

## Closure operators in convergence approach spaces

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**Abstract:** In this paper, we characterize closed and strongly closed subsets of convergence approach spaces and introduce two notions of closure in the category of convergence approach spaces which satisfy idempotent, productive and (weakly) hereditary properties. Furthermore, we explicitly characterize each of  $T_i$  convergence approach spaces,  $i = 0, 1, 2$  with respect to these closure operators and show that each of these subcategories of  $T_i$  convergence approach spaces,  $i = 0, 1, 2$  are epireflective as well as we investigate the relationship among these subcategories. Finally, we characterize connected convergence approach spaces.

**Key words:** Convergence approach space, topological category, closure operators, connectedness, initial lift, final lift

### 1. Introduction

There is absolutely no doubt that metric spaces occupy a dominant place in mathematics but it behaves badly with respect to infinite products and coproducts. To address this issue, approach spaces have been introduced by Lowen [18] which are based upon point to set distance. Since approach spaces are the generalization of metric and topological spaces, several applications in different areas of mathematics including probability theory [15], domain theory [16], group theory [20] and vector spaces [21] naturally exist. However, **App** (category of approach spaces and contraction maps) fails to enjoy some convenience categorical properties such as cartesian closedness. As a remedy to this, a bigger category **CApp** (category of convergence approach spaces and contraction maps) has been introduced by Lowen et al. [17] in 1989 which is a topological quasitopos.

In 1991, Baran [2, 4] introduced the notion of closed and strongly closed objects in set-based topological categories to generalize the notion of connectedness [8], compactness [5], perfectness and Hausdorffness [7] in a topological category. In addition to, these notion of (strongly) closedness form appropriate closure operators in the sense of Dikranjan and Giuli [12] in convergence spaces [6], preordered spaces [9], semiuniform convergence spaces [10] and constant filter convergence spaces [14].

The aim of this paper is to characterize of both closed and strongly closed subsets of convergence approach spaces and to show that they form appropriate closure operators which enjoy the basic properties like idempotency, productivity and (weak) hereditariness in the category of convergence approach spaces. Moreover, we characterize each of  $T_i$  convergence approach spaces,  $i = 0, 1, 2$  with respect to these closure operators and show that each of these subcategories of  $T_i$  convergence approach spaces,  $i = 0, 1, 2$  are epireflective

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and investigate the relationship among these subcategories. Finally, we characterize connected and strongly connected convergence approach spaces.

## 2. Preliminaries

Let  $\mathcal{E}$  and  $\mathcal{B}$  be two categories. The functor  $\mathcal{U} : \mathcal{E} \rightarrow \mathcal{B}$  is said to be a topological functor if (i)  $\mathcal{U}$  is concrete (i.e. amnestic and faithful) (ii)  $\mathcal{U}$  consists of small fibers and (iii) every  $\mathcal{U}$ -source has a unique initial lift or equivalently, each  $\mathcal{U}$ -sink has a unique final lift [1, 23, 24].

Let  $X$  be a set,  $A \subseteq X$ ,  $F(X)$  be the set of all filters and  $\mathcal{A}$  be collection of subsets of  $X$ . The stack of  $\mathcal{A}$  and the indicator map  $\theta_A : X \rightarrow [0, \infty]$  are defined by  $[\mathcal{A}] = \{B \subseteq X | \exists A \in \mathcal{A} : A \subseteq B\}$  and

$$\theta_A(x) = \begin{cases} 0, & x \in A \\ \infty, & x \notin A \end{cases}$$

respectively.

**Definition 2.1** (cf. [17, 19, 22]) A map  $\lambda : F(X) \rightarrow [0, \infty]^X$  is called a convergence approach structure on  $X$  if it fulfills the following properties:

- (i)  $\forall x \in X : \lambda[x](x) = 0$ ,
- (ii)  $\forall \alpha, \beta \in F(X) : \alpha \subset \beta \Rightarrow \lambda\beta \leq \lambda\alpha$ ,
- (iii)  $\forall \alpha, \beta \in F(X) : \lambda(\alpha \cap \beta) = \sup\{\lambda(\alpha), \lambda(\beta)\}$ .

The pair  $(X, \lambda)$  is called a convergence approach space.

**Definition 2.2** (cf. [17, 19, 22]) Let  $(X, \lambda)$  and  $(X', \lambda')$  be convergence approach spaces. The map  $f : (X, \lambda) \rightarrow (X', \lambda')$  is called a contraction map if  $\lambda'(f(\alpha)) \circ f \leq \lambda\alpha$  for any  $\alpha \in F(X)$ .

Let **CApp** denote the category with convergence approach spaces as objects and contraction maps as morphisms. Note that it is a cartesian closed topological category over **Set**, the category of sets and functions [17, 19, 22].

**Lemma 2.3** (cf. [17, 19]) Let  $X$  be a nonempty set and  $(X_i, \lambda_i)$  be the class of convergence approach spaces.

- (i) A source  $\{f_i : X \rightarrow (X_i, \lambda_i)\}$  in **CApp** is initial lift iff for all  $\alpha \in F(X)$ ,  $\lambda\alpha = \sup_{i \in I} \lambda_i(f_i(\alpha)) \circ f_i$ , where  $f_i(\alpha)$  is a filter generated by  $\{f_i(A_i), i \in I\}$ .
- (ii) A sink  $\{f_i : (X_i, \lambda_i) \rightarrow X\}$  in **CApp** is final lift iff for all  $\alpha \in F(X)$  and  $x \in X$ ,

$$\lambda(\alpha)(x) = \begin{cases} 0, & \alpha = [x] \\ \inf_{i \in I} \inf_{y \in f_i^{-1}(x)} \inf_{\substack{\beta \in F(X_i) \\ [f_i(\beta)] \subset \alpha}} \lambda_i(\beta)(y), & \alpha \neq [x] \end{cases}$$

(iii) The discrete structure  $(X, \lambda_{dis})$  on  $X$  in **CApp** is defined by for all  $\alpha \in F(X)$  and  $x \in X$ ,

$$\lambda_{dis}(\alpha) = \begin{cases} \theta_{\{x\}}, & \alpha = [x] \\ \infty, & \alpha \neq [x] \end{cases}$$

**Lemma 2.4** (cf. [3]) Let  $\emptyset \neq M \subset X$ ,  $\beta \in F(X)$ ,  $x \in X$  with  $x \notin M$ , and  $q : X \rightarrow X/M$  be the identification map identifying  $M$  to a point  $*$ .

(i) For  $x \notin M$ ,  $q\beta \subset [x]$  iff  $\beta \subset [x]$

(ii)  $q\beta \subset [*]$  iff  $\beta \cup [M]$  is proper.

### 3. Closed subsets of convergence approach space

Let  $X$  be a set,  $p \in X$  and  $X \vee_p X$  be the wedge product of  $X$ . A point  $x$  in  $X \vee_p X$  is denoted by  $x_1$  (resp.  $x_2$ ) if it is in first (resp. second component).

**Definition 3.1** (cf. [2]) A map  $S_p : X \vee_p X \rightarrow X^2$  is called a skewed  $p$  axis map if

$$S_p(x_i) = \begin{cases} (x, x), & i = 1 \\ (p, x), & i = 2 \end{cases}$$

**Definition 3.2** (cf. [2]) A map  $\nabla_p : X \vee_p X \rightarrow X$  is called a folding map at  $p$  if  $\nabla(x_i) = x$  for  $i = 1, 2$ .

The infinite wedge product  $\vee_p^\infty X$  is constructed by taking countably many disjoint copies of  $X$  and identifying them at the point  $p$ .

A point  $x$  in  $\vee_p^\infty X$  is denoted as  $x_i$  if it lies in the  $i$ -th component.

**Definition 3.3** (cf. [4]) Let  $X^\infty = X \times X \times \dots$  be the countable Cartesian product of  $X$ .

(i) The infinite principle axis map at  $p$ ,  $A_p^\infty : \vee_p^\infty X \rightarrow X^\infty$  is defined by  $A_p^\infty(x_i) = (p, p, \dots, p, x_i, p, \dots)$ .

(ii) The infinite fold map at  $p$ ,  $\nabla_p^\infty : \vee_p^\infty X \rightarrow X$  is defined by  $\nabla_p^\infty(x_i) = x$  for all  $i \in I$ .

**Definition 3.4** (cf. [4]) Let  $\mathcal{U} : \mathcal{E} \rightarrow \mathbf{Set}$  be a topological functor,  $B \in \text{Ob}(\mathcal{E})$  with  $\mathcal{U}(B) = X$  and  $p \in X$ .

(i)  $X$  is local  $T_1$  (i.e.  $T_1$  at  $p$ ) iff initial lift of the  $\mathcal{U}$ -source  $\{S_p : X \vee_p X \rightarrow \mathcal{U}(B^2) = X^2$  and  $\nabla_p : X \vee_p X \rightarrow \mathcal{UD}(X) = X\}$  is discrete, where  $\mathcal{D}$  is the discrete functor.

(ii)  $\{p\}$  is closed iff the initial lift of the  $\mathcal{U}$ -source  $\{A_p^\infty : \vee_p^\infty X \rightarrow X^\infty$  and  $\nabla_p^\infty : \vee_p^\infty X \rightarrow \mathcal{UD}(X^\infty) = X\}$  is discrete.

(iii)  $M \subset X$  is closed iff  $\{*\}$ , the image of  $M$ , is closed in  $X/M$  or  $M = \emptyset$ .

(iv)  $M \subset X$  is strongly closed iff if  $X/M$  is  $T_1$  at  $\{*\}$  or  $M = \emptyset$ .

(v) If  $X = M = \emptyset$ , then we define  $M$  to be both closed and strongly closed.

**Remark 3.5** Let  $\alpha, \beta \in F(X)$ ,  $\gamma \in F(Y)$ , and  $f : X \rightarrow Y$  be a function. Then,

$$(i) f(\alpha \cap \beta) = f(\alpha) \cap f(\beta).$$

$$(ii) f(\alpha \cup \beta) \supset f(\alpha) \cup f(\beta).$$

$$(iii) \gamma \subset f f^{-1} \gamma.$$

$$(iv) f^{-1} f \alpha \subset \alpha.$$

**Theorem 3.6** (cf. [11]) A convergence approach space  $(X, \lambda)$  is  $T_1$  at  $p$  iff for all  $x \in X$  with  $x \neq p$ ,  $\lambda([x])(p) = \infty = \lambda([p])(x)$ .

**Theorem 3.7** Let  $(X, \lambda)$  be a convergence approach space.  $\{p\}$  is closed iff for any  $x \in X$  with  $x \neq p$ ,  $\lambda([x])(p) = \infty$  or  $\lambda([p])(x) = \infty$ .

**Proof** Suppose  $\{p\}$  is closed, and  $x \in X$  with  $x \neq p$ . Let  $\alpha = [(x, p, p, \dots)] \in F(\vee_p^\infty X)$  and  $w = (p, x, p, p, \dots) \in \vee_p^\infty X$ . Note that

$$\lambda(\pi_1 A_p^\infty \alpha)(\pi_1 A_p^\infty w) = \lambda([x])(p),$$

$$\lambda(\pi_2 A_p^\infty \alpha)(\pi_2 A_p^\infty w) = \lambda([p])(x)$$

and for  $j \geq 3$ ,

$$\lambda(\pi_j A_p^\infty \alpha)(\pi_j A_p^\infty w) = \lambda([p])(p) = 0$$

and

$$\lambda_{dis}(\nabla_p^\infty \alpha)(\nabla_p^\infty w) = \lambda_{dis}([x])(x) = 0.$$

Since  $\lambda_{dis}$  is the discrete convergence approach structure on  $X$  and  $\pi_j$  is the projection map for  $j \in I$ . By Definition 3.4 (i),

$$\begin{aligned} \infty &= \sup\{\lambda_{dis}(\nabla_p^\infty \alpha)(\nabla_p^\infty w), \lambda(\pi_j A_p^\infty \alpha)(\pi_j A_p^\infty w) : j \in I\} \\ &= \sup\{\lambda([x])(p), \lambda([p])(x)\}. \end{aligned}$$

It follows that  $\lambda([x])(p) = \infty$  or  $\lambda([p])(x) = \infty$ .

Conversely, let  $\bar{\lambda}$  be the initial convergence approach structure on  $\vee_p^\infty X$  induced by  $A_p^\infty : \vee_p^\infty X \rightarrow \mathcal{U}(X^\infty, \lambda^*) = X^\infty$  and  $\nabla_p^\infty : \vee_p^\infty X \rightarrow \mathcal{U}(X, \lambda_{dis}) = X$  where  $\lambda^*$  is the product convergence approach structure on  $X^\infty$  induced by  $\pi_j : X^\infty \rightarrow X$  ( $j \in I$ ) projection maps and  $\lambda_{dis}$  is the discrete convergence approach structure on  $X$ . Suppose  $\alpha \in F(\vee_p^\infty X)$  and  $w \in \vee_p^\infty X$ . Note that

$$\lambda_{dis}(\nabla_p^\infty \alpha)(\nabla_p^\infty w) = \begin{cases} \theta_{\{x\}} \nabla_p^\infty w, & \nabla_p^\infty \alpha = [x] \\ \infty, & \nabla_p^\infty \alpha \neq [x] \end{cases}$$

$$= \begin{cases} 0, & \nabla_p^\infty \alpha = [x] \text{ and } \nabla_p^\infty w = x \\ \infty, & \nabla_p^\infty \alpha = [x] \text{ and } \nabla_p^\infty w \neq x \\ \infty, & \nabla_p^\infty \alpha \neq [x] \text{ and } \nabla_p^\infty w \neq x \end{cases}$$

Case I: If  $x = p$ , then  $\nabla_p^\infty w = x = p$  and it follows that  $w = (p, p, \dots)$  and  $\nabla_p^\infty \alpha = [x] = [p]$  implies  $\alpha = [(p, p, \dots)]$ . Note that

$$\lambda(\pi_j A_p^\infty \alpha)(\pi_j A_p^\infty w) = \lambda([p])(p) = 0,$$

consequently,

$$\bar{\lambda}(\alpha)(w) = \sup\{\lambda_{dis}(\nabla_p^\infty \alpha)(\nabla_p^\infty w), \lambda(\pi_j A_p^\infty \alpha)(\pi_j A_p^\infty w) : j \in I\} = 0.$$

Suppose  $x \neq p$ ,  $\nabla_p^\infty w = x$  and  $\nabla_p^\infty \alpha = [x]$ , it follows that  $w = x_i$  for  $i \in I$  and  $\alpha$  contains either a finite set in the form of  $U = \{x_{i_1}, x_{i_2}, \dots, x_{i_n}\}$  or an infinite set in the form of  $U = \{x_1, x_2, x_3, \dots\}$ .

If  $U = \{x_{i_1}, x_{i_2}, \dots, x_{i_n}\} \in \alpha$ , then  $\alpha$  contains a finite set and it follows that there exists some  $M_0 \in \alpha$  such that  $\alpha = [M_0]$ . If  $M_0 = \{x_k\}$ , a singleton set, then  $\alpha = [x_k]$  for some  $k \in \{i_1, i_2, \dots, i_n\}$ .

Let  $w = x_i$  and  $\alpha = [x_k]$ .

$$\lambda_{dis}(\nabla_p^\infty [x_k])(\nabla_p^\infty w) = \lambda_{dis}([x])(x) = 0.$$

For  $i \neq j = k$ , we get

$$\lambda(\pi_j A_p^\infty [x_k])(\pi_j A_p^\infty w) = \lambda([x])(p),$$

and for  $k \neq j = i$ , we have

$$\lambda(\pi_j A_p^\infty [x_k])(\pi_j A_p^\infty w) = \lambda([p])(x),$$

and for  $i \neq j \neq k$ , we have

$$\lambda(\pi_j A_p^\infty [x_k])(\pi_j A_p^\infty w) = \lambda([p])(p) = 0.$$

It follows that

$$\begin{aligned} \bar{\lambda}(\alpha)(w) &= \sup\{\lambda_{dis}(\nabla_p^\infty \alpha)(\nabla_p^\infty w), \lambda(\pi_j A_p^\infty \alpha)(\pi_j A_p^\infty w) : j \in I\} \\ &= \sup\{\lambda([x])(p), \lambda([p])(x)\} = \infty. \end{aligned}$$

By the assumption that  $\lambda([x])(p) = \infty$  or  $\lambda([p])(x) = \infty$ .

If  $\text{card } M_0 \geq 2$ , then  $[M_0] = [\{x_{i_1}, x_{i_2}, \dots, x_{i_m}\}]$  for  $m \leq n$ . Note that

$$\lambda_{dis}(\nabla_p^\infty [M_0])(\nabla_p^\infty w) = \lambda_{dis}([x])(x) = 0,$$

and for  $i \neq j$ ,

$$\lambda(\pi_j A_p^\infty [M_0])(\pi_j A_p^\infty w) = \lambda(\{[x, p]\})(p)$$

and for  $i = j$ ,

$$\lambda(\pi_j A_p^\infty [M_0])(\pi_j A_p^\infty w) = \lambda(\{x, p\})(x).$$

Note that  $\{x, p\} \subset [p]$  and  $\{x, p\} \subset [x]$ . Since  $\lambda$  is a convergence approach structure, we get  $\lambda([p])(x) \leq \lambda(\{x, p\})(x)$  and  $\lambda([x])(p) \leq \lambda(\{x, p\})(p)$ . By the assumption that  $\lambda([p])(x) = \infty$  (resp.  $\lambda([x])(p) = \infty$ ), we get  $\lambda(\{x, p\})(x) = \infty$  (resp.  $\lambda(\{x, p\})(p) = \infty$ ). It follows that

$$\begin{aligned} \bar{\lambda}(\alpha)(w) &= \sup\{\lambda_{dis}(\nabla_p^\infty [M_0])(\nabla_p^\infty w), \lambda(\pi_j A_p^\infty [M_0])(\pi_j A_p^\infty w) : j \in I\} \\ &= \sup\{0, \lambda(\{x, p\})(x), \lambda(\{x, p\})(p)\} = \infty. \end{aligned}$$

If  $U = \{x_1, x_2, \dots\}$ , then  $\alpha$  contains an infinite set  $M'_0$  such that  $\alpha = [M'_0]$ . Note that  $\lambda(\pi_j A_p^\infty [M'_0])(\pi_j A_p^\infty w) = \lambda(\{x, p\})(x)$  for  $i = j$ , otherwise  $\lambda(\pi_j A_p^\infty [M'_0])(\pi_j A_p^\infty w) = \lambda(\{x, p\})(p)$ , and it follows that

$$\begin{aligned} \bar{\lambda}(\alpha)(w) &= \sup\{\lambda_{dis}(\nabla_p^\infty [M'_0])(\nabla_p^\infty w), \lambda(\pi_j A_p^\infty [M'_0])(\pi_j A_p^\infty w) : j \in I\} \\ &= \sup\{0, \lambda(\{x, p\})(x), \lambda(\{x, p\})(p)\} = \infty \end{aligned}$$

since  $\lambda(\{x, p\})(x) = \infty$  (resp.  $\lambda(\{x, p\})(p) = \infty$ ).

Case II: Let  $p = \nabla_p^\infty w \neq x$  and  $\nabla_p^\infty \alpha = [x]$ . It follows that  $\lambda_{dis}(\nabla_p^\infty \alpha)(\nabla_p^\infty (p, p, \dots)) = \lambda_{dis}([x])(p) = \infty$  since  $\lambda_{dis}$  is a discrete convergence approach structure and  $x \neq p$ . It follows that

$$\begin{aligned} \bar{\lambda}(\alpha)(w) &= \sup\{\lambda_{dis}(\nabla_p^\infty \alpha)(\nabla_p^\infty w), \lambda(\pi_j A_p^\infty \alpha)(\pi_j A_p^\infty w) : j \in I\} \\ &= \sup\{\infty, \lambda(\pi_j A_p^\infty \alpha)(\pi_j A_p^\infty w) : j \in I\} = \infty. \end{aligned}$$

Case III: If  $y = \nabla_p^\infty w \neq x$  with  $x \neq y \neq p$  and  $\nabla_p^\infty \alpha \neq [x]$ , then  $\lambda_{dis}(\nabla_p^\infty \alpha)(\nabla_p^\infty w) = \lambda_{dis}(\nabla_p^\infty \alpha)(y) = \infty$  since  $\lambda_{dis}$  is a discrete convergence approach structure.

It follows that

$$\begin{aligned} \bar{\lambda}(\alpha)(w) &= \sup\{\lambda_{dis}(\nabla_p^\infty \alpha)(\nabla_p^\infty w), \lambda(\pi_j A_p^\infty \alpha)(\pi_j A_p^\infty w) : j \in I\} \\ &= \sup\{\infty, \lambda(\pi_j A_p^\infty \alpha)(\pi_j A_p^\infty w) : j \in I\} = \infty. \end{aligned}$$

Hence, for all  $\alpha \in F(\nabla_p^\infty X)$  and  $v \in \nabla_p^\infty X$ , we get

$$\bar{\lambda}(\alpha) = \begin{cases} \theta_{\{v\}}, & \alpha = [v] \\ \infty, & \alpha \neq [v] \end{cases}$$

i.e. by Lemma 2.3 (iii),  $\bar{\lambda}(\alpha)$  is a discrete convergence approach structure on  $\nabla_p^\infty X$ . By Definition 3.4 (i),  $\{p\}$  is closed. □

**Theorem 3.8** *Let  $(X, \lambda)$  be a convergence approach space and  $M \subset X$ .  $M$  is strongly closed if and only if the following conditions hold.*

- (i) For any  $x \in X$  with  $x \notin M$  and for any  $y \in M$ ,  $\lambda([x])(y) = \infty$

(ii) For any  $x \in X$ ,  $\beta \in F(X)$  with  $x \notin M$  and  $\beta \cup [M]$  is a proper filter, then  $\lambda(\beta)(x) = \infty$ .

**Proof** Suppose  $M$  is strongly closed,  $x \in X$  with  $x \notin M$  and  $y \in M$ . Note that  $q(x) = x$ ,  $q(y) = *$  and  $x \neq *$ . Since  $(X/M, \lambda')$  is  $T_1$  at  $*$ , where  $\lambda'$  is the quotient convergence approach structure on  $X/M$ , by Theorem 3.6,  $\lambda'([x])(*) = \infty$  and  $\lambda'([*])(x) = \infty$ . By Lemmas 2.3 and 2.4,

$$\begin{aligned} \infty = \lambda'([x])(*) &= \inf_{y \in q^{-1}(*)=M} \{\lambda(\beta)(y) : \beta \in F(X), q\beta \subset [x]\} \\ &= \inf_{y \in M} \{\lambda(\beta)(y); \beta \subset [x]\}, \end{aligned}$$

and consequently,  $\lambda(\beta)(y) = \infty$  for all  $y \in M$ ,  $x \notin M$  and  $\beta \subset [x]$ . In particular,  $\lambda([x])(y) = \infty$  for all  $y \in M$  and  $x \notin M$ .

Suppose  $x \in X$  with  $x \notin M$  and  $\beta \cup [M]$  is proper for  $\beta \in F(X)$ . Note that  $q(x) = x \notin q(M) = *$ . By Lemmas 2.3 and 2.4,

$$\infty = \lambda'([*])(x) = \inf_{x \in q^{-1}(x)=x} \{\lambda(\beta)(x) : \beta \in F(X), q\beta \subset [*]\}$$

$= \inf\{\lambda(\beta)(x); \beta \in F(X) \text{ and } \beta \cup [M] \text{ is proper}\}$ . It follows that  $\lambda(\beta)(x) = \infty$  for all  $\beta \in F(X)$  with  $\beta \cup [M]$  is proper.

Conversely, suppose the conditions hold and  $x \in X/M$  with  $x \neq *$ . Note that  $x \notin M$ , and by Lemmas 2.3 and 2.4,

$$\begin{aligned} \lambda'([x])(*) &= \inf_{y \in q^{-1}(*)=M} \{\lambda(\beta)(y) : \beta \in F(X), q\beta \subset [x]\} \\ &= \inf_{y \in M} \{\lambda(\beta)(y); \beta \in F(X), \beta \subset [x]\} \end{aligned}$$

Since  $\beta \subset [x]$  and  $\lambda([x])(y) = \infty$  for all  $y \in M$ , it follows that  $\lambda(\beta)(y) = \infty$  for all  $\beta \in F(X)$  with  $\beta \subset [x]$  and  $y \in M$ . Hence,  $\lambda'([x])(*) = \infty$ .

By Lemmas 2.3 and 2.4,

$$\lambda'([*])(x) = \inf_{x \in q^{-1}(x)=x} \{\lambda(\beta)(x) : \beta \in F(X), q\beta \subset [*]\}$$

$= \inf\{\lambda(\beta)(x); \beta \in F(X) \text{ and } \beta \cup [M] \text{ is proper}\} = \infty$  by assumption. Hence, by Theorem 3.6,  $(X/M, \lambda')$  is  $T_1$  at  $*$ , and by Definition 3.4 (iii),  $M$  is strongly closed. □

**Theorem 3.9** Let  $(X, \lambda)$  be a convergence approach space.  $M \subset X$  is closed iff for any  $x \in X$  with  $x \notin M$  and  $y \in M$ ,  $\lambda([x])(y) = \infty$  or for any  $x \in X$ ,  $\beta \in F(X)$  with  $x \notin M$  and  $\beta \cup [M]$  is a proper filter, then  $\lambda(\beta)(x) = \infty$ .

**Proof** The proof is similar to the proof of Theorem 3.8 by using Definition 3.4 (ii) and Theorem 3.4. □

**Theorem 3.10** (1) Let  $f : (X, \lambda) \rightarrow (Y, \lambda')$  be in **CApp**. If  $A \subset Y$  is strongly closed, so is  $f^{-1}(A) \subset X$ .

(2) Let  $(X, \lambda)$  be a convergence approach space. If  $M \subset N$  and  $N \subset X$  are strongly closed, so is  $M \subset X$ .

**Proof** (1) (i) Suppose for any  $x \in X$ ,  $x \notin f^{-1}(A)$  and  $b \in f^{-1}(A)$ . It follows that  $f(x) \notin A$  and  $f(b) \in A$ . Since  $A$  is strongly closed,  $\lambda'([f(x)])(f(b)) = \infty$ . Since  $f$  is contraction map,  $\lambda'([f(x)])(f(b)) \leq \lambda([x])(b)$ , and consequently,  $\lambda([x])(b) = \infty$ .

(ii) Suppose  $x \in X$  with  $x \notin f^{-1}(A)$  and  $\beta \in F(X)$  with  $\beta \cup [f^{-1}(A)]$  a proper filter. Note that, by Remark 3.5,

$$f(\beta) \cup [A] \subset f(\beta) \cup [ff^{-1}(A)] \subset f(\beta \cup [f^{-1}(A)]).$$

Since  $\beta \cup [f^{-1}(A)]$  is proper,  $f(\beta \cup [f^{-1}(A)])$  is proper [otherwise,  $\exists U \in \beta$  such that  $f(U \cap f^{-1}(A)) = \emptyset$  and consequently,  $U \cap f^{-1}(A) = \emptyset$ , a contradiction], and consequently,  $f(\beta) \cup [A]$  is proper. Since  $A$  is strongly closed, by Theorem 3.8,  $\lambda'(f(\beta))(f(x)) = \infty$ . Since  $f$  is a contraction map,  $\lambda'(f(\beta))(f(x)) \leq \lambda(\beta)(x)$ , it follows that  $\lambda(\beta)(x) = \infty$ . Thus, by Theorem 3.8,  $f^{-1}(A)$  is strongly closed.

(2) Let  $\lambda_N$  be the subspace structure on  $N$  induced by the inclusion map  $i : N \rightarrow (X, \lambda)$ . Suppose  $\lambda_M$  is the subconvergence approach structure on  $M$  which is induced by the inclusion map  $i : M \rightarrow (N, \lambda_N)$ .

(i) Let  $a \in X$ ,  $a \notin M$  and  $b \in M$ . If  $a \notin N$ ,  $\lambda([a])(b) = \infty$  since  $N \subset X$  is strongly closed. Suppose  $a \in N$  and it follows that  $\lambda_M([a])(b) = \lambda_N(i([a]))(i(b)) = \lambda_N([a])(b) = \lambda([a])(b)$ . Since  $M \subset N$  is strongly closed,  $\lambda_M([a])(b) = \infty$  for  $a \in N$  and  $a \notin M$ , and consequently,  $\lambda([a])(b) = \infty$ .

(ii) Let  $a \in X$  with  $a \notin M$ ,  $\beta \cup [M]$  is a proper filter with  $\beta \in F(X)$ . Suppose  $a \notin N$ . Since  $\beta \cup [M]$  is proper filter and  $M \subset N$ , so is  $\beta \cup [N]$ , and consequently,  $\lambda(\beta)(a) = \infty$  since  $N \subset X$  is strongly closed. Suppose  $a \in N$ , it follows that  $\lambda_M(\beta)(a) = \lambda_N(i(\beta))(i(a)) = \lambda_N(\beta)(a) = \lambda(\beta)(a)$ . Since  $M \subset N$  is strongly closed,  $\lambda_M(\beta)(a) = \infty$ , and consequently,  $\lambda(\beta)(a) = \infty$ . Thus, by Theorem 3.8,  $M \subset X$  is strongly closed.  $\square$

**Theorem 3.11** (1) Let  $f : (X, \lambda) \rightarrow (Y, \lambda')$  be in **CApp**. If  $A \subset Y$  is closed, so is  $f^{-1}(A) \subset X$ .

(2) Let  $(X, \lambda)$  be a convergence approach space. If  $M \subset N$  and  $N \subset X$  are closed, so is  $M \subset X$ .

**Proof** It is analogous to the proof of Theorem 3.10 by using Theorem 3.9 instead of Theorem 3.8.  $\square$

#### 4. Closure Operators

Let  $\mathcal{E}$  be a set based topological category,  $X$  be an object in  $\mathcal{E}$  and  $C$  be the closure operator of  $\mathcal{E}$  in sense of [12, 13].

**Definition 4.1** Let  $(X, \lambda)$  be a convergence approach space and  $M \subset X$ .

(i)  $cl^{CApp}(M) = \cap\{U \subset X : M \subset U \text{ and } U \text{ is closed}\}$  is called the closure of  $M$ .

(ii)  $scl^{CApp}(M) = \cap\{U \subset X : M \subset U \text{ and } U \text{ is strongly closed}\}$  is called the strong closure of  $M$ .

**Theorem 4.2**  $cl^{CApp}$  and  $scl^{CApp}$  are (weakly) hereditary, productive and idempotent closure operators of **CApp**.

**Proof** Combine Theorems 3.10 and 3.11, Definition 4.1, and Exercise 2.D, Theorems 2.3 and 2.4 and Proposition 2.5 of [13].  $\square$

Let  $\mathcal{E}$  be a topological category and  $C$  be a closure operator of  $\mathcal{E}$ .

(i)  $\mathcal{E}_{0C} = \{X \in \mathcal{E} : x \in C(\{y\}) \text{ and } y \in C(\{x\}) \implies x = y \text{ with } x, y \in X\}$  [13].

(ii)  $\mathcal{E}_{1C} = \{X \in \mathcal{E} : C(\{x\}) = \{x\} \text{ for each } x \in X\}$  [13].

(iii)  $\mathcal{E}_{2C} = \{X \in \mathcal{E} : C(\Delta) = \Delta, \text{ the diagonal}\}$  [13].



**Remark 4.3** For  $\mathcal{E} = \mathbf{Top}$ , and  $C = K$ , the ordinary closure,  $\mathbf{Top}_{iC}$  reduce to the class of  $T_i$  spaces for  $i = 0, 1, 2$ .

**Theorem 4.4**  $(X, \lambda) \in \mathbf{CApp}_{0cl}$  iff for any  $x, y \in X$  with  $x \neq y$ ,  $\exists M \subset X$  closed subset such that  $x \notin M$  and  $y \in M$  or  $\exists N \subset X$  closed subset such that  $x \in N$  and  $y \notin N$ .

**Proof** Suppose  $(X, \lambda) \in \mathbf{CApp}_{0cl}$  and  $x, y \in X$  with  $x \neq y$ . It follows that  $x \notin cl(\{y\})$  or  $y \notin cl(\{x\})$ . Suppose  $x \notin cl(\{y\})$ . It follows that  $\exists M \subset X$  closed such that  $y \in M$  and  $x \notin M$ . Similarly, if  $y \notin cl(\{x\})$ . It follows that  $\exists N \subset X$  closed such that  $x \in N$  and  $y \notin N$ .

Conversely, suppose for any  $x, y \in X$  with  $x \neq y$ ,  $\exists M \subset X$  closed subset such that  $x \notin M$  and  $y \in M$  or  $\exists N \subset X$  closed subset such that  $x \in N$  and  $y \notin N$ . If the first case holds, then  $x \notin cl(\{y\})$ . If the second case holds, then  $y \notin cl(\{x\})$ . Hence,  $(X, \lambda) \in \mathbf{CApp}_{0cl}$ . □

**Theorem 4.5**  $(X, \lambda) \in \mathbf{CApp}_{0scl}$  iff for any  $x, y \in X$  with  $x \neq y$ ,  $\exists M \subset X$  strongly closed subset such that  $x \notin M$  and  $y \in M$  or  $\exists N \subset X$  strongly closed subset such that  $x \in N$  and  $y \notin N$ .

**Proof** It is similar to the proof of Theorem 4.4. □

**Theorem 4.6**  $(X, \lambda) \in \mathbf{CApp}_{1cl}$  if and only if for any  $x, y \in X$  with  $x \neq y$ ,  $\lambda([x])(y) = \infty$  or  $\lambda([y])(x) = \infty$ .

**Proof** Suppose  $(X, \lambda) \in \mathbf{CApp}_{1cl}$  and  $x, y \in X$  with  $x \neq y$ . We have  $cl\{x\} = \{x\}$  for all  $x \in X$ , i.e.,  $\{x\}$  is closed. By Theorem 3.7, for any  $y \in X$  with  $y \neq x$ ,  $\lambda([x])(y) = \infty$  or  $\lambda([y])(x) = \infty$ .

Conversely, suppose for any  $x, y \in X$  with  $x \neq y$ ,  $\lambda([x])(y) = \infty$  or  $\lambda([y])(x) = \infty$ . By Theorem 3.7,  $\{x\}$  is closed, i.e.  $cl\{x\} = \{x\}$ , and consequently,  $(X, \lambda) \in \mathbf{CApp}_{1cl}$ . □

**Theorem 4.7**  $(X, \lambda) \in \mathbf{CApp}_{1scl}$  if and only if for any  $x, y \in X$  with  $x \neq y$ ,  $\lambda([x])(y) = \infty = \lambda([y])(x)$ .

**Proof** It is similar to the proof of Theorem 4.6. □

**Theorem 4.8**  $(X, \lambda) \in \mathbf{CApp}_{2scl}$  if and only if the following conditions hold.

(i) For any  $x, y \in X$  with  $x \neq y$ ,  $\lambda([x])(y) = \infty = \lambda([y])(x)$ .

(ii) For any  $x, y \in X$  with  $x \neq y$  and for any  $\alpha, \beta \in F(X)$  if  $\alpha \cup \beta$  is proper, then  $\lambda(\alpha)(x) = \infty$  or  $\lambda(\beta)(y) = \infty$ .

**Proof** Suppose  $(X, \lambda) \in \mathbf{CApp}_{2scl}$  and  $x, y \in X$  with  $x \neq y$ . Note that  $(x, y) \notin \Delta$ . Since  $\Delta$  is strongly closed, by Theorem 3.8, in particular,  $\lambda^2([(x, y)])(y, y) = \infty = \lambda^2([(x, y)])(x, x)$  where  $\lambda^2$  is the product convergence approach structure on  $X^2$ . By Lemma 2.3 (i),  $\infty = \lambda^2([(x, y)])(y, y) = \lambda([x])(y)$  and  $\infty = \lambda^2([(x, y)])(x, x) = \lambda([y])(x)$ .

Suppose  $x, y \in X$  with  $x \neq y$ ,  $\alpha, \beta \in F(X)$  and  $\alpha \cup \beta$  is proper. Let  $\sigma = \pi_1^{-1}\alpha \cup \pi_2^{-1}\beta$ . Note that  $\sigma \in F(X^2)$ ,  $\pi_1\sigma = \alpha$  and  $\pi_2\sigma = \beta$  and  $\sigma \cup [\Delta]$  is proper. Indeed, if  $W \in \sigma$ , then there exists  $U \in \alpha$  and  $V \in \beta$  such that  $W \supset U \times V$ . Since  $\alpha \cup \beta$  is proper,  $U \cap V \neq \emptyset$ . It follows that  $(U \times V) \cap \Delta \neq \emptyset$  and

$W \cap \Delta \neq \emptyset$ . Thus,  $\sigma \cup [\Delta]$  is a proper filter. Since  $\Delta$  is strongly closed, by Theorem 3.8,  $\lambda^2(\sigma)(x, y) = \infty$  and by Lemma 2.3,  $\lambda(\alpha)(x) = \infty$  or  $\lambda(\beta)(y) = \infty$ .

Conversely, suppose that the conditions hold and for any  $(x, y) \in X^2$  with  $(x, y) \notin \Delta$ . It follows that  $x \neq y$  and by assumption,  $\lambda([x])(y) = \infty = \lambda([y])(x)$ , and for any  $(a, a) \in \Delta$ ,  $\lambda^2([(x, y)])(a, a) = \sup\{\lambda([x])(a), \lambda([y])(a)\} = \infty$ .

Suppose  $(x, y) \in X^2$  with  $(x, y) \notin \Delta$ ,  $\sigma \in F(X^2)$  such that  $\sigma \cup [\Delta]$  is a proper filter. Let  $\sigma_0 = \pi_1^{-1}\pi_1\sigma \cup \pi_2^{-1}\pi_2\sigma$ . By Remark 3.5 (iii),  $\sigma_0 \subset \sigma$ ,  $\pi_i\sigma_0 = \pi_i\sigma$  for  $i = 1, 2$  and  $\sigma_0 \cup [\Delta]$  is proper. It follows that  $x, y \in X$  with  $x \neq y$ ,  $\pi_i\sigma_0 = \pi_i\sigma \in F(X)$  for  $i = 1, 2$  and  $\pi_1\sigma_0 \cup \pi_2\sigma_0$  is proper since  $\sigma_0 \cup [\Delta]$  is proper. By assumption,  $\lambda(\pi_1\sigma)(x) = \infty$  or  $\lambda(\pi_2\sigma)(y) = \infty$ . Thus, by Theorem 3.8,  $\Delta$  is strongly closed, i.e.  $scl(\Delta) = \Delta$ , i.e.  $(X, \lambda) \in \mathbf{CApp}_{2scl}$ .  $\square$

**Theorem 4.9**  $(X, \lambda) \in \mathbf{CApp}_{2cl}$  iff any of the following conditions hold.

- (i) For any  $x, y \in X$  with  $x \neq y$ ,  $\lambda([x])(y) = \infty$  or  $\lambda([y])(x) = \infty$ .
- (ii) For any  $x, y \in X$  with  $x \neq y$  and for any  $\alpha, \beta \in F(X)$  if  $\alpha \cup \beta$  is proper, then  $\lambda(\alpha)(x) = \infty$  or  $\lambda(\beta)(y) = \infty$ .

**Proof** The proof is similar to the proof of Theorem 4.8 by using Theorem 3.9.  $\square$

Let  $\mathcal{U} : \mathcal{E} \rightarrow \mathbf{Set}$  be a topological functor,  $B$  be an object in  $\mathcal{E}$  with  $\mathcal{U}(B) = X$ .

(i) If the initial lift of the  $\mathcal{U}$ -source  $\{A : X^2 \vee_{\Delta} X^2 \rightarrow \mathcal{U}(B^3) = X^3$  and  $\nabla : X^2 \vee_{\Delta} X^2 \rightarrow \mathcal{UD}(X^2) = X^2\}$  is discrete, then  $X$  is called  $\overline{T_0}$  [2].

(ii) If the initial lift of the  $\mathcal{U}$ -source  $\{S : X^2 \vee_{\Delta} X^2 \rightarrow \mathcal{U}(B^3) = X^3$  and  $\nabla : X^2 \vee_{\Delta} X^2 \rightarrow \mathcal{UD}(X^2) = X^2\}$  is discrete, then  $X$  is called  $T_1$  [2], where  $A$ ,  $\nabla$  and  $S$  are principal axis map, folding map and skewed axis map respectively defined in [2].

**Theorem 4.10** (i) A convergence approach space  $(X, \lambda)$  is  $\overline{T_0}$  iff for all  $x, y \in X$  with  $x \neq y$ ,  $\lambda([x])(y) = \infty$  or  $\lambda([y])(x) = \infty$ .

(ii) A convergence approach space  $(X, \lambda)$  is  $T_1$  iff for any  $x, y \in X$  with  $x \neq y$ ,  $\lambda([x])(y) = \infty = \lambda([y])(x)$ .

**Proof** (i) It is given in [25].

(ii) The proof is similar to the part (i) by using skewed axis map  $S$  instead of principal axis map  $A$ .  $\square$

**Remark 4.11**  $\overline{T_0}\mathbf{CApp}$  (resp.  $T_1\mathbf{CApp}$ ) is the full subcategory of  $\mathbf{CApp}$  whose objects consist of  $\overline{T_0}$  convergence approach spaces (resp.  $T_1$  convergence approach spaces).

**Theorem 4.12** Let  $(X, \lambda)$  be a convergence approach space. The following are equivalent.

- (i)  $(X, \lambda) \in \mathbf{CApp}_{1cl}$ .
- (ii) For any  $x, y \in X$  with  $x \neq y$ ,  $\lambda([x])(y) = \infty$  or  $\lambda([y])(x) = \infty$ .
- (iii)  $(X, \lambda) \in \overline{T_0}\mathbf{CApp}$

**Proof** It follows from Theorems 4.6 and 4.10 (i). □

**Theorem 4.13** Let  $(X, \lambda)$  be a convergence approach space. The following are equivalent.

- (i)  $(X, \lambda) \in \mathbf{CApp}_{1scl}$ .
- (ii) For any  $x, y \in X$  with  $x \neq y$ ,  $\lambda([x])(y) = \infty = \lambda([y])(x)$ .
- (iii)  $(X, \lambda) \in T_1\mathbf{CApp}$

**Proof** It follows from Theorems 4.7 and 4.10 (ii). □

**Theorem 4.14** Each of subcategories  $\mathbf{CApp}_{kC}$  for  $k = 0, 1, 2$  and  $C = cl$  or  $scl$  are epireflective subcategory of  $\mathbf{CApp}$ .

**Proof** It is quite easy to see that these subcategories are full and isomorphism-closed. We just need to prove that these are closed under subspaces and products.

(i) Suppose  $(X, \lambda) \in \mathbf{CApp}_{1cl}$  and  $A \subset X$ , and let  $\lambda_A$  be the subconvergence approach structure on  $A$  induced by the inclusion map  $i : A \rightarrow (X, \lambda)$ . Suppose  $\lambda_A([x])(y) < \infty$  and  $\lambda_A([y])(x) < \infty$  for some  $x, y \in A$  with  $x \neq y$ . By Lemma 2.3 (i),  $\lambda_A([x])(y) = \lambda(i([x]))(i(y)) = \lambda([x])(y)$  and  $\lambda_A([y])(x) = \lambda([y])(x)$ . Since  $\lambda_A([x])(y) < \infty$  and  $\lambda_A([y])(x) < \infty$ ,  $\lambda([x])(y) < \infty$  and  $\lambda([y])(x) < \infty$  for some  $x, y \in X$  with  $x \neq y$ , a contradiction. Thus,  $\lambda_A([x])(y) = \infty$  or  $\lambda_A([y])(x) = \infty$ . By Theorem 4.6,  $(A, \lambda_A) \in \mathbf{CApp}_{1cl}$ .

Let  $(X_j, \lambda_j) \in \mathbf{CApp}_{1cl}$  for all  $j \in I$  and  $X = \prod_{j \in I} X_j$ . We show that  $(X, \lambda) \in \mathbf{CApp}_{1cl}$ , where  $\lambda$  is the product convergence approach structure on  $X$ . Suppose  $x = (x_1, x_2, \dots)$ ,  $y = (y_1, y_2, \dots) \in X$  and  $x \neq y$ . It follows that  $\exists n \in I$  such that  $x_n \neq y_n$ . Since  $(X_n, \lambda_n) \in \mathbf{CApp}_{1cl}$ , by Theorem 4.6,  $\lambda_n([x_n])(y_n) = \infty$  or  $\lambda_n([y_n])(x_n) = \infty$ . By Lemma 2.3 (i),  $\lambda([x])(y) = \sup_{j \in I} \lambda_j([x_j])(y_j) = \infty$  or  $\lambda([y])(x) = \sup_{j \in I} \lambda_j([y_j])(x_j) = \infty$ , and by Theorem 4.6,  $(X, \lambda) \in \mathbf{CApp}_{1cl}$ . Hence,  $\mathbf{CApp}_{1cl}$  is an epireflective subcategory of  $\mathbf{CApp}$ .

(ii) Suppose  $(X, \lambda) \in \mathbf{CApp}_{2scl}$ ,  $A \subset X$  and  $x, y \in A$  with  $x \neq y$ . By argument above  $\lambda_A([x])(y) = \infty = \lambda_A([y])(x)$ , where  $\lambda_A$  is the subconvergence approach structure on  $A$ . Suppose for any  $x, y \in A$  with  $x \neq y$  and  $\alpha \cup \beta$  is proper for any  $\alpha, \beta \in F(A)$ . Note that  $x, y \in X$ ,  $i(\alpha), i(\beta) \in F(X)$  and  $i(\alpha \cup \beta) = i(\alpha) \cup i(\beta)$  is proper. Since  $(X, \lambda) \in \mathbf{CApp}_{2scl}$ , by Theorem 4.8,  $\lambda(i(\alpha))(x) = \infty$  or  $\lambda(i(\beta))(y) = \infty$ . By Lemma 2.3 (i),  $\lambda_A(\alpha)(x) = \lambda(i(\alpha))(x) = \infty$  or  $\lambda_A(\beta)(y) = \lambda(i(\beta))(y) = \infty$  and by Theorem 4.8,  $(A, \lambda_A) \in \mathbf{CApp}_{2scl}$ .

Suppose for  $i \in I$ ,  $(X_i, \lambda_i) \in \mathbf{CApp}_{2scl}$  and  $X = \prod_{i \in I} X_i$ . We show that  $(X, \lambda) \in \mathbf{CApp}_{2scl}$ . By the same argument used in (i),  $\lambda([x])(y) = \infty = \lambda([y])(x)$ . Suppose for any  $x, y \in X$  with  $x \neq y$  and  $\alpha \cup \beta$  is proper for any  $\alpha, \beta \in F(X)$ . Let  $\alpha_0 = \{U \subset X : U \supset \prod_{i \in I} A_i, A_i \in \pi_i \alpha\}$  and  $\beta_0 = \{V \subset X : V \supset \prod_{i \in I} B_i, B_i \in \pi_i \beta\}$ . Note that  $\pi_i \alpha_0 = \pi_i \alpha$  and  $\pi_i \beta_0 = \pi_i \beta$  for all  $i \in I$ . Indeed, suppose  $A \in \pi_i \alpha_0$ . It follows that  $\exists W \in \alpha_0$  such that  $W \supset \prod_{i \in I} A_i$ ,  $A_i \in \pi_i \alpha$  and  $A \supset \pi_i W \supset \pi_i \prod_{i \in I} A_i = A_i \in \pi_i \alpha$ . Therefore,  $A \in \pi_i \alpha$ . Now, let  $A_i \in \pi_i \alpha$ . It follows that  $\prod_{i \in I} A_i \in \alpha_0$  and  $\pi_i(\prod_{i \in I} A_i) = A_i \in \pi_i \alpha_0$ . Thus,  $A_i \in \pi_i \alpha_0$ . Hence,  $\pi_i \alpha_0 = \pi_i \alpha$  for all  $i \in I$ . Since  $\alpha \cup \beta$  is proper,  $\pi_i(\alpha \cup \beta) = \pi_i(\alpha_0 \cup \beta_0)$  are proper for all  $i \in I$  and it follows that  $\alpha_0 \cup \beta_0$  is proper. If not,  $\exists U \in \alpha_0$  and  $\exists V \in \beta_0$  such that  $\emptyset = U \cap V \supset \prod_{i \in I} A_i \cap \prod_{i \in I} B_i = \prod_{i \in I} (A_i \cap B_i)$  implies  $\prod_{i \in I} (A_i \cap B_i) = \emptyset$ , i.e.  $\exists k \in I$  such

that  $A_k \cap B_k = \emptyset$  and therefore,  $\pi_k \alpha \cup \pi_k \beta$  is improper. It follows that  $\pi_k \alpha_0 \cup \pi_k \beta_0 = \pi_k \alpha \cup \pi_k \beta \subset \pi_k(\alpha \cup \beta)$  is improper.  $(X_i, \lambda_i) \in \mathbf{CApp}_{2scl}$  implies  $\lambda_i(\pi_i \alpha_0)(x_i) = \infty$  or  $\lambda_i(\pi_i \beta_0)(x_i) = \infty$ . By Lemma 2.3 (i),  $\lambda(\alpha_0)(x) = \sup_{i \in I} \lambda_i(\pi_i \alpha_0)(\pi_i(x)) = \infty$  or  $\lambda(\beta_0)(y) = \sup_{i \in I} \lambda_i(\pi_i \beta_0)(\pi_i(y)) = \infty$ , and consequently,  $\lambda(\alpha)(x) = \infty$  or  $\lambda(\beta)(y) = \infty$ . Therefore, by Theorem 4.8,  $(X, \lambda) \in \mathbf{CApp}_{2scl}$ . Hence,  $\mathbf{CApp}_{2scl}$  is an epi-reflective subcategory of  $\mathbf{CApp}$ .

The proof for other cases is similar.

□

**Remark 4.15** (i) In  $\mathbf{Top}$  (category of topological spaces and continuous maps),  $\mathbf{Top}_{2scl} = \mathbf{Top}_{2cl} \subset \mathbf{Top}_{1scl} = \mathbf{Top}_{1cl} \subset \mathbf{Top}_{0scl} = \mathbf{Top}_{0cl}$ .

(ii) In  $\mathbf{Born}$  (category of bornological spaces and bounded maps), by Lemma 2.11 of [6],  $\mathbf{Born}_{kcl} \subset \mathbf{Born}_{kscl}$  for  $k = 0, 1, 2$ .

(iii) In  $\mathbf{Prord}$  (category of preordered spaces and order preserving maps), by Theorem 4.5 of [9],  $\mathbf{Prord}_{kscl} = \mathbf{Prord}_{kcl} \subset \mathbf{Prord}_{0scl} = \mathbf{Prord}_{0cl}$  for  $k = 1, 2$ .

(iv) In  $\mathbf{FCO}$  (category of filter convergence spaces and filter convergence maps), by Theorem 2.9 of [6],  $\mathbf{FCO}_{2scl} \subset \mathbf{FCO}_{2cl} = \mathbf{FCO}_{1scl} = \mathbf{FCO}_{1cl} \subset \mathbf{FCO}_{0scl} = \mathbf{FCO}_{0cl}$ .

(v) In  $\mathbf{ConFCO}$  (category of constant filter convergence spaces and filter convergence maps), by Theorems 4.3, 4.4 and 4.5 of [14],  $\mathbf{ConFCO}_{2scl} = \mathbf{ConFCO}_{2cl} \subset \mathbf{ConFCO}_{kscl} = \mathbf{ConFCO}_{kcl}$  for  $k = 0, 1$ .

(vi) By Theorems 4.8, 4.9, 4.12 and 4.13, we have  $\mathbf{CApp}_{2scl} \subset \mathbf{CApp}_{1scl} \subset \mathbf{CApp}_{0scl}$  and  $\mathbf{CApp}_{2cl} \subset \mathbf{CApp}_{1cl} \subset \mathbf{CApp}_{0cl}$ .

### 5. Connected convergence approach spaces

**Definition 5.1** (cf. [8]) Let  $\mathcal{E}$  be a set based topological category,  $X$  be an object in  $\mathcal{E}$  and  $M$  be a nonempty subset of  $X$ .

(i)  $M$  is open iff  $M^c$ , the complement of  $M$ , is closed in  $X$ .

(ii)  $M$  is strongly open iff  $M^c$  is strongly closed in  $X$ .

**Definition 5.2** (cf. [8]) Let  $U : \mathcal{E} \rightarrow \mathbf{Set}$  be a topological functor,  $B$  be an object of  $\mathcal{E}$  with  $U(B) = X$ .

(i)  $B$  is connected iff the only subsets of  $B$  both strongly open and strongly closed are  $B$  and  $\emptyset$ .

(ii)  $B$  is strongly connected iff the only subsets of  $B$  both open and closed are  $B$  and  $\emptyset$ .

**Remark 5.3** For  $\mathbf{Top}$ , the notion of strongly connectedness coincides with the usual connectedness. Moreover, if a topological space is  $T_1$ , then the notions of connectedness and strongly connectedness coincide.

**Theorem 5.4** A convergence approach space  $(X, \lambda)$  is connected iff there exists a proper subset  $M$  of  $X$  such that either the statement (I) or (II) holds, where

(I)  $\lambda([x])(y) < \infty$  for some  $x \in X$ ,  $x \notin M$  and  $y \in M$  or  $\lambda(\beta)(x) < \infty$  for some  $\beta \in F(X)$  with  $\beta \cup [M]$  is proper and  $x \notin M$ .

(II)  $\lambda([x])(y) < \infty$  for some  $x \in M$  and  $y \in M$  or  $\lambda(\beta)(x) < \infty$  for some  $\beta \in F(X)$  with  $\beta \cup [M^c]$  is proper and  $x \in M$ .

**Proof** It follows from Definition 5.2 and Theorem 3.8. □

**Theorem 5.5** A convergence approach space  $(X, \lambda)$  is strongly connected iff there exists a proper subset  $M$  of  $X$  such that either the statement (I) or (II) holds, where

(I)  $\lambda([x])(y) < \infty$  for some  $x \in X$ ,  $x \notin M$  and  $y \in M$  and  $\lambda(\beta)(x) < \infty$  for some  $\beta \in F(X)$  with  $\beta \cup [M]$  is proper and  $x \notin M$ .

(II)  $\lambda([x])(y) < \infty$  for some  $x \in M$  and  $y \in M$  and  $\lambda(\beta)(x) < \infty$  for some  $\beta \in F(X)$  with  $\beta \cup [M^c]$  is proper and  $x \in M$ .

**Proof** It follows from Definition 5.2 and Theorem 3.9. □

**Lemma 5.6** Let  $(X, \lambda)$  be a convergence approach space. If  $(X, \lambda)$  is strongly connected, then  $(X, \lambda)$  is connected.

**Proof** It follows from Theorems 5.4 and 5.5. □

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