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On rings whose Jacobson radical coincides with upper nilradical

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Abstract: We call a ring R is JN if whose Jacobson radical coincides with upper nilradical, and R is right SR if each element $r \in R$ can be written as $r = s + r$ where s is an element from the right socle and r is a regular element of R. SR rings is a class of special subrings of JN rings, which is the extension of soclean rings. We give their some characterizations and examples, and investigate the relationship between JN rings, SR rings and related rings, respectively.

Key words: Radical, JN ring, right SR ring, soclean ring

1. Introduction

Throughout this paper, all rings are associative with identity unless otherwise stated. For a ring R, $J(R)$, $Nil(R)$, $U(R)$, $Id(R)$, $P(R)$, $L(R)$ and $Nil^*(R)$ denote the Jacobson radical, the set of all nilpotent elements, the group of all units, the set of all idempotents, prime radical, Levitzki radical and upper nilradical of R, respectively. It is well known that $P(R) \subseteq L(R) \subseteq Nil^*(R) \subseteq Nil(R) \cap J(R) \subseteq Nil(R)$, and the equality occurs provided R is either commutative or left (respectively, right) Artinian. Peoples are interested in such rings that their radical is equal to the set of all nilpotent elements. For example, a ring R is called 2-primal in [4] if $P(R) = Nil(R)$; a ring R is called weakly 2-primal in [13] if $L(R) = Nil(R)$; a ring R is called NI in [30] if $Nil^*(R) = Nil(R)$ and a ring R is called NJ ring in [25] if $J(R) = Nil(R)$. In the past few decades, these rings have been investigated by many authors [5, 24, 35, 38]. Motivated by these developments, this paper investigates the rings that their Jacobson radical is equal to the upper nilradical, and we call a ring R is JN rings if $J(R) = Nil^*(R)$. We will give some characterizations and examples of JN rings and study the relationship between JN rings, SR rings and soclean rings.

2. JN rings and related rings

Recall that a ring R is called semiprimitive if $J(R) = 0$, a ring R is called J-reduced in [10] if $Nil(R) \subseteq J(R)$, and a ring R is said to be UU in [14] if $1 + Nil(R) = U(R)$. Clearly, every semiprimitive ring is JN. UU rings are JN by [14, Proposition 2.6]. The following is obvious.

Proposition 2.1 *A ring R is NJ if and only if R is J-reduced and JN.*

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Example 2.2 *The following three items are ture:*

(1) There is a JN ring that is not J-reduced ring, hence not NJ ring.

(2) There is a JN ring that is not UU ring.

(3) There is a JN ring that is not semiprimitive ring.

Proof (1) Let F be a field and $R = M_n(F)$ with $n \geq 2$. Since $J(R) = M_n(J(F)) = 0$, we have R is semiprimitive and hence JN. However, R cannot be a J-reduced ring because $Nil(R) \neq 0$.

(2) Assume that $R = \mathbb{Z}$. Then R is JN since $J(R) = Nil(R) = 0$. However, R is not a UU ring by $U(R) = \{1, -1\}$.

(3) Assume that $R = \mathbb{Z}_4$. Then $J(R) = Nil(R) = \{0, 2\}$, so R is JN but not semiprimitive. □

Imitating [16], a ring R called a weakly UU ring if $U(R) = Nil(R) \pm 1$ (denoted as WUU ring). Clearly, UU rings are themselves WUU rings but the converse is not ture. For instance, the ring $\mathbb{Z}$ is WUU but not UU.

Proposition 2.3 *Every WUU ring is JN, but the converse is not ture.*

Proof If R is WUU, then R is JN by [16, Proposition 2.6]. Assume that $R = T_n(\mathbb{Z}_3)$. Then $J(R) = Nil(R)$ and so R is JN. According to [16, Corollary 2.28], R is not a WUU ring. □

In [15], a ring R is called a nil-good ring if every element $r \in R$ can be represented as $r = a + u$, where $a \in Nil(R)$ and $u \in U(R) \cup \{0\}$. The concept of nil-good rings is a nontrivial generalization to fine rings that are rings for which each nonzero element can be written as the sum of a unit and a nilpotent (see [7]).

Proposition 2.4 *Every nil-good ring is JN, but the converse is false.*

Proof According to [15, Propositon 2.5], every nil-good ring is JN. Assume that $R = \mathbb{Z} \times \mathbb{Z}$. Then R is JN. But R is not nil-good since $(1, 0) \in R$ cannot written as either a sum of a nilpotent and a unit or a sum of a nilpotent and 0. □

Corollary 2.5 *Every fine ring is JN, but the converse is false.*

A ring R is called a nil-clean ring in [21] if every $r \in R$ can be represented as $r = e + b$, where $e \in Id(R)$ and $b \in Nil(R)$. In [6] the authors called a ring R weakly nil-clean if each $r \in R$ can be written as $r = n + e$ or $r = n - e$, where $n \in Nil(R)$ and $e \in Id(R)$. Clearly, nil-clean rings are weakly nil-clean. The ring $\mathbb{Z}_3$ is an example of a weakly nil-clean ring that is not nil-clean. If a ring R has characteristic 2, then R is weakly nil-clean if and only if it is nil-clean by a simple computation.

Proposition 2.6 *Every weakly nil-clean ring is JN, but the converse is false.*

Proof According to [6, Theorem 2], every weakly nil-clean ring is JN. Assume that $R = \mathbb{Z}_2[C_3]$, where C_3 is the multiplicative cyclic group of order 3. By Proposition 3.19 below, R is JN. But R is not weakly nil-clean due to [20, Corollary 2.2]. □

Corollary 2.7 *Every nil-clean ring is JN, but the converse is not ture.*

Recall that a ring R is said to be semi-Boolean if, for each $r \in R$, there exist an element $j \in J(R)$ and an idempotent e such that $r = j + e$. In the terminology of [11], semi-Boolean rings are just called J-clean rings. Later, the notion of a weakly semi-Boolean ring was introduced in [19], a ring R is called weakly semi-Boolean if, for every $r \in R$, there exist an element $j \in J(R)$ and an idempotent e such that $r = j + e$ or $r = j - e$. Semi-Boolean rings are weakly semi-Boolean but the converse is wrong. For instance, $\mathbb{Z}_9$ is weakly semi-Boolean but not semi-Boolean because 5 and 8 cannot be represented as a sum of an idempotent and an element from $J(R)$.

Proposition 2.8 *Let R is a JN ring. Then the following are hold:*

(1) If R is weakly semi-Boolean, then R is weakly nil-clean. The converse holds for J-reduced rings.

(2) If R is semi-Boolean, then R is nil-clean. The converse holds for J-reduced rings.

Proof Since $Nil(R) \subseteq J(R)$ in J-reduced ring, the proof of (1) and (2) is obvious. □

As usual, an involution in a ring R means an element $a \in R$ satisfying $a^2 = 1$ and $Inv(R)$ the subset of $U(R)$ consisting of all involutions of R. Mimicking [17], a ring R is called a invo-clean ring if every $r \in R$ can be written as $r = v + e$, where $v \in Inv(R)$ and $e \in Id(R)$. It was established in [17] that a ring R is invo-clean with $2 \in Nil(R)$, then R is nil-clean with bounded index of nilpotence not exceeding 3. The concept of invo-cleanness was extended in [18], respectively, by defining the notion of weak invo-cleanness. A ring R is said to be weakly invo-clean if every $r \in R$ can be presented as $r = v + e$ or $r = v - e$, where $v \in Inv(R)$ and $e \in Id(R)$. The ring $\mathbb{Z}_5$ is weakly invo-clean that is not invo-clean and not weakly nil-clean. It was shown in [18] that if R is weakly invo-clean and 4=0, then $Z(R)$(the center of R) is invo-clean.

Proposition 2.9 *Every weakly invo-clean ring is JN, but the converse is false.*

Proof Owing to [18, Corollary 4.4], every weakly invo-clean ring is JN. The direct product $R = \mathbb{Z}_5 \times \mathbb{Z}_5$ is JN but not weakly invo-clean although $\mathbb{Z}_5$ is. Since (2,3) and (3,2) cannot be represented as a sum or difference of an involution and an idempotent. Therefore R is not weakly invo-clean. □

Corollary 2.10 *Every invo-clean ring is JN, but the converse is not ture.*

Recall that the right (left) socle of a ring R is the sum of all minimal right (left) ideals of R and is denoted by $Soc_r(R)$ (repectively, by $Soc_l(R)$). As in [26] a ring R is called right soclean if each element $r \in R$ is written as $r = s + e$, where $s \in Soc_r(R)$ and $e \in Id(R)$. A left soclean ring is defined similarly. A ring is called soclean if it is both right and left soclean. Weakly right soclean rings (those in which every element is a sum or difference of an element of the right socle and an idempotent) are also tackled in [26].

Proposition 2.11 *Every weakly right soclean ring is JN, but the converse is false.*

Proof If R is weakly right soclean, according to [26, Corollary 4.3], $J(R)$ are contained in $Soc_r(R)$ and $J(R)^2 = 0$. However, consider the countably infinite direct product $R = \prod_{i=1}^{\infty} R_i$, where each R_i is a copy of the ring $\mathbb{Z}_4$. Clearly, R is JN. But the tuple $x = (2, 2, \cdots) \in R$ does not satisfy the condition of Theorem 4.2 of [26]. So R is not weakly right soclean. □

Corollary 2.12 *Every right soclean ring is JN, but the converse is false.*

3. More examples of JN rings

Proposition 3.1 *The following are equivalent:*

(1) R is a JN ring;

(2) Every ideal I of R is a JN ring;

(3) Every proper ideal of R is a JN ring.

Proof (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3) are trivial by $J(I) = J(R) \cap I \subseteq Nil(R) \cap I = Nil(R)$. So it only remains to show that (3) $\Rightarrow$ (1). Let $a \in J(R)$, it follows that $J(RaR) \subseteq Nil(RaR)$ by hypothesis. On the other hand, $a \in J(R) \cap RaR = J(RaR)$ which leads to $a \in Nil(RaR) = Nil(R) \cap RaR$. Therefore $a \in Nil(R)$, as desired. $\square$

Lemma 3.2 *Let I be a ideal of a ring R. If $I \subseteq J(R)$, then $a \in J(R)$ if and only if $\bar{a} \in J(R/I)$.*

Proof Let $\bar{a} \in J(R/I)$. Then for any $r \in R$, $\bar{1} + \bar{a}\bar{r} \in U(R/I)$. Therefore, there exists $\bar{b} \in U(R/I)$ such that $\bar{b}(\bar{1} + \bar{a}\bar{r}) = \bar{1}$, that is $b(1 + ar) - 1 \in I$. Since $I \subseteq J(R)$ and $1 + J(R) \subseteq U(R)$, we have $b(1 + ar) \in U(R)$. Therefore $1 + ar \in U(R)$ and hence $a \in J(R)$. The converse is trivial. $\square$

Proposition 3.3 *Let I be a nil ideal of a ring R. Then the following are equivalent:*

(1) R is a JN ring;

(2) R/I is a JN ring.

Proof (1)$\Rightarrow$(2) Let $\bar{a} \in J(R/I)$. Since $I \subseteq Nil^*(R) \subseteq J(R)$, we get $a \in J(R)$ by Lemma 3.2. Therefore $a \in Nil(R)$ and so $\bar{a} \in Nil(R/I)$.

(2)$\Leftarrow$(1) Let $a \in J(R)$. Then $\bar{a} \in J(R)/I \subseteq J(R/I)$. By hypothesis, $\bar{a} \in Nil(R/I)$. Therefore $a^n \in I$ for some $n \in \mathbb{N}$. But I is nil which leads to $a \in Nil(R)$, as desired. $\square$

Lemma 3.4 *A ring R is JN if and only if so is eRe for all idempotent $e \in R$.*

Proof If $eae \in J(eRe)$, then $eae \in J(R) \cap eRe$. Since R is JN, $eae \in Nil(R)$, and so $eae \in eRe \cap Nil(R) = Nil(eRe)$. Therefore eRe is JN. The converse is trivial. $\square$

Let α be an endomorphism of R and n a positive integer. Nasr-Isfahani [31] defined skew triangular matrix ring

$$T_n(R,\alpha) = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \mid a_i \in R \right\}$$

with addition pointwise and multiplication given by:

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \\ 0 & b_0 & b_1 & \cdots & b_{n-2} \\ 0 & 0 & b_0 & \cdots & b_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_0 \end{pmatrix} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ 0 & c_0 & c_1 & \cdots & c_{n-2} \\ 0 & 0 & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_0 \end{pmatrix}$$

where $c_i = a_0\alpha^0(b_i) + a_1\alpha^1(b_{i-1}) + \cdots + a_i\alpha^i(b_0)$, $0 \le i \le n-1$. We denote elements of $T_n(R,\alpha)$ by $(a_0, a_1, \cdots, a_{n-1})$. If α is an identity, then $T_n(R,\alpha)$ is a subring of upper triangular matrix ring $T_n(R)$.

Theorem 3.5 *Let R be a ring. Then the following are equivalent:*

(1) R is JN;

(2) $T_n(R,\alpha)$ is JN.

Proof Choose

$$I = \left\{ \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \mid a_{ij} \in R \ (i \le j) \right\}.$$

Then $I^n = 0$ and $T_n(R,\alpha)/I \cong R$. We get the result by Proposition 3.3. □

Let α be an endomorphism of R. We denote by $R[x;\alpha]$ the skew polynomial ring whose elements are the polynomials over R, the addition is defined as usual, and the multiplication subject to the reaction $xr = \alpha(r)x$ for any $r \in R$. There is a ring isomorphism $\varphi : R[x;\alpha]/(x^n) \to T_n(R,\alpha)$, given by $\varphi(a_0 + a_1 x + \cdots a_{n-1}x^{n-1} + (x^n)) = (a_0, a_1, \cdots, a_{n-1})$, with $a_i \in R$, $0 \le i \le n-1$. So $T_n(R,\alpha) \cong R[x;\alpha]/(x^n)$, where (x^n) is the ideal generated by x^n.

Corollary 3.6 *Let R be a ring. Then the following are equivalent:*

(1) R is JN;

(2) $R[x;\alpha]/(x^n)$ is JN.

Let S and T be any rings, ${}_SM_T$ a bimodule and the formal triangular matrix $R = \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$. It is well-known that $J(R) = \begin{pmatrix} J(S) & M \\ 0 & J(T) \end{pmatrix}$.

Proposition 3.7 *Let $R = \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$. Then R is JN if and only if S and T are JN.*

Proof Take $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $S \cong eRe$, $T \cong fRf$. It follows from Lemma 3.4 that S and T are JN.

Conversely, let S and T are JN and $A = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in J(R)$. Then $a \in J(S), b \in J(T)$. By hypothesis, $a \in Nil(S), b \in Nil(T)$. Then there exist $n \in \mathbb{N}$ and $m \in \mathbb{N}$ such that $a^n = 0 = b^m$. Therefore $A^{n+m} = \begin{pmatrix} a^{n+m} & * \\ 0 & b^{n+m} \end{pmatrix} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \in Nil(R)$. Thus $A \in Nil(R)$ and so R is JN. □

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R,M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operation are used.

Corollary 3.8 *Let R be a ring. Then R is JN if and only if $T(R,M)$ is JN.*

Let R be a ring and a bimodule ${}_RV_R$ which is a general ring (possibly with no unity) in which $(vw)r = v(wr), (vr)w = v(rw)$ and $(rv)w = r(vw)$ hold for all $v, w \in V$ and $r \in R$. The ideal extension $I(R;V)$ of R by V is defined to be the additive abelian group $I(R;V) = R \oplus V$ with multiplication $(r,v)(s,w) = (rs, rw + vs + vw)$.

Theorem 3.9 *Suppose that for any $v \in V$ there exists $w \in V$ such that $v + w + vw = 0$. Then the following are equivalent for a ring R:*

(1) R and V are JN;

(2) An ideal extension $S = I(R;V)$ is JN.

Proof (1) $\Longrightarrow$ (2) Note that $V = J(V)$ and $(0,V) \subseteq J(S)$ by hypothesis. Let $s = (r,v) \in J(S)$, since $(r,v) = (r,0) + (0,v)$, we have $(r,0) \in J(S)$. For any $a \in R$, $1 - ra \in U(R)$ because $(1,0) - (r,0)(a,0) = (1 - ra, 0) \in U(S)$ and hence $r \in J(R)$. Thus $r \in Nil(R)$. Therefore, there exist $n \in \mathbb{N}$ and $x \in V$ such that $s^n = (r,v)^n = (r^n, x) = (0,x)$. As V is JN, we get $x \in V = J(V) \subseteq Nil(V)$. Hence we write $x^m = 0$ for some $m \in \mathbb{N}$. Thus $s^{n+m} = (0,x)^m = (0,x^m) = 0$ and so $s \in Nil(S)$. Accordingly, $S = I(R;V)$ is JN.

(2) $\Longrightarrow$ (1) Suppose S is JN and let $a \in J(R)$, we first show that $(a,0) \in J(S)$. For any $(r,v) \in S$, $(1,0) - (a,0)(r,v) = (1 - ar, -av) = (1 - ar, 0)(1, (1 - ar)^{-1}(-av))$. Since $(0,V) \in J(S)$, we get $(1, (1 - ar)^{-1}(-av)) = (1,0) + (0, (1 - ar)^{-1}(-av)) \in U(S)$. Hence $(1,0) - (a,0)(r,v) \in U(S)$ and so $(a,0) \in J(S) \subseteq Nil(S)$. Therefore, there exists $n \in \mathbb{N}$ such that $(a,0)^n = (a^n, 0) = 0$. It follows that $a^n = 0$ and hence $a \in Nil(R)$. Thus R is JN. If $v \in J(V)$, then $(0,v) \in J(S)$. Since S is JN, we have $(0,v) \in Nil(S)$ and hence $v \in Nil(V)$. Thus V is JN. □

Let D be a ring and C a subring of D with $1_D \in C$. We set $R\{D,C\} = \{(d_1, \cdots, d_n, c_{n+1}, c_{n+2}, \cdots) \mid d_i \in D, c_j \in C, n \geq 1\}$, $R(D,C) = \{(d_1, \cdots, d_n, c_{n+1}, c_{n+2}, \cdots) \mid d_i \in D, c_j \in C, n \geq 1$, and only a finite number of j are not zero$\}$, $R[D,C] = \{(d_1, \cdots, d_n, c, c, \cdots) \mid d_i \in D, c \in C, n \geq 1\}$. On the above three sets, we define addition and multiplication by components, it is easy to see that they are all rings. Also $J(R\{D,C\}) = R\{J(D), J(D) \cap J(C)\}$ (see [39]).

Recall that a ring R is of bounded index of nilpotency if there exists a number n such that $x^n = 0$ for every nilpotent element $x \in R$.

Proposition 3.10 *Let D be a ring and a subring C of D. Then the following two implications hold:*

(1) If D and C are JN and C is of bounded index of nilpotency, then $R\{D,C\}$ is JN;

(2) If $R\{D,C\}$ is JN and $J(C) \subseteq J(D)$, then D and C are JN.

Proof (1) Let $(d_1, \cdots, d_n, c_{n+1}, c_{n+2}, \cdots) \in J(R\{D,C\}) = R\{J(D), J(D) \cap J(C)\}$. Then $d_i, c_j \in J(D), i = 1, 2, \cdots, n$, $j = n+1, n+2, \cdots,$. By assumption, $d_i, c_j \in Nil(D)$ and $c_j \in Nil(C)$. Since C is of bounded index of nilpotency, this implies that $(d_1, \cdots, d_n, c_{n+1}, c_{n+2}, \cdots) \in Nil(R\{D,C\})$. Thus $R\{D,C\}$ is JN.

(2) Let $a \in J(D)$. Then $(a, 0, 0, \cdots) \in J(R\{D,C\}) \subseteq Nil(R\{D,C\})$. Therefore, there exists $n \in \mathbb{N}$ such that $(a, 0, 0, \cdots)^n = (a^n, 0, 0, \cdots) = 0$. It follows that $a \in Nil(R)$ and hence D is JN. Let $b \in J(C)$. Since $J(C) \subseteq J(D)$ and $J(R\{D,C\}) \subseteq Nil(R\{D,C\})$, we have $(0, b, b, \cdots) \in J(R\{D,C\})$ and $(0, b, b, \cdots)^m = (0, b^m, b^m, \cdots) = 0$ for some $m \in \mathbb{N}$. Hence $b \in Nil(C)$, as desired. □

Corollary 3.11 *Let D be a ring and a subring C of D. If D and C are JN, then $R(D,C)$ and $R[D,C]$ are JN. The converse holds if $J(C)\subseteq J(D)$.*

Proof The proof is similar to Proposition 3.10. □

Proposition 3.12 *Let $R=\prod_{\gamma\in\Gamma} R_\gamma$ be a direct product of rings R_γ and Γ an indexed set. If R is of bounded index of nilpotency, then R is JN if and only if R_γ is JN for every $\gamma\in\Gamma$.*

Proof Since R is of bounded index of nilpotency, we get $Nil(\prod_{\gamma\in\Gamma} R_\gamma)=\prod_{\gamma\in\Gamma} Nil(R_\gamma)$. On the other hand, $J(\prod_{\gamma\in\Gamma} R_\gamma)=\prod_{\gamma\in\Gamma} J(R_\gamma)$. Then the assertion follows easily. □

Let α be an injective homomorphism of a ring R and A an extension ring of R. If α can be extended to an isomorphism of A and $A=\bigcup_{n=0}^{\infty}\alpha^{-n}(R)$, then we call this extension ring A the Jordan extension of R by α.

Proposition 3.13 *Let α be an injective homomorphism of R. If R is a JN ring, then the Jordan extension A of R by α is also JN.*

Proof Let $a\in J(A)$ with $\alpha^n(a)\in R$ for some $n\in\mathbb{N}$. For any $r\in R$, $a\alpha^{-n}(r)\in J(A)$. Then there exists $b\in A$ such that $a\alpha^{-n}(r)+b+a\alpha^{-n}(r)b=0$. For $b\in A$, we get $\alpha^m(b)\in R$ for some $m\in\mathbb{N}$. Therefore $\alpha^n(a\alpha^{-n}(r)+b+a\alpha^{-n}(r)b)=0$ and hence $\alpha^n(a)r+\alpha^n(b)+\alpha^n(a)r\alpha^n(b)=0$. Since $\alpha^n(b)=\alpha^{n-m}(\alpha^m(b))\in R$, we have $\alpha^n(a)\in J(R)\subseteq Nil(R)$. Therefore $[\alpha^n(a)]^k=\alpha^n(a^k)=0$ for some $k\in\mathbb{N}$. This implies $a^k=0$ and so $a\in Nil(A)$. Thus A is JN. □

Proposition 3.14 *Let $(I,\leqslant)$ be a strictly ordered set and $\{A_\alpha|\alpha\in I\}$ a family of JN rings. Suppose that $(A_\alpha,(\varphi_{\alpha\beta})_{\alpha\leqslant\beta})$ is a direct system over I and $(A,(\eta_\alpha)_{\alpha\in I})$ is a direct limit of the direct system. If $\varphi_{\alpha\beta}:\ A_\alpha\to A_\beta$ is an isomorphism for all $\alpha\leqslant\beta$ and $\eta_\alpha:\ A_\alpha\to A$ is a monomorphism for all $\alpha\in I$, then the direct limit $A=\lim\limits_{\to} A_\alpha$ is also JN.*

Proof Let $a\in J(A)$. Then there exists $a_\alpha\in A_\alpha$ such that $\eta_\alpha(a_\alpha)=a$. First we prove $a_\alpha\in J(A_\alpha)$. For any $r_\alpha\in A_\alpha$, we have $\eta_\alpha(a_\alpha r_\alpha)=a\eta_\alpha(r_\alpha)$ is quasi-regular in A and so $a\eta_\alpha(r_\alpha)+b+a\eta_\alpha(r_\alpha)b=0$ for some $b\in A$. For $b\in A$, we can find $b_\beta\in A_\beta$ such that $\eta_\beta(b_\beta)=b$. Then, by the condition, there is $k\in I$ such that $\alpha\leqslant k$ and $\beta\leqslant k$ and we gain $\eta_\alpha(a_\alpha r_\alpha)=\eta_k\varphi_{\alpha k}(a_\alpha r_\alpha)$ and $\eta_\beta(b_\beta)=\eta_k\varphi_{\beta k}(b_\beta)$. Since $\eta_\alpha(a_\alpha r_\alpha)+b+\eta_\alpha(a_\alpha r_\alpha)b=0$, we have the equation $\eta_k\varphi_{\alpha k}(a_\alpha r_\alpha)+\eta_k\varphi_{\beta k}(b_\beta)+\eta_k\varphi_{\alpha k}(a_\alpha r_\alpha)\eta_k\varphi_{\beta k}(b_\beta)=0$ and hence $\varphi_{\alpha k}(a_\alpha r_\alpha)+\varphi_{\beta k}(b_\beta)+\varphi_{\alpha k}(a_\alpha r_\alpha)\varphi_{\beta k}(b_\beta)=0$. Since I is a strictly ordered set, there are the following two cases:

If $\beta\leqslant\alpha\leqslant k$, then $\varphi_{\beta k}=\varphi_{\alpha k}\varphi_{\beta\alpha}$. By the fact that $\varphi_{\alpha k}$ is isomorphism, we get $a_\alpha r_\alpha+\varphi_{\beta\alpha}(b_\beta)+a_\alpha r_\alpha\varphi_{\beta\alpha}(b_\beta)=0$. Hence, $a_\alpha\in J(A_\alpha)$ since $\varphi_{\beta\alpha}(b_\beta)\in A_\alpha$.

If $\alpha\leqslant\beta\leqslant k$, then $\varphi_{\alpha k}=\varphi_{\beta k}\varphi_{\alpha\beta}$. Similarly, we gain $\varphi_{\alpha\beta}(a_\alpha r_\alpha)+b_\beta+\varphi_{\alpha\beta}(a_\alpha r_\alpha)b_\beta=0$. Since $\varphi_{\alpha\beta}$ is isomorphism, we have $a_\alpha r_\alpha+\varphi_{\alpha\beta}^{-1}(b_\beta)+a_\alpha r_\alpha\varphi_{\alpha\beta}^{-1}(b_\beta)=0$. So $a_\alpha\in J(A_\alpha)$ by $\varphi_{\alpha\beta}^{-1}(b_\beta)\in A_\alpha$.

In conclusion, we always have $a_\alpha \in J(A_\alpha) \subseteq Nil(A_\alpha)$. Then $a_\alpha^n = 0$ for some $n \in \mathbb{N}$. It is further implied that $a^n = (\eta_\alpha(a_\alpha))^n = \eta_\alpha(a_\alpha^n) = 0$ and so $a \in Nil(A)$. This proves that $J(A) \subseteq Nil(A)$. Therefore $A = \lim\limits_{\rightarrow} A_\alpha$ is JN. □

Recall that a ring R is called Armendariz if $f(x)g(x) = 0$ for any $f(x) = \sum\limits_{i=0}^{n} a_i x^i, g(x) = \sum\limits_{j=0}^{m} b_j x^j \in R[x]$, then $a_i b_j = 0$ for any i, j. Assume that $R = \mathbb{Z}[[x]]$, R is domain (hence Armendariz) with $J(R) = x\mathbb{Z}[[x]]$, then it is not JN. We can also find a JN ring such that it is not Armendariz (see [28, Example 1.7]). Therefore, there is no relationship between Armendariz rings and JN rings.

Proposition 3.15 *If R is a Armendariz ring, then $R[x]$ is JN.*

Proof Let R be a Armendariz ring, By [28, Theorem 1.3], $J(R[x]) = Nil^*(R[x])$. Therefore $R[x]$ is JN. □

Let α be an endomorphism of R. According to Annin [2], a ring R is said to be α-compatible if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. In [12] the authors called a ring R nil-semicommutative if for any $a, b \in Nil(R)$, $ab = 0$ implies $aRb = 0$. Following [3], an automorphism α of R is said to be of locally finite order if for every $r \in R$ there exists integer $n(r) \geqslant 1$ such that $\alpha^{n(r)}(r) = r$.

Lemma 3.16 *([3], Corollary 3.3) If α is an automorphism of R of locally finite order and $J(R)$ is locally nilpotent, then $J(R[x;\alpha]) = J(R)[x;\alpha]$.*

Proposition 3.17 *Let α be an automorphism of R of locally finite order and $J(R)$ locally nilpotent. If R is a nil-semicommutative α-compatible, then $R[x;\alpha]$ is JN.*

Proof According to Lemma 3.16, we have $J(R[x;\alpha]) = J(R)[x;\alpha]$. Moreover, it follows that $Nil(R)[x;\alpha] = Nil(R[x;\alpha])$ by [37, Theorem 2.5]. Therefore $J(R[x;\alpha]) = J(R)[x;\alpha] \subseteq Nil(R)[x;\alpha] = Nil(R[x;\alpha])$ and so $R[x;\alpha]$ is JN. □

Let δ be a derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + a\delta(b)$, for $a, b \in R$. We denote by $R[x;\delta]$ the differential polynomial ring whose elements are the polynomials over R, the addition is defined as usual, and the multiplication subject to the reaction $xr = rx + \delta(r)$ for any $r \in R$. In [32], a ring R is called δ-Armendariz if for each $f(x) = \sum\limits_{i=0}^{n} a_i x^i$, $g(x) = \sum\limits_{j=0}^{m} b_j x^j \in R[x;\delta]$, $f(x)g(x) = 0$ implies $a_i b_j = 0$ for each $0 \leqslant i \leqslant n$, $0 \leqslant j \leqslant m$.

Proposition 3.18 *Let R be a ring and δ a derivation of R. If R is a δ-Armendariz ring, then $R[x;\delta]$ is JN.*

Proof Let R be a δ-Armendariz ring. By [32, Corollary 3.4], we have $J(R[x;\delta]) = Nil^*(R[x;\delta])$. Hence $R[x;\delta]$ is JN. □

Let G be a group. An element g of G is called a torsion element if g has finite order. If all elements of G are torsion, then G is called a torsion group. For a ring R, we let $R[G]$ denote the group ring of an abelian group G over R. By [27, Corollary 2], we yield:

Proposition 3.19 *Suppose that R is a commutative ring and G is an Abeian group. Then $R[G]$ is JN if and only if one of the following two conditions hold:*

(1) G not torsion;

(2) G torsion and R is JN.

Let Δ denote a multiplicatively closed subset of a ring R consisting of central regular elements. Let $\Delta^{-1}R$ be the localization of R at Δ. Then we have:

Proposition 3.20 *For a ring R, R is JN if and only if $\Delta^{-1}R$ is JN.*

Proof We first show that $J(\Delta^{-1}R) = \Delta^{-1}J(R)$. Assume that $\Delta^{-1}I$ is a maximal left ideal of $\Delta^{-1}R$. Then I is a left ideal of R. For any proper left ideal J of R, if $I \subseteq J$, we have $\Delta^{-1}I \subseteq \Delta^{-1}J$ and $\Delta^{-1}J$ is a proper left ideal of $\Delta^{-1}R$. By the maximality of $\Delta^{-1}I$, we get $\Delta^{-1}I = \Delta^{-1}J$. Thus $I = J$. Therefore I is a maximal left ideal of R. Conversely, assume that I is a maximal left ideal of R, Then $\Delta^{-1}I$ is a left ideal of $\Delta^{-1}R$. For any proper left ideal $\Delta^{-1}J$ of $\Delta^{-1}R$, if $\Delta^{-1}I \subseteq \Delta^{-1}J$, we have $I \subseteq J$ and J is a proper left ideal of R. By the maximality of I, we have $I = J$. Hence $\Delta^{-1}I = \Delta^{-1}J$. Thus $\Delta^{-1}I$ is a maximal left ideal of $\Delta^{-1}R$. Therefore, it is implied that $J(\Delta^{-1}R) = \Delta^{-1}J(R)$. The rest proof is obvious since $\Delta^{-1}Nil(R) = Nil(\Delta^{-1}R)$. □

4. A class of special subrings of JN rings

Inspired by the notion of soclean rings, the present section deals with a subclass of JN rings, we call right SR rings. The socle $Soc(M)$ of a left module M over a ring R is defined to be the sum of all simple submodule of M (with $Soc(M) = 0$ if there are no simple submodules). By [29, Exercise 4.18], $Soc(M) \subseteq \{m \in M \mid J(R)m = 0\}$, with equality if $R/J(R)$ is an Artinian ring. For any ring R, let $Soc_r(R)$ be the socle of R as a right R module, i.e. $Soc_r(R)$ is the sum of all minimal right ideal of R, We call $Soc_r(R)$ the right socle of R, and define the left socle $Soc_l(R)$ similarly. It is easy see that both socles are ideals of R. In general, these may be different ideals. But for semiprime rings R, they are equal in view of [29, Lemma 11.9]. We use $Reg(R)$ denotes the set of all (von Neumann) regular elements of R, $ann_R^r(M)$ ($ann_R^l(M)$) denote the right (left) annihilator of M in R. A ring R is called right soclean in [26] if each element $r \in R$ is represented as $r = s + e$, where $s \in Soc_r(R)$ and $e \in Id(R)$. As an extension of soclean rings, we introduce the following concept.

Definition 4.1 *We call a ring R is right SR if each element $r \in R$ is written as $r = s + r$, where $s \in Soc_r(R)$ and $r \in Reg(R)$. A left SR ring is defined similarly. A ring is called SR if it is both right and left SR.*

It is clearly that the class of SR rings contains both regular rings and semisimple rings.

Proposition 4.2 *Every right soclean ring is a right SR ring.*

Proof This is obvious because every idempotent element is regular. □

Recall that a ring R is said to be local if $R/J(R)$ is a division ring. The converse of Proposition 4.2 need not be ture in general. For example, if R is local JN and $J(R) = Soc_r(R)$, then R is right SR by Proposition 4.9. But it is not right soclean unless $R/J(R) \cong \mathbb{Z}_2$.

Proposition 4.3 *Every right SR ring is a JN ring. In particular, $J(R)^2 = 0$.*

Proof Let R be a right SR ring. Then $R/Soc_r(R)$ is a regular ring. Since every regular ring is semiprimitive, we see that $J(R) \subseteq Soc_r(R)$, and hence, $J(R)^2 = 0$ since $Soc_r(R) \subseteq \{r \in R \mid rJ(R) = 0\}$. □

The following example implies the converse of Proposition 4.3 is wrong.

Example 4.4 *Assume that* $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, *where* $\mathbb{Z}$ *is the integral ring. Clearly* R *is JN. Since* $Soc_r(R) = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, *we get* $R/Soc_r(R) \cong \mathbb{Z}$ *is not regular. Therefore* R *is not right SR by Proposition 4.7.*

Proposition 4.5 *Homomorphic images of right SR rings are again right SR.*

Proof Let $\varphi : R \to B$ be a ring epimorphism and suppose R be a right SR ring. Let $b \in B$ and choose $a \in R$ such that $\varphi(a) = b$. Then we can write $a = r + s$ for some $r \in Reg(R)$ and $s \in Soc_r(R)$. Hence $b = \varphi(a) = \varphi(r) + \varphi(s)$ where clearly $\varphi(r) \in Reg(B)$ and $\varphi(s) \in Soc_r(B)$. Therefore B is a right SR ring. □

For a ring R, an idempotent $e \in R$ is called left semicentral if $ae = eae$, for all $a \in R$.

Proposition 4.6 *Let* $e \in Id(R)$ *be left semicentral,and* $f = 1 - e$. *If* R *is right SR, then the corner rings* eRe *and* fRf *are right SR.*

Proof Let R is right SR and $e \in Id(R)$ is left semicentral. It is easy to check that the map $g : R \to eRe$ sending r to ere is a surjective ring homomorphism. Therefore eRe is a right SR by Proposition 4.5. Similarly, fRf is a homomorphic image of R, hence fRf is also right SR. □

For an one-side ideal I of R, we say that regular elements lift module I if whenever $a - aba \in I$, with $a, b \in R$, there exists a regular element d of R such that $a - d \in I$.

Proposition 4.7 *Let* R *be a ring. Then the following are equivalent:*

(1) R *is a right SR ring;*

(2) $R/Soc_r(R)$ *is regular and regular elements lift modulo* $Soc_r(R)$.

Proof (1) ⇒ (2) let $\bar{a} \in R/Soc_r(R)$. By hypothesis, there exists a regular element $r \in R$ such that $a - r \in Soc_r(R)$. Then $\bar{a} = \bar{r} \in R/Soc_r(R)$ is regular. Clearly, every regular element lifts modulo $Soc_r(R)$.

(2) ⇒ (1) Assume that $R/Soc_r(R)$ is a regular ring and $a \in R$. Then $\bar{a} \in R/Soc_r(R)$. Since $\bar{a}$ is regular and regular elements lift modulo $Soc_r(R)$, there exists $r \in Reg(R)$ such that $a - r \in Soc_r(R)$. Hence R is right SR. □

Lemma 4.8 *Let* I *be an ideal of* R *with* $I \subseteq J(R)$. *Then idempotent elements lift module* I *if and only if regular elements lift module* I.

Proof See Lemma 2.4 of [40]. □

Proposition 4.9 *Let* R *be a local ring and* $Soc_r(R) = J(R)$. *Then* R *is JN if and only if* R *is right* SR.

Proof Since R is local, we have $R/J(R)$ is a division ring and hence a regular ring. Also, $J(R)$ is nil and $Soc_r(R) = J(R)$ imply that regular elements lift modulo $Soc_r(R)$ by Lemma 4.8. Therefore R is right SR by Proposition 4.7. The converse is obvious by Proposition 4.3. □

A ring R is called unique right soclean in [26] if each element $r \in R$ is represented uniquely as $r = s + e$, where $s \in Soc_r(R)$ and $e \in Id(R)$. A uniquely left soclean ring is defined similarly. A ring is called uniquely soclean if it is both uniquely right and uniquely left soclean.

Proposition 4.10 *Let R be an unique right soclean ring. Then the following are equivalent:*

(1) R is right SR;

(2) $R/Soc_r(R)$ is regular.

Proof (1) $\Rightarrow$ (2) That is evident by Proposition 4.7.

(2) $\Rightarrow$ (1) Suppose R is an uniquely right soclean ring. Then R is exchange by [26, Theorem 2.23], hence idempotents can be lifted module every left ideal by [34, Corllary 1.3]. According to [26, Theorem 3.3], R is uniquely right soclean imlpies $Soc_r(R) = J(R)$. By Lemma 4.8, we have every regular element lifts module $Soc_r(R)$. Hence, R is right SR ring. $\square$

A ring R with an ideal I is called right I-semiregular in [41] if every principal right ideal K of R has a decomposition $K = eR \bigoplus S$ where $e \in Id(R)$ and $S \subseteq I$.

Proposition 4.11 *For any right SR ring R, the following hold:*

(1) R is $Soc_r(R)$-semiregular.

(2) For any $a \in R$, there exist $e \in Id(R)$ and a right ideal $U \subseteq J(R)$ such that $aR = eR \bigoplus U$.

(3) If X is a finitely generated submodule of a (finitely generated) projective module P, then $X = A \bigoplus B$, where A is a summand of P and $B \subseteq Soc(P)$.

Proof Assume that R is right SR. Then $R/Soc_r(R)$ is regular by Proposition 4.7. Now (1),(2) and (3) are direct consequences of [41, Theorem 1.6]. $\square$

Recall that a ring R is semiperfect if $R/J(R)$ is semisimple and the idempotent can be lifted module $J(R)$. A semiperfect ring with $J(R)$ nilpotent is called semiprimary.

Lemma 4.12 *A ring R is semiperfect if and only if any right (left) ideal can be decomposed as a sum of a right (left) direct summand and a right (left) ideal of R contained in $J(R)$.*

Proof See [33, Theorem 4.3] . $\square$

Theorem 2.21 of [26] states that every right Noetherian right soclean is right Artinian. We can generalize this to the following statement.

Theorem 4.13 *Every right Noetherian right SR ring is right Artinian.*

Proof Let R be a right SR ring. According to Propositon 4.11 (2), every right ideal I of R can be written as $I = eR \bigoplus U$, where $e \in Id(R)$ and $U \subseteq J(R)$. It follows from Lemma 4.12 that R is semiperfect. By Proposition 4.3, $J(R)$ is nilpotent. Therefore R is semiprimary. Thus R is right Artinian by [29, 4.15]. $\square$

Recall that a ring is said to be prime in case the zero ideal is a prime ideal.

Proposition 4.14 *Let R be a prime right SR. Then, for any nonzero ideal I of R, the ring R/I is regular.*

Proof We first show that $Soc_r(R) \subseteq I$ for any nonzero ideal I of R. For this purpose, we only need to show that I contains any minimal right ideal A of R. Since R is a prime ring, we have $0 \neq AI \subseteq A$. By the minimality of A, we see that $A = AI \subseteq I$. Therefore the ring R/I is a homomorphisic image of regular $R/Soc_r(R)$. Thus R/I is regular. □

The following criterion, which enlarges [26, Theorem 2.19], is valid:

Proposition 4.15 *In every right SR ring, each right ideal is a sum of an idempotent right ideal and a nilpotent right dieal.*

Proof The proof is similar to [26, Theorem 2.19]. □

Recall that a ring R is said to be semiregular if $R/J(R)$ is von Neumann regular and idempotents lift module $J(R)$, so that R is semiregular if and only if R is right $J(R)$-semiregular (see [36, 2.3]).

Theorem 4.16 *Every right SR ring is semiregular.*

Proof By Proposition 4.11(2), R is right $J(R)$-semiregular and hence semiregular. □

Recall that a ring R is called exchange in [34] if idempotent can be lifted module every left ideal. Following [34, Proposition 1.6], every semiregular ring is exchange. Consequently, the next is immediate.

Corollary 4.17 *Every right SR ring is exchange.*

By [9, Corollary 2], every exchange ring is either semiperfect, or else contains an infinite set of orthogonal idempotents. Consequently, the following corollary is direct.

Corollary 4.18 *Every right SR ring with no infinite sets of orthogonal idempotents is semiprimary.*

A ring R is called abelian if all idempotent of R are central. By [34, Proposition 1.8], a abelian ring R is clean if and only if R is exchange. Now we turn to clean rings and seek to find conditions for a right SR ring.

Proposition 4.19 *If R is a right SR ring and $R/J(R)$ is abelian, then R is clean.*

Proof By hypothesis and [22, Corollary 4.2], R is unit regular. Therefore R is clean by [8, Theorem 1]. □

Proposition 4.20 *For any ring $R \neq 0$ and any ring endomorphism $\alpha : R \to R$, the skew polynomial ring $R[x;\alpha]$ and skew power series ring $R[[x;\alpha]]$ are never right SR.*

Proof We prove the claim for the skew polynomial ring $S = R[x;\alpha]$, and the argument for the skew formal power series ring $R[[x;\alpha]]$ is similar.

Let I be any minimal right ideal of S and $0 \neq f = a_k x^k + a_{k+1}x^{k+1} + \cdots a_n x^n \in I$, where k is the smallest nonzero index with $a_k \neq 0$. Then $0 \neq fx \in I$. By the minimality of I, we gain $I = fxS$. Hence $f = fxg$ for some $g = b_0 + b_1 x + \cdots + b_t x^t \in S$. This entails $a_k x^k + a_{k+1}x^{k+1} + \cdots + a_n x^n = a_k \alpha^{k+1}(b_0)x^{k+1} + \cdots + a_n \alpha^{n+1}(b_t)x^{n+t+1}$, a contradiction. Therefore the ring S has no minimal right ideals, so $Soc_r(S) = 0$. Thus S is not right SR, as $x \notin Reg(S)$. □

Corollary 4.21 *For any ring $R \neq 0$, the rings $R[x]$ and $R[[x]]$ are never right SR.*

Assume that R is a commutative ring and M is an R-module. The idealization $S = R(+)M$ with element-wise addition and the multiplication rule $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ is a commutative ring with identity $(1, 0)$ and having $0(+)M$ as a square zero ideal.

Lemma 4.22 *If I is an ideal of a commutative ring R and N is a submodule of an R-module M, then $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$. As such, M/N is an R/I-module and $(R(+)M)/(I(+)N) \cong (R/I)(+)(M/N)$. In particular, $(R(+)M)/(0(+)M) \cong R$.*

Proof See [1, Theorem 3.1]. □

Lemma 4.23 *If M is a module over a commutative ring R, and $S = R(+)M$, then*

$$Soc(S) = [Soc(R) \cap ann_R(M)](+)Soc(M)$$

Proof See [26, Proposition 5.3]. □

We now arrange to prove the following:

Theorem 4.24 *Let R be a commutative ring and M be an R-module. If the idealization $S = R(+)M$ is a right SR ring, then M is semisimple and the ring $R/[Soc(R) \cap ann_R(M)]$ is regular.*

Proof Assume that S is a right SR ring. Since $0(+)M$ is a square zero ideal, it is contained in $J(R)$. Also, since $J(R) \subseteq Soc_r(R)$ by the proof of Proposition 4.3, we have $0(+)M \subseteq Soc_r(S)$. Therefore, by Lemma 4.23, we get $M \subseteq Soc(M)$, that is, M is semisimple. Hence, by $Soc(S) = [Soc(R) \cap ann_R(M)](+)M$ and Lemma 4.22, we conclude that $R/[Soc(R) \cap ann_R(M)] \cong S/Soc(S)$ is a regular ring.

□

Suppose that S and T are arbitrary rings and ${}_SM_T$ is a bimodule. To settle the question of when the formal triangular matrix $R = \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$ is right SR we need the following result whose proof can be found in [23, Corollary 2.2].

Lemma 4.25 *The right socle of the triangular matrix ring $R = \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$ is given by*

$$Soc_r(R) = \begin{pmatrix} Soc(S) \cap ann^l_S(M) & Soc(M_T) \\ 0 & Soc_r(T) \end{pmatrix}.$$

Theorem 4.26 *Let the triangular matrix ring $R = \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$ be a right SR ring. Then the following conditions hold:*

(1) M_T is semisimple;

(2) $T/Soc_r(T)$ is a regular ring;

(3) $S/(Soc(S) \cap ann^l_S(M))$ is a regular ring.

Proof Suppose that R is a right SR ring. Since $\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ is a square zero ideal, it is contained in $J(R)$ and hence it is falls into $Soc_r(R)$. Therefore $M \subseteq Soc(M_T)$ by Lemma 4.25 and so M_T is semisimple. Thus, we have $R/Soc_r(R) \cong S/(Soc(S) \cap ann_S^l(M)) \times T/Soc_r(T)$. Since $R/Soc_r(R)$ is regular, we conclude that $T/Soc_r(T)$ and $S/(Soc(S) \cap ann_S^l(M))$ are regular as homomorphic images of $R/Soc_r(R)$. $\square$

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