. Diaz et al. [8] were the first to characterize κ-gamma function, κ-beta function and κ-Pochhammer symbol.

Definition 2.1 *The κ-gamma function is defined as*

$$\Gamma_\kappa(z) = \int_0^\infty t^{z-1} e^{-\frac{t^\kappa}{\kappa}}\,\mathrm{d}t, \tag{2.1}$$

with $z, \kappa > 0$.
It has the following relationships

$$\Gamma_\kappa(z) = \kappa^{\frac{z}{\kappa}-1}\Gamma\left(\frac{z}{\kappa}\right), \quad \text{and} \quad \Gamma_\kappa(\kappa) = 1, \tag{2.2}$$

when $\kappa \to 1$ then $\Gamma_\kappa(z) = \Gamma(z)$.

Definition 2.2 *The κ-Pochhammer symbol is defined by*

$$(z)_{m,\kappa} = \begin{cases} 1, & \text{for} \quad m = 0. \\ z\,(z+\kappa)\ldots(z+(m-1)\,\kappa)\,, & \text{for} \quad m \in \mathbb{R}, \quad z \in, \quad \kappa > 0. \end{cases} \tag{2.3}$$

Or as to a quotient of function κ-gamma,

$$(z)_{m,\kappa} = \frac{\Gamma_\kappa (z + m\kappa)}{\Gamma_\kappa (z)}. \tag{2.4}$$

Definition 2.3 *Finally, the function κ-beta is described by*

$$\beta_\kappa (y, z) = \frac{1}{\kappa} \int_0^1 u^{\frac{y}{\kappa}-1} (1-u)^{\frac{z}{\kappa}-1} du, \quad y > 0, \quad z > 0, \quad \kappa > 0, \tag{2.5}$$

when $\kappa \to 1$ then $\beta_\kappa (y, z) = \beta (y, z)$.
In terms of κ-gamma function, the κ-beta function can be written as follows:

$$\beta_\kappa (y, z) = \frac{\Gamma_\kappa (y) \Gamma_\kappa (z)}{\Gamma_\kappa (y + z)}, \quad \text{and} \quad \beta_\kappa (y, z) = \frac{1}{\kappa} \beta(\frac{y}{\kappa}, \frac{z}{\kappa}). \tag{2.6}$$

Mittag–Leffler function plays a very significant role in solving integral equations and linear differential equation [16, 17].

Definition 2.4 *Dorrego and Cerutti [9] defined the so-called κ-Mittag–Leffler function as follows for the generalization of these functions:*

$$F^{\delta}_{\kappa,\alpha,\eta} (x) = \sum_{m=0}^{\infty} \frac{(\delta)_{m,\kappa}}{\Gamma_\kappa (\alpha n + \eta)} \frac{x}{m!}, \quad x \in \mathbb{R}, \quad \alpha > 0, \quad \eta > 0, \tag{2.7}$$

where $m \in \mathbb{N}$ and $(\delta)_{m,\kappa}$ is the κ-Pochhammer symbol defined in equation (2.3).

Definition 2.5 *Gupta and Parihar [12] used the following sequence to describe the so-called κ-new generalized Mittag–Leffler function.*

$$F_{\kappa,\beta,\gamma} (x) = \sum_{m=0}^{\infty} \frac{x^m}{\Gamma_\kappa (\beta m + \gamma)}, \quad x \in \mathbb{R}, \quad \beta > 0, \quad \gamma > 0, \tag{2.8}$$

where $m \in \mathbb{N}$.

Until presenting the definition of κ-fractional integrals and their generalization. We characterize the specific function spaces for such definitions and Lipschitz condition for the function $g(z, \psi)$.

Definition 2.6 *[16] Let $[a, b]$ be a finite or infinite interval on the real axis $\mathbb{R} = (-\infty, \infty)$. By $M_q = (a, b)$, we denote the set of the complex-valued Lebesgue measurable function ψ on $[a, b]$,*

$$M_q (a, b) = \left\{ \psi : \psi_q = \sqrt[q]{\int_a^b |\psi (z)|^q dz} < +\infty \right\}, \quad 1 \leq q < \infty. \tag{2.9}$$

In case if $q = 1$, we have $M_q (a, b) = M (a, b)$.

Definition 2.7 *[7] Suppose that $g(z,\psi)$ is set to the collection $(a,b]\times H$ and $H\subset\mathbb{R}$. A function $g(z,\psi)$ fulfills the condition of Lipschitz with respect to ψ, if $\forall z\in(a,b]$ and for $\psi_1,\psi_2\in\mathbb{R}$ then,*

$$\left|g(z,\psi_1)-g(z,\psi_2)\right|\leq C\left|\psi_1-\psi_2\right|,$$

where $C>0$ and it does not depend on z.

Definition 2.8 *Katugampola [14] presented the alleged κ-Riemann–Liouville fractional integral, a generalization of the Riemann–Liouville fractional integral, got for $\kappa=1$. Left sided (right sided) integral is defined for $\psi(z)\in M(a,b)$ as,*

$$\left({}^{\rho}\Im^{\omega}_{a\pm}\psi\right)(z)=\pm\frac{1}{\Gamma(\omega)}\int\limits_{a}^{z}\left(\frac{z^{\rho}-y^{\rho}}{\rho}\right)^{\omega-1}y^{\rho-1}\psi(y)\,dy,\quad\omega>0,\quad x>a. \tag{2.10}$$

Definition 2.9 *Recently, Sarikaya et al. [27] presented κ-fractional integral that recovers the κ-Riemann–Liouville at adequate limits. The left and right sided operator is defined with $m-1<\omega\leq m$, $m\in\mathbb{N}$, $\rho>0$, $\kappa>0$ as*

$$\left({}^{\rho}_{\kappa}\Im^{\omega}_{a\pm}\psi\right)(z)=\pm\frac{1}{\kappa\Gamma_{\kappa}(\omega)}\int_{a}^{z}\left(\frac{z^{\rho}-y^{\rho}}{\rho}\right)^{\frac{\omega}{\kappa}-1}y^{\rho-1}\psi(y)\,\mathrm{d}y,\quad\omega>0,\quad x>a. \tag{2.11}$$

3. Auxiliary results

We currently present a few properties of the fractional integrals characterized in the previous section, so as to utilize them all through this work. We start by introducing the semigroup property for the κ-fractional integral and an application to the force work, the two theorems are found in [27].

Theorem 3.1 *Let $\omega>0$, $\theta>0$, $\kappa>0$ and $\psi\in M_q(a,b)$, then*

$$\left({}^{\rho}_{\kappa}\Im^{\omega}_{a+}{}^{\rho}_{\kappa}\Im^{\theta}_{a+}\psi\right)(z)=\left({}^{\rho}_{\kappa}\Im^{\omega+\theta}_{a+}\psi\right)(z)=\left({}^{\rho}_{\kappa}\Im^{\theta}_{a+}{}^{\rho}_{\kappa}\Im^{\omega}_{a+}\psi\right)(z).$$

Theorem 3.2 *Let $\omega>0$, $\theta>0$, $\kappa>0$ and $\psi\in M_q(a,b)$, then*

$${}^{\rho}_{\kappa}\Im^{\omega}_{a+}(z^{\rho}-a^{\rho})^{\frac{\gamma}{\kappa}-1}(z)=\frac{\Gamma_{\kappa}(\gamma)}{\rho^{\frac{\omega}{\kappa}}\Gamma_{\kappa}(\omega+\gamma)}(z^{\rho}-a^{\rho})^{\frac{\omega+\gamma}{\kappa}-1}.$$

Lemma 3.3 *[27] Let $\psi\in M(a,b)$, the κ-Riemann–Liouville fractional integral of order $\omega>0$ is bounded in the space $M(a,b)$,*

$$\left\|{}^{\rho}_{\kappa}\Im^{\omega}_{a+}\psi\right\|\leq N\left\|\psi\right\|, \tag{3.1}$$

where

$$N=\frac{1}{\omega\Gamma_{\kappa}(\omega)}\left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{\frac{\omega}{\kappa}}.$$

In terms of some corresponding fractional integral, most fractional differentiation operators are defined. We now present the definition of Hilfer–Katugampola fractional derivative which is associated with the Riemann–Liouville fractional integral.

Definition 3.4 *[22] Let $m-1<\omega\le m,\ 0\le\theta\le 1,\ m\in\mathbb{N},\ \rho>0,\ \kappa>0$ and $\psi\in M(a,b)$, where ω is the order and θ is the type, the Hilfer–Katugampola fractional derivative (left sided and right sided) is defined as,*

$$
{}^{\rho}D_{a\pm}^{\omega,\theta}\psi(z)=\pm\left(\Im_{a\pm}^{\theta(m-\omega)}\left(z^{1-\rho}\frac{d}{dz}\right)^{m}{}^{\rho}\Im_{a\pm}^{(1-\theta)(m-\omega)}\psi\right)(z) \tag{3.2}
$$

$$
=\pm\left(\Im_{a\pm}^{\theta(m-\omega)}\delta_{\rho}^{m}{}^{\rho}\Im_{a\pm}^{(1-\theta)(m-\omega)}\psi\right)(z), \tag{3.3}
$$

where $\delta_{\rho}^{m}=\left(z^{1-\rho}\frac{d}{dz}\right)^{m}$ and ${}^{\rho}\Im_{a\pm}^{\omega}$ is the integral defined in Equation (2.10).

4. Generalized k-fractional Hilfer–Katugampola derivative

In this section, generalization of Hilfer–Katugampola derivative is presented. Here we consider that $\omega\in\mathbb{R}^{+}$ and $m-1<\omega\le m$ and $m\in\mathbb{N}$. The fractional integral aligned with this differentiation operator is given in Equation (2.11).

Definition 4.1 *Let $m-1<\omega\le m,\ 0\le\theta\le 1,\ m\in\mathbb{N},\ \rho>0,\ \kappa>0$ and $\psi\in M_q(a,b)$, the generalized κ-Hilfer–Katugampola fractional derivative (left sided and right sided) is defined as*

$$
\left({}_{\kappa}^{\rho}D_{a\pm}^{\omega,\theta}\psi\right)(z)=\pm\left({}_{\kappa}^{\rho}\Im_{a\pm}^{\theta(\kappa m-\omega)}\left(z^{1-\rho}\frac{d}{dz}\right)^{m}\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a\pm}^{(1-\theta)(\kappa m-\omega)}\psi\right)\right)(z) \tag{4.1}
$$

$$
=\pm\left({}_{\kappa}^{\rho}\Im_{a\pm}^{\theta(\kappa m-\omega)}\delta_{\rho}^{m}\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a\pm}^{(1-\theta)(\kappa m-\omega)}\psi\right)\right)(z), \tag{4.2}
$$

where $\delta_{\rho}^{m}=\left(z^{1-\rho}\frac{d}{dz}\right)^{m}$.

Theorem 4.2 *Let $\omega\in\mathbb{R}^{+}$ and $\rho,\kappa>0$. For $\psi\in M_q(a,b)$ and $1\le q<\infty$, then we have*

$$
\left({}_{\kappa}^{\rho}D_{a+}^{\omega,\theta}{}_{\kappa}^{\rho}\Im_{a+}^{\omega}\psi\right)(z)=\psi(z). \tag{4.3}
$$

Proof To simplify the notation, we define

$$
\Omega=\frac{\theta(\kappa m-\omega)}{\kappa}\quad\text{and}\quad\Lambda=m-\Omega. \tag{4.4}
$$

From Definition 4.1 and Theorem 3.1, we have

$$
\left({}_{\kappa}^{\rho}D_{a+}^{\omega,\theta}{}_{\kappa}^{\rho}\Im_{a+}^{\omega}\psi\right)(z)=\left({}_{\kappa}^{\rho}\Im_{a+}^{\theta(\kappa m-\omega)}\delta_{\rho}^{m}\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)+\omega}\psi\right)\right)(z) \tag{4.5}
$$

$$
=\frac{\kappa^{m-2}\rho^{2-\Omega-\Lambda}}{\Gamma_{\kappa}[k\Omega]\,\Gamma_{\kappa}[k\Lambda]}\int_{a}^{z}(z^{\rho}-y^{\rho})^{\Omega-1}y^{\rho-1}\delta_{\rho}^{m}\underbrace{\left[\int_{a}^{y}(y^{\rho}-x^{\rho})^{\Lambda-1}x^{\rho-1}\psi(x)\,dx\right]}_{G(y)}dy. \tag{4.6}
$$

For the integral within the bracket, let $\psi(x)$ as first term and $(y^{\rho}-x^{\rho})^{\Lambda-1}$ as second term and integrate by parts, we yield

$$
G(y)=\frac{\rho^{-1}}{\Lambda}\left\{\psi(a)(y^{\rho}-a^{\rho})^{\Lambda}+\int_{a}^{y}(y^{\rho}-x^{\rho})^{\Lambda}\psi'(x)\,dx\right\}. \tag{4.7}
$$

Now, apply the operator δ_ρ^m to the equation (4.7). By mathematical induction, we have the following expression

$$\delta_\rho^m G(y) = \frac{\rho^{m-1}\Gamma(\Lambda+1)}{\Lambda\Gamma(\Lambda-m+1)}\left\{\psi(a)(y^\rho-a^\rho)^{\Lambda-m} + \int_a^y (y^\rho-x^\rho)^{\Lambda-m}\psi'(x)\,dx\right\}. \tag{4.8}$$

Substitute Equation (4.8) into Equation (4.6) and utilize the first expression of Equation (2.2), to get

$$\begin{aligned}\left({}_\kappa^\rho D_{a+}^{\omega,\theta}\,{}_\kappa^\rho\Im_{a+}^\omega\psi\right)(z) &= \frac{\rho}{\kappa\kappa^{\Omega-1}\Gamma[\Omega]\,\kappa^{(1-\Omega)-1}\Gamma[1-\Omega]}\left\{\psi(a)\int_a^z (z^\rho-y^\rho)^{\Omega-1}y^{\rho-1}(y^\rho-a^\rho)^{-\Omega}dy\right.\\ &\left.+\int_a^z \psi'(z)\,dz\int_x^z (z^\rho-y^\rho)^{\Omega-1}y^{\rho-1}(y^\rho-x^\rho)^{-\Omega}dx\right\}.\end{aligned}$$

In the integral from a to z change the variable $u=\frac{y^\rho-a^\rho}{z^\rho-a^\rho}$, similarly changing the variable in the integral from y to z, we get

$$\left({}_\kappa^\rho D_{a+}^{\omega,\theta}\,{}_\kappa^\rho\Im_{a+}^\omega\psi\right)(z) = \frac{\rho}{\Gamma_K[\kappa\Omega]\,\Gamma_\kappa[\kappa(1-\Omega)]}\left\{\frac{1}{\kappa}\int_0^1 (1-u)^{\Omega-1}u^{(1-\Omega)-1}du\right\}\left\{\psi(a)+\int_a^z\psi'(z)\,dz\right\}.$$

Use the expression two of Equation (2.2), we obtain

$$\begin{aligned}\left({}_\kappa^\rho D_{a+}^{\omega,\theta}\,{}_\kappa^\rho\Im_{a+}^\omega\psi\right)(z) &= \frac{\rho}{\Gamma_K[\kappa\Omega]\,\Gamma_\kappa[\kappa(1-\Omega)]}\left\{\frac{\Gamma_K[\kappa\Omega]\,\Gamma_\kappa[\kappa(1-\Omega)]}{\rho}\right\}\left\{\psi(a)+\int_a^z\psi'(z)\,dz\right\}\\ &= \psi(a)+\int_a^z\psi'(z)\,dz.\end{aligned}$$

Eventually, by fundamental theorem of calculus, from which it immediately follows

$$\left({}_\kappa^\rho D_{a+}^{\omega,\theta}\,{}_\kappa^\rho\Im_{a+}^\omega\psi\right)(z) = \psi(z),$$

which shows that ${}_\kappa^\rho D_{a+}^{\omega,\theta}$ and ${}_\kappa^\rho\Im_{a+}^\omega$ are inverse operator of each other. □

Theorem 4.3 *Let $\omega>0$, $m=[\omega]+1$, where $m\in\mathbb{N}$. If $\psi\in M_q(a,b)$ and $\left({}_\kappa^\rho\Im_{a+}^{(1-\theta)(\kappa m-\omega)-\kappa(m-n)}\psi\right)(z)\in AC_\delta^m[a,b]$, then*

$$\left({}_\kappa^\rho\Im_{a+}^\omega\,{}_\kappa^\rho D_{a+}^{\omega,\theta}\psi\right)(z) = \psi(z) - \sum_{n=1}^m \frac{\left({}_\kappa^\rho\Im_{a+}^{(1-\theta)(\kappa m-\omega)-\kappa(m-n)}\psi\right)(a)}{\Gamma_\kappa[\kappa(\nabla-n+1)]}\left(\frac{z^\rho-a^\rho}{\rho}\right)^{\nabla-n}, \tag{4.9}$$

where

$$\nabla = \frac{\theta(\kappa m-\omega)+\omega}{\kappa}. \tag{A}$$

and in particular, if $0<\omega<1$ then

$$\left({}_\kappa^\rho\Im_{a+}^\omega\,{}_\kappa^\rho D_{a+}^{\omega,\theta}\psi\right)(z) = \psi(z) - \frac{\left({}_\kappa^\rho\Im_{a+}^{(1-\theta)(\kappa-\omega)-\kappa(1-n)}\psi\right)(a)}{\Gamma_\kappa[\theta(\kappa-\omega)+\omega-\kappa(n-1)]}\left(\frac{z^\rho-a^\rho}{\rho}\right)^{\nabla-n}. \tag{4.10}$$

Proof We can write from Definition 4.1,

$$\begin{aligned}\left({}_{\kappa}^{\rho}\Im_{a+\kappa}^{\omega}{}^{\rho}D_{a+}^{\omega,\theta}\psi\right)(z) &= \left({}_{\kappa}^{\rho}\Im_{a+\kappa}^{\omega}{}^{\rho}\Im_{a+}^{\theta(\kappa m-\omega)}\delta_{\rho}^{m}\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)}\psi\right)\right)(z) \\ &= \left({}_{\kappa}^{\rho}\Im_{a+}^{\theta(\kappa m-\omega)+\omega}\delta_{\rho}^{m}\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)}\psi\right)\right)(z) \\ &= \frac{\rho^{1-\nabla}}{\Gamma_{\kappa}[k\nabla]}\int_{a}^{z}(z^{\rho}-y^{\rho})^{\nabla-1}y^{\rho-1}\left\{\delta_{\rho}^{m}\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)}\psi(y)\right)\right\}dy.\end{aligned}$$

Integrating by parts the last expression, we obtain

$$\begin{aligned}\left({}_{\kappa}^{\rho}\Im_{a+\kappa}^{\omega}{}^{\rho}D_{a+}^{\omega,\theta}\psi\right)(z) &= \frac{-\rho^{1-\nabla}(z^{\rho}-a^{\rho})^{\nabla-1}}{\kappa^{\nabla}\Gamma(\nabla)}\left\{\delta_{\rho}^{m-1}\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)}\psi\right)(a)\right\} \\ &+ \frac{\rho^{2-\nabla}(z^{\rho}-a^{\rho})^{\nabla-1}}{\kappa^{\nabla}\Gamma(\nabla-1)}\int_{a}^{z}(z^{\rho}-y^{\rho})^{\nabla-1}y^{\rho-1}\delta_{\rho}^{m}\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)}\psi\right)(x)\,dy.\end{aligned}$$

Now integrating by parts $(m-1)$-times, we yield

$$\begin{aligned}\left({}_{\kappa}^{\rho}\Im_{a+\kappa}^{\omega}{}^{\rho}D_{a+}^{\omega,\theta}\psi\right)(z) &= -\sum_{n=1}^{m-1}\frac{\delta_{\rho}^{m-n-1}\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)}\psi\right)(a)}{\kappa^{n+1}\Gamma_{\kappa}[\kappa(\nabla-n)]}\left(\frac{z^{\rho}-a^{\rho}}{\rho}\right)^{\nabla-n-1} \\ &+ \frac{1}{\kappa\Gamma_{\kappa}[\kappa(\nabla-m)]}\int_{a}^{z}\left(\frac{z^{\rho}-y^{\rho}}{\rho}\right)^{\nabla-m-1}y^{\rho-1}\left({}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)}\psi\right)(y)\,dy \\ &= -\sum_{n=1}^{m-1}\frac{\delta_{\rho}^{m-n-1}\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)}\psi\right)(a)}{\kappa^{n+1}\Gamma_{\kappa}[\kappa(\nabla-n)]}\left(\frac{z^{\rho}-a^{\rho}}{\rho}\right)^{\nabla-n-1} \\ &+ \left({}_{\kappa}^{\rho}\Im_{a+}^{\theta(\kappa m-\omega)+\omega-m\kappa}\psi{}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)}\psi\right)(z) \\ &= \psi(z)-\sum_{n=1}^{m}\frac{\delta_{\rho}^{m-n}\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)}\psi\right)(a)}{\kappa^{n}\Gamma_{\kappa}[\kappa(\nabla-n+1)]}\left(\frac{z^{\rho}-a^{\rho}}{\rho}\right)^{\nabla-n} \\ &= \psi(z)-\sum_{n=1}^{m}\frac{\left({}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)-\kappa(m-n)}\psi\right)(a)}{\Gamma_{\kappa}[\kappa(\nabla-n+1)]}\left(\frac{z^{\rho}-a^{\rho}}{\rho}\right)^{\nabla-n}.\end{aligned}$$

□

Theorem 4.4 *Let $\omega,\theta\in\mathbb{R}$, $m-1<\omega\leq m$, $m\in\mathbb{N}$, $0\leq\theta\leq 1$ and $\kappa>0$ then $\forall n=1,2,3,\ldots,m$, we have*

$$\left[{}_{\kappa}^{\rho}D_{a+}^{\omega,\theta}(y^{\rho}-a^{\rho})^{\nabla-n}\right](z)=0. \tag{4.11}$$

Proof To simplify the notation, we suppose that $\chi=\frac{(1-\theta)(\kappa m-\omega)}{\kappa}$, so from Definition 4.1 and Equation (2.11), we obtain

$$\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)}(y^{\rho}-a^{\rho})^{\nabla-n}\right)(z)=\frac{\kappa^{m}\rho^{1-\chi}}{\kappa\Gamma_{\kappa}[k\chi]}\int_{a}^{z}(z^{\rho}-y^{\rho})^{\chi-1}y^{\rho-1}(y^{\rho}-a^{\rho})^{\nabla-n}dy.$$

Now, change the variable in the above expression by $v=\frac{y^\rho-a^\rho}{z^\rho-a^\rho}$ and from Equation (2.5), we get

$$\left(\kappa^m{}_\kappa^\rho\Im_{a+}^{(1-\theta)(\kappa m-\omega)}(y^\rho-a^\rho)^{\nabla-n}\right)(z)=\frac{\kappa^m\rho^{-\chi}}{\Gamma_\kappa[k\chi]}(z^\rho-a^\rho)^{m-n}\left\{\frac{1}{\kappa}\int_0^1(1-v)^{\chi-1}v^{\nabla-n}dv\right\}$$
$$=\frac{\kappa^m\rho^{-\chi}\Gamma_\kappa[\kappa(\nabla-n+1)]}{\Gamma_\kappa[\kappa(m-n-1)]}(z^\rho-a^\rho)^{m-n}.$$

Calculating the $\delta_\rho^m\left(\kappa^m{}_\kappa^\rho\Im_{a+}^{(1-\theta)(\kappa m-\omega)}(z^\rho-a^\rho)^{m-n}\right)(z)$, i.e.,

$$\left(z^{1-\rho}\frac{d}{dz}\right)^m(z^\rho-a^\rho)^{m-n}=\left(z^{1-\rho}\frac{d}{dz}\right)^{m-1}\left(z^{1-\rho}\frac{d}{dz}\right)(z^\rho-a^\rho)^{m-n}$$
$$=\rho(m-n)\left(z^{1-\rho}\frac{d}{dz}\right)^{m-1}(z^\rho-a^\rho)^{m-n-1}.$$

Differentiating $(m-1)$-times, we yield

$$\left(z^{1-\rho}\frac{d}{dz}\right)^m(z^\rho-a^\rho)^{m-n}=\rho^m(m-n)(m-n-1)\dots(2-n)(1-n)(z^\rho-a^\rho)^{-n}=0. \tag{4.12}$$

as $n=1,2,\dots,m$, for each value of n, the product in equation (4.12) in null. Which shows that the generalized k-fractional Hilfer–Katugampola derivative of order ω of a polynomial $(y^\rho-a^\rho)^{\nabla-n}$ is null. □

5. Equivalence between Cauchy problem and Volterra integral equation

In this section, we will show the equivalence between Cauchy problem and Volterra integral equation of second kind. Following theorem will prove the required result.

Theorem 5.1 *Let $\omega>0$ and $m=[\omega]+1$, where $m\in\mathbb{N}$. Let H is an open set in $\mathbb{R}$ such that $g:(a,b]\times H\to\mathbb{R}$ be a function such that $g(z,\psi(z))\in M(a,b)$ for any $\psi\in H$. If $\psi\in H$ then H satisfies the relation*

$$\left({}_\kappa^\rho D_{a+}^{\omega,\theta}\psi\right)(z)=g(z,\psi(z)) \tag{5.1}$$

$$\left({}_\kappa^\rho\Im_{a+}^{(1-\theta)(\kappa m-\omega)-\kappa(m-n)}\psi\right)(a^+)=b_n,\quad b_n\in\mathbb{R},\quad n=1,2,\dots,m. \tag{5.2}$$

Iff ψ satisfies the Volterra integral equation,

$$\psi(z)=\sum_{n=1}^m\frac{b_n}{\Gamma_\kappa[\kappa(\nabla-n+1)]}\left(\frac{z^\rho-a^\rho}{\rho}\right)^{\nabla-n}+\frac{1}{\kappa\Gamma_\kappa(\omega)}\int_a^z\left(\frac{z^\rho-y^\rho}{\rho}\right)^{\frac{\omega}{\kappa}-1}y^{\rho-1}g(z,\psi(z))\,dy, \tag{5.3}$$

where ∇ is defined in equation (A).

Proof ⇒ We suppose that $\psi\in M(a,b)$ satisfy Equations (5.1) and (5.2). As $\psi\in M(a,b)$ exists and $\left({}_\kappa^\rho D_{a+}^{\omega,\theta}\psi\right)(z)\in M(a,b)$, now applying the operator ${}_\kappa^\rho\Im_{a+}^\omega$ on both sides of Equation (5.1) and utilize Theorem 4.3 and Equation (5.2), we yield

$$\left({}_\kappa^\rho\Im_{a+}^\omega{}_\kappa^\rho D_{a+}^{\omega,\theta}\psi\right)(z)=\left({}_\kappa^\rho\Im_{a+}^\omega g(z,\psi(z))\right)(z)$$

$$\psi(z) - \sum_{n=1}^{m} \frac{\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)-\kappa(m-n)}\psi\right)(a)}{\Gamma_\kappa\left[\kappa(\nabla-n+1)\right]}\left(\frac{z^\rho-a^\rho}{\rho}\right)^{\nabla-n} = \left({}_{\kappa}^{\rho}\Im_{a+}^{\omega} g(z,\psi(z))\right)(z)$$

$$\psi(z) = \sum_{n=1}^{m} \frac{\left(\kappa^{m}{}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)-\kappa(m-n)}\psi\right)(a)}{\Gamma_\kappa\left[\kappa(\nabla-n+1)\right]}\left(\frac{z^\rho-a^\rho}{\rho}\right)^{\nabla-n} + \left({}_{\kappa}^{\rho}\Im_{a+}^{\omega} g(z,\psi(z))\right)(z).$$

Using Lemma 3.1, the integral $\left({}_{\kappa}^{\rho}\Im_{a+}^{\omega} g(z,\psi(z))\right)(z) \in M(a,b)$, so Equation (5.3) follows.

$\Leftarrow$ Suppose that $\psi \in M(a,b)$ satisfies Equation (5.3). Now applying the operator ${}_{\kappa}^{\rho}D_{a+}^{\omega,\theta}$ on both sides of the Equation (5.3), we have

$$\left({}_{\kappa}^{\rho}D_{a+}^{\omega,\theta}\psi\right)(z) = \sum_{n=1}^{m}\frac{b_n}{\Gamma_\kappa\left[\kappa(\nabla-n+1)\right]}\left[{}_{\kappa}^{\rho}D_{a+}^{\omega,\theta}(z^\rho-a^\rho)^{\nabla-n}\right](z) + \left({}_{\kappa}^{\rho}D_{a+}^{\omega,\theta}\left({}_{\kappa}^{\rho}\Im_{a+}^{\omega} g(z,\psi(z))\right)\right)(z).$$

From Theorems 4.2 and 4.4, Equation (5.1) follows, apply the operator ${}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)-\kappa(m-n)}$ with $n=1,2,\ldots,m$ to prove the validity of Equation (5.2) on both sides of Equation (5.3),

$$\begin{aligned}\left({}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)-\kappa(m-n)}\psi\right)(z) &= \sum_{n=1}^{m}\frac{b_n}{\Gamma_\kappa\left[\kappa(\nabla-n+1)\right]}\left[{}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)-\kappa(m-n)}\left(\frac{z^\rho-a^\rho}{\rho}\right)^{\nabla-n}\right]\\ &\quad+\left({}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)-\kappa(m-n)}\left({}_{\kappa}^{\rho}\Im_{a+}^{\omega} g(z,\psi(z))\right)\right)(z)\\ &= \sum_{n=1}^{m}\frac{b_n}{\Gamma_\kappa\left[\kappa(\nabla-n+1)\right]}\left(\frac{z^\rho-a^\rho}{\rho}\right)^{m-n}\\ &\quad+\left({}_{\kappa}^{\rho}\Im_{a+}^{\kappa(n-\theta m)+\omega\theta}\left({}_{\kappa}^{\rho}\Im_{a+}^{\omega} g(z,\psi(z))\right)\right)(z).\end{aligned}$$

Let $z \to a_{a+}$, we have

$$\left({}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)-\kappa(m-n)}\psi\right)(a^+) = b_n, \quad \text{with} \quad n=1,2,\ldots,m.$$

□

6. Linear differential equations of noninteger order

In this section, we discuss some special cases of function $g(z,\psi(z))$ that appear in Theorem 5.1. As a consequence, we suggest to apply the method of successive approximations to obtain an empirical solution for the resulting linear fractional differential equations. Let us first consider $g(z,\psi(z))$ in Theorem 5.1.

Theorem 6.1 *Let $\omega,\eta \in \mathbb{R}^+$ with $m-1<\omega\le m-1$ with $m\in\mathbb{N}$. If $\psi\in M(a,b)$, then the Cauchy problem*

$$\left({}_{\kappa}^{\rho}D_{a+}^{\omega,\theta}\psi\right)(z) = g(z,\psi(z)) \tag{6.1}$$

$$\left({}_{\kappa}^{\rho}\Im_{a+}^{(1-\theta)(\kappa m-\omega)-\kappa(m-n)}\psi\right)(a^+) = b_n, \quad b_n\in\mathbb{R}, \quad n=1,2,\ldots,m, \tag{6.2}$$

concedes a unique solution in the space $M(a,b)$, given by

$$\psi(z) = \sum_{n=1}^{m}\left(\frac{z^\rho-a^\rho}{\rho}\right)^{\nabla-n} F_{\kappa,\omega,\kappa(\nabla-n+1)}\left[\eta\left(\frac{z^\rho-a^\rho}{\rho}\right)^{\frac{\omega}{\kappa}}\right], \tag{6.3}$$

where $F_{\kappa,\beta,\gamma}$ is defined in Equation (2.8).

Proof Thus, according to Theorem 5.1, it is only necessary to solve the integral equation of Volterra Equation (5.1), with $g(z,\psi(z))$. The uniqueness of Equation (5.1) is assured, as the Volterra integral equation of second kind admits a special solution (6.3). We use the form of successive approximations to find the exact solution, that is to say we consider

$$\psi_0(z) = \sum_{n=1}^{m} \frac{b_n}{\Gamma_\kappa[\kappa(\nabla - n + 1)]} \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla - n} \tag{6.4}$$

$$\psi_j(z) = \psi_0(z) + \frac{\eta}{\kappa\Gamma_\kappa(\omega)} \int_a^z \left(\frac{z^\rho - y^\rho}{\rho}\right)^{\frac{\omega}{\kappa}-1} y^{\rho-1}\psi_{j-1}(z)\,dy. \tag{6.5}$$

Define the parameter

$$\nabla_n = \frac{\theta(\kappa m - \omega) + \omega n}{k}, \quad \text{with} \quad n = 1, 2, \dots, j+1. \tag{6.6}$$

When $n = 1$, we get $\nabla_1 = \nabla$ as given by equation (A), so from Equation (6.1), we have

$$\begin{aligned} \psi_1(z) &= \psi_0(z) + \frac{\eta}{\kappa\Gamma_\kappa(\omega)} \int_a^z \left(\frac{z^\rho - y^\rho}{\rho}\right)^{\frac{\omega}{\kappa}-1} y^{\rho-1}\psi_0(z)\,dy \\ &= \psi_0(z) + \sum_{n=1}^{m} \frac{\eta b_n}{\Gamma_\kappa[\kappa(\nabla - n + 1)]} \left[{}^{\rho}_{\kappa}\Im^{\omega}_{a+}\left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla - n}\right](z). \end{aligned}$$

By using Theorem 5.1, we have

$$\psi_1(z) = \sum_{n=1}^{m} \frac{b_n}{\Gamma_\kappa[\kappa(\nabla - n + 1)]} \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla - n} + \sum_{n=1}^{m} \frac{b_n}{\Gamma_\kappa[\kappa(\nabla_2 - n + 1)]} \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla_2 - n} \tag{6.7}$$

$$= \sum_{n=1}^{m} b_n \sum_{n=1}^{2} \frac{\eta^{n-1}}{\Gamma_\kappa[\kappa(\nabla_n - n + 1)]} \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla_n - n}. \tag{6.8}$$

Likewise, with Equation (6.8) and Theorem 5.2, we get the expression for $\psi_2(z)$, i.e.

$$\begin{aligned} \psi_2(z) &= \psi_0(z) + \frac{\eta}{\kappa\Gamma_\kappa(\omega)} \int_a^z \left(\frac{z^\rho - y^\rho}{\rho}\right)^{\frac{\omega}{\kappa}-1} y^{\rho-1}\psi_1(z)\,dy \\ &= \psi_0(z) + \eta \sum_{n=1}^{m} b_n \sum_{n=1}^{2} \frac{\eta^{n-1}}{\Gamma_\kappa[\kappa(\nabla_n - n + 1)]} \left[{}^{\rho}_{\kappa}\Im^{\omega}_{a+}\left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla - n}\right](z) \\ &= \sum_{n=1}^{m} b_n \sum_{n=1}^{3} \frac{\eta^{n-1}}{\Gamma_\kappa[\kappa(\nabla_n - n + 1)]} \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla_n - n}. \end{aligned}$$

Continuing the process, we get the expression for $\psi_j(z)$ for $j \in \mathbb{N}$

$$\psi_j(z) = \sum_{n=1}^{m} b_n \sum_{n=1}^{j+1} \frac{\eta^{n-1}}{\Gamma_\kappa[\kappa(\nabla_n - n + 1)]} \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla_n - n}.$$

Let $j \to \infty$, the solution for $\psi(z)$ is

$$\psi(z) = \sum_{n=1}^{m} b_n \sum_{n=1}^{\infty} \frac{\eta^{n-1}}{\Gamma_\kappa\left[\kappa\left(\nabla_n - n + 1\right)\right]} \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla_n - n}.$$

We have modified the summation index by $n = n + 1$

$$\psi(z) = \sum_{n=1}^{m} b_n \sum_{n=1}^{\infty} \frac{\eta^{n-1}}{\Gamma_\kappa\left[\kappa\left(\nabla_{n+1} - n + 1\right)\right]} \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla_{n+1} - n}.$$

However, in terms of the κ-new generalized Mittag–Leffler function, we can rewrite this last expression as

$$\psi(z) = \sum_{n=1}^{m} \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla - n} F_{\kappa,\omega,\kappa(\nabla - n + 1)}\left[\eta\left(\frac{z^\rho - a^\rho}{\rho}\right)^{\frac{\omega}{\kappa}}\right]. \tag{6.9}$$

□

Theorem 6.2 *Let $\omega, \mu \in \mathbb{R}$, $\omega > \mu > 0$, $m - 1 < \omega \le m$, where $m \in \mathbb{N}$ and $\eta \in \mathbb{R}$. Then the Cauchy problem is*

$$\left({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}\psi\right)(z) = \eta\left({}^{\rho}_{\kappa}D^{\gamma,\theta}_{a+}\psi\right)(z)$$

$$\left({}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m - \omega) - \kappa(m-n)}_{a+}\psi\right)\left(a^+\right) = b_n, \quad b_n \in \mathbb{R}, \quad n = 1, 2, \ldots, m,$$

concedes a unique solution in the space $M(a,b)$, given by

$$\psi(z) = \sum_{n=1}^{m} b_n \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\zeta - n} F_{\kappa,\omega,\kappa(\xi - n + 1)}\left[\eta\left(\frac{z^\rho - a^\rho}{\rho}\right)^{\frac{\omega-\gamma}{\kappa}}\right],$$

$$\textit{where} \quad \xi = \frac{\omega + \theta\left(m\kappa - \omega + \gamma\right)}{\kappa}.$$

Proof Suppose that the solution $\psi(z) = \left({}^{\rho}_{\kappa}\Im^{\gamma}_{a+}f\right)(z) \in M(a,b)$, then we have

$$\left({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}{}^{\rho}_{\kappa}\Im^{\gamma}_{a+}f\right)(z) = \eta\left({}^{\rho}_{\kappa}D^{\gamma,\theta}_{a+}{}^{\rho}_{\kappa}\Im^{\gamma}_{a+}f\right)(z).$$

By using Theorem 4.2, we obtain

$$\left({}^{\rho}_{\kappa}D^{\gamma,\theta}_{a+}{}^{\rho}_{\kappa}\Im^{\gamma}_{a+}f\right)(z) = f(z)$$

$$\left({}^{\rho}_{\kappa}D^{\omega,\theta}_{a+}{}^{\rho}_{\kappa}\Im^{\gamma}_{a+}f\right)(z) = \left({}^{\rho}_{\kappa}D^{\omega-\gamma,\theta}_{a+}f\right)(z)$$

$$\left({}^{\rho}_{\kappa}D^{\omega-\gamma,\theta}_{a+}f\right)(z) = \eta f(z).$$

So let $\lambda_n = \frac{(\omega-\gamma)+\omega+\gamma+\theta(n\kappa-\omega+\gamma)}{\kappa}$, when $n = 1$, we have $\lambda_n = \lambda$.
Now let $\omega \to \omega - \gamma$ in Theorem 6.1, we obtain

$$g(z) = \sum_{n=1}^{m} \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\lambda - n} F_{\kappa,\omega-\gamma,\kappa(\lambda - n + 1)}\left[\eta\left(\frac{z^\rho - a^\rho}{\rho}\right)^{\frac{\omega-\gamma}{\kappa}}\right]. \tag{6.10}$$

$$\psi(z) = \left({}^{\rho}_{\kappa}\Im^{\gamma}_{a+} f\right)(z)$$

Applying the operator ${}^{\rho}_{\kappa}\Im^{\gamma}_{a+}$ on both sides of Equation (6.10), we obtain

$$\left({}^{\rho}_{\kappa}\Im^{\gamma}_{a+} g\right)(z) = \sum_{n=1}^{m} b_n \sum_{n=0}^{\infty} \frac{\eta^n}{\Gamma_\kappa\left[\kappa\left(\lambda_m - n + 1\right)\right]} \left[{}^{\rho}_{\kappa}\Im^{\gamma}_{a+}\left(\frac{z^\rho - a^\rho}{\rho}\right)^{\lambda_m - n}\right](z).$$

By using the Theorem 3.2 and rewriting the expression, we get

$$\psi(z) = \sum_{n=1}^{m} b_n \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\xi - n} F_{\kappa, \omega-\gamma, \kappa(\xi - n + 1)}\left[\eta\left(\frac{z^\rho - a^\rho}{\rho}\right)^{\frac{\omega-\gamma}{\kappa}}\right].$$

□

We consider a sequence of linear fractional differential equations of order ωm in the next theorem.

Theorem 6.3 *Let $\omega, \mu \in \mathbb{R}$, $\omega > \mu > 0$, $m - 1 < \omega \leq m$, where $m \in \mathbb{N}$ and $\eta \in \mathbb{R}$. Then the Cauchy problem is*

$$\left({}^{\rho}_{\kappa}D^{m\omega,\theta}_{a+}\psi\right)(z) = \eta\psi(z) \tag{6.11}$$

$$\left({}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m - m\omega) - \kappa(m-n)}_{a+}\psi\right)\left(a^+\right) = b_n, \quad b_n \in \mathbb{R}, \quad n = 1, 2, \ldots, m, \tag{6.12}$$

concedes a unique solution in the space $M(a, b)$, given by

$$\psi(z) = \sum_{n=1}^{m} b_n \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla_m - n} F_{\kappa, \omega m, \kappa(\xi_m - n + 1)}\left[\eta_m\left(\frac{z^\rho - a^\rho}{\rho}\right)^{\frac{\omega m}{\kappa}}\right], \tag{6.13}$$

where $\nabla_m = \frac{\theta(\mathrm{m}\kappa - \omega m) + \omega m}{\kappa}$.

Proof $\omega \to \omega m$ in Theorem 6.1, we obtain the solution (6.13). □

7. Reliance on initial conditions

We present in this section the changes in a solution involving minor changes in initial conditions. Consider Equation (5.1) with the following modifications in the initial conditions in Equation (5.2),

$$\left({}^{\rho}_{\kappa}\Im^{(1-\theta)(\kappa m - \omega) - \kappa(m-n)}_{a+}\psi\right)\left(a^+\right) = b_n + \mu_n, \quad b_n \in \mathbb{R}, \quad n = 1, 2, \ldots, m, \tag{7.1}$$

where μ_n, $\forall n = 1, 2, \ldots, m$ are arbitrary constants.

Theorem 7.1 *Assume that hypothesis in Theorem 5.1 is fulfilled. Let $\psi(z)$ and $\tilde{\psi}(z)$ be the initial value problem solutions of Equations (5.1) and (5.2). Then*

$$\left|\psi(z) - \tilde{\psi}(z)\right| \leq \sum_{n=1}^{m} |\mu_n| \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\frac{\omega + \theta(m\kappa - \omega)}{\kappa} - n} F_{\kappa, \omega, \omega + \theta(m\kappa - \omega) - \kappa(n-1)}\left[C\left(\frac{z^\rho - a^\rho}{\rho}\right)^{\frac{\omega}{\kappa}}\right],$$

where $z \in (a, b]$ and $F_{\kappa,\beta,\gamma}(z)$ is defined in Equation (2.8).

Proof From Theorem 5.1, we have

$$\psi(z) = \lim_{j\to\infty} \psi_j(z),$$

where $\psi_0(x)$ is given in Equation (6.4) and

$$\psi_j(z) = \psi_0(z) + \frac{1}{\kappa\Gamma_\kappa(\omega)} \int_a^z \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\frac{\omega}{\kappa}-1} y^{\rho-1} g(z, \psi_{j-1}(z)) dy. \tag{7.2}$$

We have,

$$\tilde{\psi}(z) = \lim_{j\to\infty} \tilde{\psi}_j(z) \tag{7.3}$$

$$\tilde{\psi}_0(z) = \sum_{n=1}^{m} \frac{b_n + \mu_n}{\Gamma_\kappa[\kappa(\nabla - n + 1)]} \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla - n} \tag{7.4}$$

$$\tilde{\psi}_j(z) = \tilde{\psi}_0(z) + \frac{1}{\kappa\Gamma_\kappa(\omega)} \int_a^z \left(\frac{z^\rho - y^\rho}{\rho}\right)^{\frac{\omega}{\kappa}-1} y^{\rho-1} g(z, \tilde{\psi}_{j-1}(z)) dy, \quad j = 1, 2, \ldots. \tag{7.5}$$

From Equations (6.4) and (7.4), we have

$$\left|\psi_0(z) - \tilde{\psi}_0(z)\right| \le \sum_{n=1}^{m} \frac{|\mu_n|}{\Gamma_\kappa[\kappa(\nabla - n + 1)]} \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla - n}. \tag{7.6}$$

Eventually, we consider Equations (7.2) and (7.5) with $j = 1$, the Lipschitz function condition $g(z, \psi(z))$, Definition 2.2, Equation (7.6) and Theorem 3.2 inequality, in order to obtain

$$\begin{aligned}
\left|\psi_1(z) - \tilde{\psi}_1(z)\right| &\le \sum_{n=1}^{m} \frac{|\mu_n|}{\Gamma_\kappa[\kappa(\nabla - n + 1)]} \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla - n} \\
&\quad + \frac{C}{\kappa\Gamma_\kappa(\omega)} \int_a^z \left(\frac{z^\rho - y^\rho}{\rho}\right)^{\frac{\omega}{\kappa}-1} y^{\rho-1} \left[g(z, \psi_0(z)) - g\left(z, \tilde{\psi}_0(z)\right)\right] dy \\
&\le \sum_{n=1}^{m} \frac{|\mu_n|}{\Gamma_\kappa[\kappa(\nabla - n + 1)]} \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla - n} \\
&\quad + \frac{C}{\kappa\Gamma_\kappa(\omega)} \int_a^z \left(\frac{z^\rho - y^\rho}{\rho}\right)^{\frac{\omega}{\kappa}-1} y^{\rho-1} \left|\psi_0(z) - \tilde{\psi}_0(z)\right| dy \\
&\le \sum_{n=1}^{m} \frac{|\mu_n|}{\Gamma_\kappa[\kappa(\nabla - n + 1)]} \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla - n} \\
&\quad + \frac{C}{\kappa\Gamma_\kappa(\omega)} \int_a^z \left(\frac{z^\rho - y^\rho}{\rho}\right)^{\frac{\omega}{\kappa}-1} y^{\rho-1} \left(\frac{y^\rho - a^\rho}{\rho}\right)^{\nabla - n} dy \\
&= \sum_{n=1}^{m} |\mu_n| \sum_{n=1}^{2} \frac{C^{n-1}}{\Gamma_\kappa[\kappa(\nabla_n - n + 1)]} \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla_n - n},
\end{aligned}$$

where ∇_n is defined in Equation (6.6), repeating the process we have,

$$\left|\psi_j(z) - \tilde{\psi}_j(z)\right| \le \sum_{n=1}^{m} |\mu_n| \sum_{n=1}^{j+1} \frac{C^{n-1}}{\Gamma_\kappa[\kappa(\nabla_n - n + 1)]} \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla_n - n}.$$

Now let $j \to \infty$ and $n \to n+1$, so it follow that

$$\left|\psi(z) - \tilde{\psi}(z)\right| \leq \sum_{n=1}^{m} |\mu_n| \left(\frac{z^\rho - a^\rho}{\rho}\right)^{\nabla - n} F_{\kappa,\omega,\kappa(\nabla-n+1)} \left[C\left(\frac{z^\rho - a^\rho}{\rho}\right)^{\frac{\omega}{\kappa}}\right].$$

□

8. Conclusion

For a fractional derivative recently discussed in [20], we proposed a generalization obtained by introducing a new parameter in its definition. This generalization retrieves a wide collection of definitions of classical fractional derivatives for adequate values of its parameters. We introduced a few properties of this generalized κ-fractional Hilfer–Katugampola derivative. We also discussed the equivalence between a Cauchy problem, using this fractional differentiation operator and a second kind of Volterra integral equation. We have found several particular cases for this Cauchy problem and have shown that minor changes to initial conditions require minor changes in the solution of the problem.

Acknowledgment

The authors would like to express profound gratitude to the anonymous referees and the editor for deeper review of this paper and their valuable advice.

Conflict of interest

The authors declare that they have no conflicts of interest.

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