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The Lebesgue constants on projective spaces

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Abstract: We give the solution of a classical problem of Approximation Theory on sharp asymptotic of the Lebesgue constants or norms of the Fourier-Laplace projections on the real projective spaces $\mathrm{P}^d(\mathbb{R})$. In particular, these results extend sharp asymptotic found by Fejer [2] in the case of $\mathbb{S}^1$ in 1910 and by Gronwall [4] in 1914 in the case of $\mathbb{S}^2$. The case of spheres, $\mathbb{S}^d$, complex and quaternionic projective spaces, $\mathrm{P}^d(\mathbb{C})$, $\mathrm{P}^d(\mathbb{H})$ and the Cayley elliptic plane $\mathrm{P}^{16}(\mathrm{Cay})$ was considered by Kushpel [8].

Key words: Lebesgue constant, Fourier-Laplace projection, Jacoby polynomial

1. Introduction

Let $\mathbb{P}^d(\mathbb{R})$ be the real d-dimensional projective space, ν its normalized volume element, Δ its Laplace-Beltrami operator. It is well-known that the eigenvalues θ_m, $m = 2k$, $k = 0, 1, 2, \cdots$ of Δ are discrete, nonnegative, and form an increasing sequence $0 \le \theta_0 \le \theta_2 \le \cdots \le \theta_{2k} \le \cdots$ with $+\infty$ as the only accumulation point. Corresponding eigenspaces H_{2k}, are finite dimensional, $d_{2k} = \dim \mathrm{H}_{2k} < \infty$, orthogonal, and $L_2(\mathbb{P}^d(\mathbb{R}), \nu) = \oplus_{2k=0}^{\infty} \mathrm{H}_{2k}$. Let $\{Y_j^{2k}\}_{j=1}^{d_{2k}}$ be an orthonormal basis of H_{2k}. Let φ be a continuous function on $\mathbb{P}^d(\mathbb{R})$, $\varphi \in C(\mathbb{P}^d(\mathbb{R}))$ with the formal Fourier expansion

$$\varphi \sim c_0 + \sum_{k \in \mathbb{N}} \sum_{j=1}^{d_{2k}} c_{2k,j}(\varphi) Y_j^{2k}, \;\; c_{2k,j}(\varphi) = \int_{\mathbb{P}^d(\mathbb{R})} \varphi \; \overline{Y_j^{2k}} d\nu.$$

Consider the sequence of Fourier sums

$$S_{2n}(\varphi, x) = c_0 + \sum_{k=1}^{n} \sum_{j=1}^{d_{2k}} c_{2k,j}(\varphi) Y_j^{2k}(x), \;\; n \in \mathbb{N}.$$

The main aim of this article is to establish sharp asymptotic for the sequence of Lebesgue constants defined as

$$L_{2n}\left(\mathbb{P}^d(\mathbb{R})\right) := \| S_{2n} \left| C\left(\mathbb{P}^d(\mathbb{R})\right) \to C\left(\mathbb{P}^d(\mathbb{R})\right) \right\|, \, n \to \infty.$$

In the case of the circle, $\mathbb{S}^1$, the following result has been found by Fejer [2] in 1910

$$L_n\left(\mathbb{S}^1\right) = \frac{1}{\pi} \int_{-\pi}^{\pi} |D_n(t)| dt \; = \frac{4}{\pi^2} \ln n + O(1), n \to \infty,$$

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where $D_n(t) = 1/2 + \sum_{k=1}^{n} \cos\ kt$ is the Dirichlet kernel. In the case of $\mathbb{S}^2$, the two-dimensional unit sphere in $\mathbb{R}^3$, the estimates of $L_n\left(\mathbb{S}^2\right)$ have been established by Gronwall [4]. Namely, it was shown that

$$L_n\left(\mathbb{S}^2\right) = n^{1/2}\ \frac{2}{\pi^{3/2}} \int_0^{\pi} \sqrt{\cot\left(\frac{\eta}{2}\right)}\ d\eta + O(1)$$

$$= n^{1/2}\frac{2^{3/2}}{\pi^{1/2}} + O(1),\ n \to \infty.$$

Lebesgue constants on more general manifolds, $\mathbb{M}^d$, were considered by Kushpel [8]. Namely, in the case of the real spheres $\mathbb{S}^d$, $d \geq 3$, complex and quaternionic projective spaces, $\mathrm{P}^d(\mathbb{C})$ and $\mathrm{P}^d(\mathbb{H})$ respectively, and the Cayley elliptic plain $\mathrm{P}^{16}(\mathrm{Cay})$ it was shown that

$$L_n\left(\mathbb{M}^d\right) = \mathcal{K}(\mathbb{M}^d) n^{(d-1)/2} + O\left\{\begin{array}{cc} 1, & d = 2,3 \\ n^{(d-3)/2}, & d \geq 4 \end{array}\right\},$$

where

$$\mathcal{K}(\mathbb{S}^d) = \frac{2\,\Gamma\left(\frac{d-1}{4}\right)\Gamma\left(\frac{d+1}{4}\right)}{\pi^{3/2}\left(\Gamma\left(\frac{d}{2}\right)\right)^2},\ \ d = 2,3,4,\cdots,$$

$$\mathcal{K}(\mathrm{P}^d(\mathbb{C})) = \frac{2\,\Gamma\left(\frac{d-1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\pi^{3/2}\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+2}{4}\right)},\ \ d = 4,6,8,\cdots,$$

$$\mathcal{K}(\mathrm{P}^d(\mathbb{H})) = \frac{\Gamma\left(\frac{d-1}{4}\right)}{\pi\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+5}{4}\right)},\ \ d = 8,12,16,\cdots,$$

$$\mathcal{K}(\mathrm{P}^{16}(\mathrm{Cay})) = \frac{11\cdot 2^{1/2}}{2949120\cdot\pi^{1/2}}.$$

2. Elements of harmonic analysis

The real projective spaces $\mathbb{P}^d(\mathbb{R})$ can be obtained by identifying the antipodal points on $\mathbb{S}^d$. This quotient space of the sphere is homeomorphic with the collection of all lines passing through the origin in $\mathbb{R}^d$. Also, $\mathbb{P}^d(\mathbb{R})$ can be defined as the cosets of the orthogonal group $\mathbf{O}(d+1)$, i.e.

$$\mathbb{P}^d(\mathbb{R}) = \frac{\mathbf{O}(d+1)}{\mathbf{O}(1)\times\mathbf{O}(d)}.$$

Let

$$\pi:\ \mathbf{O}(d+1) \to \frac{\mathbf{O}(d+1)}{\mathbf{O}(1)\times\mathbf{O}(d)}$$

be the natural mapping and $\mathbf{e}$ be the identity of $\mathbf{O}(d+1)$. The point $\mathbf{o} = \pi(\mathbf{e})$, which is invariant under all motions of $\mathbf{O}(1)\times\mathbf{O}(d)$ is called the pole (or the north pole) of $\mathbb{P}^d(\mathbb{R})$. On $\mathbb{P}^d(\mathbb{R})$ there is an invariant Riemannian metric $d(\cdot,\cdot)$, an invariant Haar measure $d\nu$ and an invariant second order differential operator, the Laplace-Beltrami operator Δ. A function $Z(\cdot) : \mathbb{P}^d(\mathbb{R}) \to \mathbb{R}$ is called zonal if $Z(h^{-1}\cdot) = Z(\cdot)$ for any $h \in \mathbf{O}(1)\times\mathbf{O}(d)$. For more details see, e.g., Cartan [1], Gangolli [3], and Helgason [5, 6].

A function on $\mathbb{P}^d(\mathbb{R})$ is invariant under the left action of $\mathbf{O}(1)\times\mathbf{O}(d)$ on $\mathbb{P}^d(\mathbb{R})$ if and only if it depends only the distance of its argument from $\mathbf{o}$. Since the distance of any point of $\mathbb{P}^d(\mathbb{R})$ from $\mathbf{o}$ is at most $\pi/2$, it follows that a spherical function Z on $\mathbb{P}^d(\mathbb{R})$ can be identified with a function $\tilde{Z}$ on $[0,\pi/2]$. Let θ be the distance of a point from $\mathbf{o}$. We may choose a geodesic polar coordinate system $(\theta,\mathbf{u})$, where $\mathbf{u}$ is an angular parameter. In this coordinate system, the radial part Δ_θ of the Laplace-Beltrami operator Δ has the expression

$$\Delta_\theta = \frac{1}{A(\theta)}\ \frac{d}{d\theta}\left(A(\theta)\frac{d}{d\theta}\right),$$

where $A(\theta)$ is the area of the sphere of radius θ in $\mathbb{P}^d(\mathbb{R})$. It is interesting to remark that an explicit form the function $A(\theta)$ can be computed using methods of Lie algebras (see Helgason [6], p.251, [5], p.168 for the details). It can be shown that

$$A(\theta) = \omega_d(\sin\theta)^{d-1},$$

where ω_d is the area of the unit sphere in $\mathbb{R}^d$. Now we can write the operator Δ_θ (up to some numerical constant) in the form

$$\Delta_\theta = \frac{1}{(\sin\theta)^{d-1}}\ \frac{d}{d\theta}(\sin\theta)^{d-1}\ \frac{d}{d\theta}.$$

Using a simple change of variables $t=\cos\theta$, this operator takes the form (up to a positive multiple),

$$\Delta_t = (1-t^2)^{-(d-2)/2}\ \frac{d}{dt}(1-t^2)^{d/2}\ \frac{d}{dt}. \tag{2.1}$$

We will need the following statement Szegö [9], p.60:

Lemma 2.1 *The Jacobi polynomials $y = P_k^{(\alpha,\beta)}$ satisfy the following linear homogeneous differential equation of the second order:*

$$\frac{d}{dt}((1-t)^{\alpha+1}(1-t)^{\beta+1}y^{'}) + k(k+\alpha+\beta+1)(1-t)^{\alpha}(1+t)^{\beta}y = 0.$$

Hence, the eigenfunctions of the operator Δ_t, which has been defined in (2.1) are well-known Jacobi polynomials $P_k^{(\alpha,\beta)}(t)$, and the corresponding eigenvalues are $\theta_k = -k(k+\alpha+\beta+1)$, where $\alpha=\beta=(d-2)/2$. In this way, zonal functions on $\mathbb{P}^d(\mathbb{R})$ can be easily identified since the elementary zonal functions are eigenfunctions of the Laplace–Beltrami operator. Note that, on the real projective spaces, $\mathbb{P}^d(\mathbb{R})$, the only polynomials of even degree, appear because, due to the identification of antipodal points on $\mathbb{S}^d$, only the even order polynomials $P_{2k}^{(\alpha,\alpha)}$, $k=0,1,2,\cdots$ can be lifted to be functions on $\mathbb{P}^d(\mathbb{R})$. Let Z_{2k}, $k\in\mathbb{N}$, with $Z_0\equiv 1$ be a zonal function corresponding to the eigenvalue $\theta_{2k} = -2k(2k+d-1)$ and $\tilde{Z}_{2k}$ be the corresponding functions induced on $[0,\pi/2]$ by Z_{2k}. Then, Koornwinder [7],

$$\tilde{Z}_{2k}(\theta) = C_{2k}\left(\mathbb{P}^d(\mathbb{R})\right)P_{2k}^{((d-2)/2,(d-2)/2)}(\cos\theta). \tag{2.2}$$

Remark that, for any $k\in\mathbb{N}$, the polynomial $P_k^{((d-2)/2,(d-2)/2)}$ is just a multiple of the Gegenbauer polynomial $P_k^{(d-1)/2}$. A detailed treatment of the Jacobi polynomials can be found in Szegö [9]. In particular, the Jacobi polynomials $P_k^{(\alpha,\beta)}(t)$, $\alpha > -1$, $\beta > -1$ are orthogonal with respect to $\omega^{\alpha,\beta}(t) = c^{-1}(1-t)^{\alpha}(1+t)^{\beta}$ on $(-1,1)$. The above constant c can be found using the normalization condition $\int_{\mathbb{P}^d(\mathbb{R})} d\nu = 1$ for the invariant measure $d\nu$ on $\mathbb{P}^d(\mathbb{R})$ and a well-known formula for the Euler integral of the first kind

$$\mathrm{B}(p,q) = \int_0^1 \xi^{p-1}(1-\xi)^{q-1}d\xi = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \ \ p>0, \ q>0. \tag{2.3}$$

Applying (2.3) and a simple change of variables we get

$$1 = \int_{\mathbb{P}^d(\mathbb{R})} d\nu = \int_0^1 \omega^{(d-2)/2,(d-2)/2}(t)dt = c^{-1}\int_0^1 (1-t^2)^{(d-2)/2}dt,$$

so that,

$$c = \int_0^1 (1-t^2)^{(d-2)/2}dt = 2^{d-2}\frac{(\Gamma(d/2))^2}{\Gamma(d)}. \tag{2.4}$$

We normalize the Jacobi polynomials as follows:

$$P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)\Gamma(k+1)}.$$

This way of normalization is coming from the definition of Jacoby polynomials using the generating function Szegö [9], p.69. In particular,

$$P_{2k}^{((d-2)/2,(d-2)/2)}(1) = \frac{\Gamma(2k+d/2)}{\Gamma(d/2)\Gamma(2k+1)}.$$

The Hilbert space $L_2(\mathbb{P}^d(\mathbb{R}))$ with usual scalar product

$$\langle f,g\rangle = \int_{\mathbb{P}^d(\mathbb{R})} f(x)\overline{g(x)}d\nu(x)$$

has the decomposition

$$L_2(\mathbb{P}^d(\mathbb{R})) = \bigoplus_{k=0}^{\infty} \mathrm{H}_{2k},$$

where H_{2k} is the eigenspace of the Laplace–Beltrami operator corresponding to the eigenvalue $\theta_{2k} = -2k(2k+\alpha+\beta+1)$. Let $\{Y_j^{2k}\}_{j=1}^{d_{2k}}$ be an orthonormal basis of H_{2k}. The following addition formula is known, Koornwinder [7],

$$\sum_{j=1}^{d_{2k}} Y_j^{2k}(x)\overline{Y_j^{2k}(y)} = \tilde{Z}_{2k}(\cos\theta), \tag{2.5}$$

where $\theta = d(x,y)$ or comparing (2.5) with (2.2) we get

$$\sum_{j=1}^{d_{2k}} Y_j^{2k}(x)\overline{Y_j^{k}(y)} = \tilde{Z}_k(\cos\theta) = C_{2k}(\mathbb{P}^d(\mathbb{R}))P_{2k}^{(\alpha,\beta)}(\cos\theta). \tag{2.6}$$

See Helgason [5, 6], Cartan [1], Koornwinder [7], and Gangolli [3] for more information concerning the harmonic analysis on homogeneous spaces.

3. The result

Theorem 3.1 *In our notations*

$$L_{2n}\left(\mathbb{P}^d(\mathbb{R})\right) = n^{(d-1)/2}\frac{2\Gamma\left(\frac{d-1}{4}\right)}{\pi\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+1}{4}\right)} + O\left\{\begin{array}{ll} n^{(d-2)/2}, & d=2, \\ n^{(d-3)/2}, & d\geq 3 \end{array}\right\},\ d=2,3,4,\cdots.$$

Proof We will need an explicit representation for the constant $C_{2k}(\mathbb{P}^d(\mathbb{R}))$ defined in (2.6). Putting $y=x$ in (2.6) and then integrating both sides with respect to $d\nu(x)$ we get

$$d_{2k} = \dim \mathrm{H}_{2k} = \sum_{j=1}^{d_{2k}} \int_{\mathbb{P}^d(\mathbb{R})} |Y_j^{2k}(x)|^2 d\nu(x)$$

$$= C_{2k}(\mathbb{P}^d(\mathbb{R}))P_{2k}^{((d-2)/2,(d-2)/2)}(1). \tag{3.1}$$

Taking the square of both sides of (2.6) and then integrating with respect to $d\nu(x)$ we find

$$\sum_{j=1}^{d_{2k}} |Y_j^{2k}(y)|^2 = C_{2k}^2(\mathbb{P}^d(\mathbb{R})) \int_{\mathbb{P}^d(\mathbb{R})} \left(P_{2k}^{((d-2)/2,(d-2)/2)}(\cos d(x,y))\right)^2 d\nu(x). \tag{3.2}$$

Since $d\nu$ is shift invariant then

$$\int_{\mathbb{P}^d(\mathbb{R})} \left(P_{2k}^{((d-2)/2,(d-2)/2)}(\cos(d(x,y)))\right)^2 d\nu(x) = c^{-1}\left\|P_{2k}^{((d-2)/2,(d-2)/2)}\right\|_2^2,$$

where the constant c is defined by (2.4) and (see Szegö [9], p.68)

$$\left\|P_{2k}^{((d-2)/2,(d-2)/2)}\right\|_2^2 = \int_0^1 \left(P_{2k}^{((d-2)/2,(d-2)/2)}(t)\right)^2 (1-t^2)^{(d-2)/2} dt$$

$$= \frac{2^{d-2}}{4k+d-1}\ \frac{(\Gamma(2k+d/2))^2}{\Gamma(2k+1)\Gamma(2k+d-1)}.$$

So that, (3.2) can be written in the form

$$\sum_{j=1}^{d_{2k}} |Y_j^{2k}(y)|^2 = c^{-1}C_{2k}^2(\mathbb{P}^d(\mathbb{R}))\left\|P_{2k}^{((d-2)/2,(d-2)/2)}\right\|_2^2.$$

Integrating the last line with respect to $d\nu(y)$ we obtain

$$d_{2k} = c^{-1}C_{2k}^2(\mathbb{P}^d(\mathbb{R}))\left\|P_{2k}^{(\alpha,\beta)}\right\|_2^2.$$

It is sufficient to compare this with (3.1) to obtain

$$C_{2k}(\mathbb{P}^d(\mathbb{R})) = \frac{cP_{2k}^{((d-2)/2,(d-2)/2)}(1)}{\left\|P_{2k}^{((d-2)/2,(d-2)/2)}\right\|_2^2}. \tag{3.3}$$

We get now an integral representation for the Fourier sums $S_{2n}(\varphi,x)$ of a function $\varphi\in L_\infty(\mathbb{P}^d(\mathbb{R}))$,

$$S_{2n}(\varphi,x)=c_0(\varphi)+\sum_{k=1}^{n}\sum_{j=1}^{d_{2k}}c_{2k,j}(\varphi)Y_j^{2k}(x)$$

$$=\int_{\mathbb{P}^d(\mathbb{R})}\varphi(y)\overline{Y_1^0(y)}d\nu(y)+\sum_{k=1}^{n}\sum_{j=1}^{d_{2k}}\left(\int_{\mathbb{P}^d(\mathbb{R})}\varphi(y)\overline{Y_j^{2k}(y)}d\nu(y)\right)Y_j^{2k}(x)$$

$$=\int_{\mathbb{P}^d(\mathbb{R})}\sum_{k=0}^{n}\left(\sum_{j=1}^{d_{2k}}\overline{Y_j^{2k}(y)}Y_j^{2k}(x)\right)\varphi(y)d\nu(y)$$

$$=\int_{\mathbb{P}^d(\mathbb{R})}\sum_{k=0}^{n}Z_{2k}^x(y)\varphi(y)d\nu(y)$$

$$=\int_{\mathbb{P}^d(\mathbb{R})}K_{2n}(x,y)\varphi(y)d\nu(y), \tag{3.4}$$

where

$$K_{2n}(x,y)=\sum_{k=0}^{n}Z_{2k}^x(y). \tag{3.5}$$

By (2.2) and (3.3) we have

$$K_{2n}(x,y)=c\sum_{k=0}^{n}\frac{P_{2k}^{((d-2)/2,(d-2)/2)}(1)}{\left\|P_{2k}^{((d-2)/2,(d-2)/2)}\right\|_2^2}\;P_{2k}^{((d-2)/2,(d-2)/2)}(\cos d(x,y)).$$

Let us denote

$$G_n^{(\alpha,\beta)}(\gamma,\delta)=\sum_{k=0}^{n}\frac{P_k^{(\alpha,\beta)}(\gamma)P_k^{(\alpha,\beta)}(\delta)}{\left\|P_k^{(\alpha,\beta)}\right\|_{2,*}^2},$$

where

$$\left\|P_k^{(\alpha,\beta)}\right\|_{2,*}^2=\int_{-1}^{1}\left(P_k^{(\alpha,\beta)}(t)\right)^2(1-t)^\alpha(1+t)^\beta dt$$

Then by Szegö [9], p.71,

$$G_n^{(\alpha,\beta)}(\gamma,1)=\sum_{k=0}^{n}\frac{P_k^{(\alpha,\beta)}(\gamma)P_k^{(\alpha,\beta)}(1)}{\left\|P_k^{(\alpha,\beta)}\right\|_{2,*}^2}$$

$$=2^{-\alpha-\beta-1}\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)}P_n^{(\alpha+1,\beta)}(\gamma). \tag{3.6}$$

Remark that, Szegö [9],

$$P_k^{(\alpha,\beta)}(\gamma)=(-1)^kP_k^{(\beta,\alpha)}(-\gamma) \tag{3.7}$$

for any $\gamma \in \mathbb{R}$ and $k \in \mathbb{N}$. By the definitions of the norms $\|\cdot\|_2$ and $\|\cdot\|_{2,*}$

$$\left\| P_{2k}^{((d-2)/2,(d-2)/2)} \right\|_{2,*}^2 = 2 \left\| P_{2k}^{((d-2)/2,(d-2)/2)} \right\|_2^2, \tag{3.8}$$

for any $k \in \mathbb{N}$ since $P_{2k}^{((d-1)/2,(d-1)/2)}$ is an even function. Comparing (3.6) - (3.8) we get an explicit representation for the kernel function (3.5) in the integral representation (3.4), i.e.,

$$K_{2n}(x,y) = c2^{-\alpha-\beta-1}2\frac{\Gamma(2n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(2n+\beta+1)} \times \frac{P_{2n}^{(\alpha+1,\beta)}(\cos d(x,y)) + P_{2n}^{(\beta,\alpha+1)}(\cos d(x,y))}{2}$$

$$= c2^{-d+1}\frac{\Gamma(2n+d)}{\Gamma(d/2)\Gamma(2n+d/2)} \times \left(P_{2n}^{(d/2,(d-2)/2)}(\cos d(x,y)) + P_{2n}^{((d-2)/2,d/2)}(\cos d(x,y))\right) \tag{3.9}$$

since $\alpha = \beta = (d-2)/2$. It is known, Szegö [9], p.196, that for $0 < \eta < \pi$,

$$P_n^{(\alpha,\beta)}(\cos\eta) = n^{-1/2}\ \kappa^{(\alpha,\beta)}(\eta)\cos(N\eta+\gamma) + O(n^{-3/2}), \tag{3.10}$$

where

$$\kappa^{(\alpha,\beta)}(\eta) = \pi^{-1/2}\left(\sin\frac{\eta}{2}\right)^{-\alpha-1/2}\left(\cos\frac{\eta}{2}\right)^{-\beta-1/2},$$

$$N = n + \frac{\alpha+\beta+1}{2} = n + \frac{d-1}{2},$$

and

$$\gamma = -\frac{\alpha+1/2}{2}\pi.$$

Let $\eta = d(x,y)$ and $\mathbf{o}$ be the north pole of $\mathbb{P}^d(\mathbb{R})$, then from (3.9), (3.10) and since K_{2n} is a zonal function and $d\nu$ is shift invariant we get

$$\|S_{2n}\left|C\left(\mathbb{P}^d(\mathbb{R})\right) \to C\left(\mathbb{P}^d(\mathbb{R})\right)\right\| = \sup\left\{\int_{\mathbb{P}^d(\mathbb{R})} |K_{2n}(x,y)| d\nu(y) :\ x \in \mathbb{P}^d(\mathbb{R})\right\}$$

$$= \int_{\mathbb{P}^d(\mathbb{R})} |K_{2n}(\mathbf{o},y)| d\nu(y)$$

$$= \frac{c2^{-d+1}\Gamma(2n+d)}{\Gamma(d/2)\Gamma(2n+d/2)} \times \int_{\mathbb{P}^d(\mathbb{R})} \left| P_{2n}^{(d/2,(d-2)/2)}(\cos(d(\mathbf{o},y))) + P_{2n}^{((d-2)/2,d/2)}(\cos(d(\mathbf{o},y)))\right| d\nu(y)$$

$$= \frac{2^{-d+1}\Gamma(2n+d)}{\Gamma(d/2)\Gamma(2n+d/2)} I_n$$

where

$$I_n := \int_0^1 \left| P_{2n}^{(d/2,(d-2)/2)}(t) + P_{2n}^{((d-2)/2,d/2)}(t)\right| (1-t^2)^{(d-2)/2} dt$$

$$= \int_0^{\pi/2} \left| P_{2n}^{(d/2,(d-2)/2)}(\cos\eta) + P_{2n}^{((d-2)/2,d/2)}(\cos\eta)\right| (\sin\eta)^{d-1} dt$$

$$= \frac{2^{d/2+1/2}}{\pi^{1/2}(2n)^{1/2}} \int_0^{\pi/2} (\sin\eta)^{(d-3)/2} \left| \cos\left(\left(2n + \frac{d-1}{2} \right) \eta - \frac{(d+1)\pi}{4} \right) \right| d\eta + O(n^{-3/2}).$$

Applying a simple Tylor series arguments and an elementary estimates of the derivative of the function $(\sin\eta)^{(d-3)/2}$, we get

$$I_n = \frac{2^{d/2+1}}{\pi^{3/2} n^{1/2}} \int_0^{\pi/2} (\sin\eta)^{(d-3)/2} d\eta + \left\{ \begin{array}{ll} O(n^{-1/2}), & d = 2, \\ O(n^{-1}), & d \geq 3 \end{array} \right\}.$$

□

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