

Existence of nonnegative solutions for discrete Robin boundary value problems with sign-changing weight

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Abstract: In this paper, we are concerned with the following discrete problem first

$$\begin{cases} -\Delta^2 u(t-1) = \lambda p(t)f(u(t)), & t \in [1, N-1]_{\mathbb{Z}}, \\ \Delta u(0) = u(N) = 0, \end{cases}$$

where $N > 2$ is an integer, $\lambda > 0$ is a parameter, $p : [1, N-1]_{\mathbb{Z}} \rightarrow \mathbb{R}$ is a sign-changing function, $f : [0, +\infty) \rightarrow [0, +\infty)$ is a continuous and nondecreasing function. $\Delta u(t) = u(t+1) - u(t)$, $\Delta^2 u(t) = \Delta(\Delta u(t))$. By using the iterative method and Schauder's fixed point theorem, we will show the existence of nonnegative solutions to the above problem. Furthermore, we obtain the existence of nonnegative solutions for discrete Robin systems with indefinite weights.

Key words: Robin boundary value problems, Green's function, iterative method, nonnegative solutions, system

1. Introduction

The boundary value problems with sign-changing weight arise from a selection migration model in population genetics, see Fleming [7]. The weight changes sign corresponds to the fact that an allele A1 holds an advantage over a rival allele A2 at the same points and is at a disadvantage at others. Therefore, it is of great significance to study the existence and multiplicity of positive solutions of the problems with sign-changing weight in continuous functions, which has attracted the attention of many scholars. They obtain a lot of meaningful results, see, for example, [1,5,10,11,13] and the references therein.

For the discrete case, many authors have studied the existence of positive solutions to boundary value problems in various boundary conditions when the weight function is invariant. By using different methods, such as the Krasnosel'skii's fixed point theorem in a cone, bifurcation methods, and fixed point theory, the authors obtain the existence of positive solutions of the boundary value problems for second-order discrete equations. We refer to [2-4,6,8,9,12] and the references therein.

In [12], Ma et al. study the global structure of positive solutions of the discrete problem

$$\begin{cases} -\Delta^2 u(t-1) = \lambda p(t)f(u(t)), & t \in [1, N]_{\mathbb{Z}}, \\ u(0) = u(N+1) = 0, \end{cases} \quad (1.1)$$

where λ is a positive parameter, $p : [0, N+1] \rightarrow [0, \infty)$ and $f : [0, \infty) \rightarrow [0, \infty)$ is continuous. The authors obtained the existence of positive solutions of (1.1) by using the Rabinowitz's global bifurcation theorem.

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There are few studies on the existence of positive solutions for boundary value problems when the weight function is sign-changing, since the strong maximum principle dose not be applied. In [2], Bai et al. study the existence of positive solutions of the discrete Neumann boundary value problem

$$\begin{cases} -\Delta^2 u(t-1) = f(t, u(t)), & t \in [1, N]_{\mathbb{Z}}, \\ \Delta u(0) = \Delta u(N) = 0, \end{cases} \tag{1.2}$$

where $f : [1, N]_{\mathbb{Z}} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is a sign-changing function. They first assume that

(C1) there exists a function $h : [1, N]_{\mathbb{Z}} \rightarrow \mathbb{R}^+$ with $h(t) \not\equiv 0$ on $[1, N]_{\mathbb{Z}}$, and a constant number $L > 0$ such that

$$f(t, z) + Lz + h(t) \geq 0, \quad (t, z) \in [1, N]_{\mathbb{Z}} \times \mathbb{R}^+.$$

Furthermore, they use the Guo–Krasnosel’skii fixed point theorem to obtain the existence of positive solutions to (1.2) under some complex conditions including (C1). In [9], Jiang et al. study the positive solutions for a discrete system

$$\begin{cases} -\Delta^2 u_i(t-1) = f_i(t, u_1(t), u_2(t)), & t \in [1, N]_{\mathbb{Z}}, \\ \Delta u_i(0) = \Delta u_i(N) = 0, & i = (1, 2) \end{cases} \tag{1.3}$$

under similar conditions.

However, the conditions in [2] are complex. When we consider the boundary value problem with sign-changing weight, can we obtain the existence of positive solutions to the boundary value problem under some simple conditions? It is the purpose of the present paper is to study the existence of nonnegative solutions for the following Robin boundary value problems

$$\begin{cases} -\Delta^2 u(t-1) = \lambda p(t)f(u(t)), & t \in [1, N-1]_{\mathbb{Z}}, \\ \Delta u(0) = u(N) = 0, \end{cases} \tag{1.4}$$

where $N > 2$ is an integer, $\lambda > 0$ is a parameter, $p : [1, N-1]_{\mathbb{Z}} \rightarrow \mathbb{R}$ is a sign-changing function, $f : [0, +\infty) \rightarrow [0, +\infty)$ is a continuous and nondecreasing function. By using the iterative method, we will show the existence of nonnegative solutions to the above problem. Furthermore, we also obtain the existence of nonnegative solutions for following discrete coupled systems

$$\begin{cases} -\Delta^2 u(t-1) = \lambda p(t)f(v(t)), & t \in [1, N-1]_{\mathbb{Z}}, \\ -\Delta^2 v(t-1) = \lambda q(t)g(u(t)), & t \in [1, N-1]_{\mathbb{Z}}, \\ \Delta u(0) = u(N) = 0, \\ \Delta v(0) = v(N) = 0. \end{cases} \tag{1.5}$$

Denote

$$p^+ = \begin{cases} p(t), & p(t) \geq 0, \quad t \in [1, N-1], \\ 0, & p(t) < 0, \quad t \in [1, N-1], \end{cases} \quad p^- = \begin{cases} 0, & p(t) > 0, \quad t \in [1, N-1], \\ -p(t), & p(t) \leq 0, \quad t \in [1, N-1]. \end{cases}$$

There are similar definitions for q^+ , q^- .

The paper is organized as follows. In Section 2 we introduce some preliminary results. In Section 3 we state and prove our main results on the existence of nonnegative solution of (1.4). Finally, in Section 4, we will prove our main results on the existence of nonnegative solution of (1.5).

2. Preliminary results

Let

$$X = \{u|u : [0, N]_{\mathbb{Z}} \rightarrow \mathbb{R}, \Delta u(0) = u(N) = 0\}$$

be a Banach space with the norm $\|u\| = \max_{i \in [0, N+1]_{\mathbb{Z}}} |u(i)|$.

It is easy to see that a solution to (1.4) is a fixed point of the operator

$$T\theta(t) = \lambda \sum_{s=1}^{N-1} G(t, s)p(s)f(\theta(s)), \quad t \in [0, N]_{\mathbb{Z}}, \tag{2.1}$$

where

$$G(t, s) = \begin{cases} T - t, & 1 \leq s \leq t \leq N, \\ T - s, & 0 \leq t \leq s \leq N - 1. \end{cases}$$

It is easy to check that $G(t, s) \geq 0$.

We will make use of the following assumptions:

- (H1) $p : [1, N - 1]_{\mathbb{Z}} \rightarrow \mathbb{R}$ is a sign-changing function;
- (H2) $f : [0, +\infty) \rightarrow [0, +\infty)$ is a continuous and nondecreasing function;
- (H3) there is an $\mu > 0$ so that

$$\sum_{s=1}^t p^+(s) \geq (1 + \mu) \sum_{s=1}^t p^-(s), \quad t \in [1, N - 1]_{\mathbb{Z}}.$$

Lemma 2.1 *Assume that (H1) and (H2) hold. Let $A = \{t|p(t) \geq 0\}$ and $B = \{t|p(t) < 0\}$. Suppose we have bounded functions $\varphi_0(t), \psi_0(t), t \in [0, N]_{\mathbb{Z}}$ such that they satisfy*

- (i) $0 \leq \varphi_0 \leq \psi_0$ on A , $0 \leq \psi_0 \leq \varphi_0$ on B ;
- (ii) $T\psi_0 \leq \psi_0$ on A , $T\psi_0 \leq \varphi_0$ on B ;
- (iii) $T\varphi_0 \geq \varphi_0$ on A , $T\varphi_0 \geq \psi_0$ on B .

Define

$$(iv) \quad \varphi_1(t) = \begin{cases} T\varphi_0(t) & \text{on } A, \\ T\psi_0(t) & \text{on } B; \end{cases} \quad \psi_1(t) = \begin{cases} T\psi_0(t) & \text{on } A, \\ T\varphi_0(t) & \text{on } B. \end{cases}$$

Then φ_1, ψ_1 also satisfy (i), (ii), (iii).

Proof We note that the operator T can be written as

$$T\varphi(t) = \lambda \sum_{s \in A} G(t, s)p^+(s)f(\varphi(s)) - \lambda \sum_{s \in B} G(t, s)p^-(s)f(\varphi(s)). \tag{2.2}$$

For convenience we write

$$T\varphi(t) = T_1\varphi(t) - T_2\varphi(t), \tag{2.3}$$

where $T_i\varphi$, $i = 1, 2$ are both monotone in the sense that $\varphi \leq \psi$ implies $T_i\varphi \leq T_i\psi$.

According to (i), note that

$$T\varphi_0 = T_1\varphi_0 - T_2\varphi_0 \leq T_1\psi_0 - T_2\psi_0 = T\psi_0. \tag{2.4}$$

From this we can directly get that φ_1 and ψ_1 satisfies (i).

Using (ii) and (iii), we can get that

$$T\psi_1 = T_1(T\psi_0) - T_2(T\varphi_0) \leq T_1(\psi_0) - T_2(\varphi_0) = T\psi_0. \tag{2.5}$$

For ψ_1 , when it satisfies condition (ii), it means that on A , $T\psi_1 \leq \psi_1 = T\psi_0$ and on B , $T\psi_1 \leq \varphi_1 = T\psi_0$, i.e. $T\psi_1 \leq T\psi_0$. Obviously, this inequality can be satisfied.

Similarly, conditions (iii) for φ_1 is that $T\varphi_1 \geq \varphi_1 = T\varphi_0$ on A and $T\varphi_1 \geq \psi_1 = T(\varphi_0)$ on B . Using (ii) and (iii) again, we can get that

$$T\varphi_1 = T_1(T\varphi_0) - T_2(T\psi_0) \geq T_1(\varphi_0) - T_2(\psi_0) = T\varphi_0. \tag{2.6}$$

□

3. Existence of nonnegative solution for (1.4)

Theorem 3.1 Assume (H1)–(H2) hold, and suppose there are functions φ_0, ψ_0 that satisfy (i), (ii), and (iii). Then the problem (1.4) has a solution.

Proof We define

$$\varphi_{n+1} = \begin{cases} T\varphi_n & \text{on } A, \\ T\psi_n & \text{on } B, \end{cases} \quad \psi_{n+1} = \begin{cases} T\psi_n & \text{on } A, \\ T\varphi_n & \text{on } B. \end{cases} \tag{3.1}$$

By the Lemma 2.1 and induction, (φ_n, ψ_n) satisfies (i), (ii), and (iii). Hence, we can get the following inequality through (2.4), (2.5), (2.6). i.e.

$$0 \leq T\varphi_n \leq T\varphi_{n+1} \leq T\psi_{n+1} \leq T\psi_n \leq T\psi_0.$$

Thus $T\varphi_n \uparrow \bar{\varphi}$ and $T\psi_n \downarrow \bar{\psi}$ pointwise and $\bar{\varphi} \leq \bar{\psi}$.

Since

$$T\varphi_{n+1} = T_1(T\varphi_n) - T_2(T\psi_n),$$

we can easily have

$$\bar{\varphi} = T_1(\bar{\varphi}) - T_2(\bar{\psi}). \tag{3.2}$$

Similarly,

$$\bar{\psi} = T_1(\bar{\psi}) - T_2(\bar{\varphi}). \tag{3.3}$$

We have $\varphi_{n+1} \rightarrow \bar{\varphi}$ on A by definition of $\bar{\varphi}$. On the other hand, from (3.1), $\varphi_{n+1} = T\varphi_n$ on A , so $\varphi_{n+1} \rightarrow T\bar{\varphi}$. Thus $\bar{\varphi} = T\bar{\varphi}$ on A . On B we have $\psi_{n+1} = T\varphi_n \rightarrow \bar{\varphi}$ by definition of $\bar{\varphi}$. But $\varphi_{n+1} = T\psi_n$ on B , so $\varphi_{n+1} \rightarrow T\bar{\psi}$ on B . Thus $T\bar{\psi} = \bar{\varphi}$ on B . In a similar way we have

$$\bar{\varphi} = \begin{cases} T\bar{\varphi} & \text{on } A, \\ T\bar{\psi} & \text{on } B; \end{cases} \quad \bar{\psi} = \begin{cases} T\bar{\psi} & \text{on } A, \\ T\bar{\varphi} & \text{on } B, \end{cases}$$

and $\bar{\varphi}$ and $\bar{\psi}$ are fixed points of T^2 .

Now consider the convex region defined by

$$K = \{u(i) \in X | \bar{\varphi}(i) \leq u(i) \leq \bar{\psi}(i)\}.$$

From (3.3), we have

$$Tu = T_1(u) - T_2(u) \leq T_1(\bar{\psi}) - T_2(\bar{\varphi}) = \bar{\psi}.$$

Similarly, we get

$$Tu = T_1(u) - T_2(u) \geq T_1(\bar{\varphi}) - T_2(\bar{\psi}) = \bar{\varphi}$$

by (3.2).

These imply that $T(K) \subset K$. Because X is a finite-dimensional space, it is easy to show T restricted to K is a compact operator. Thus we know that T has a fixed point by Schauder's fixed point theorem. \square

Remark 3.2 For order interval $\{u \in X | T\varphi_0 \leq u \leq T\psi_0\}$, it is already invariant and we could use Schauder's fixed point theorem directly. But the iteration improves the estimates. In general, the functions $\bar{\varphi}$ and $\bar{\psi}$ are the same. So we could actually construct the solution numerically.

Remark 3.3 If $f \geq 0$ and nonincreasing, then we take the following modified versions:

(i)*

$$0 \leq \varphi \leq \psi \quad \text{on } A, \quad 0 \leq \psi \leq \varphi \quad \text{on } B;$$

(ii)*

$$T\varphi \leq \psi \quad \text{on } A, \quad T\varphi \leq \varphi \quad \text{on } B;$$

(iii)*

$$T\psi \geq \varphi \quad \text{on } A, \quad T\psi \geq \psi \quad \text{on } B.$$

Define

(iv)*

$$\varphi_1(t) = \begin{cases} T\psi_0(t) & \text{on } A, \\ T\varphi_0(t) & \text{on } B, \end{cases} \quad \psi_1(t) = \begin{cases} T\varphi_0(t) & \text{on } A, \\ T\psi_0(t) & \text{on } B. \end{cases}$$

We can prove the theorem of existence where now $T\varphi_n \geq T\psi_n$, then $T\varphi_n \downarrow \bar{\varphi}$ and $T\psi_n \uparrow \bar{\psi}$. The details are the same.

Theorem 3.4 If $f(0) > 0$, Let (H1)-(H3) hold, then the problem (1.4) has a nonnegative solution for $0 \leq \lambda \leq \lambda_0$, where

$$\lambda_0 = \frac{\alpha}{f(0)[\mu \sum_{s=1}^{N-1} G(1, s)p^-(s) + (1 + \mu) \sum_{s=1}^{N-1} (N - s)p(s)]}$$

if $f(\alpha) \leq f(0)(1 + \mu)$.

Proof We will seek φ_0 and ψ_0 so that (i), (ii), and (iii) are satisfied in Lemma 2.1.

Let $\varphi_0(t) = \alpha$ on B and $\varphi_0(t) = 0$ on A , $\psi_0(t) = \alpha$ on A and $\psi_0(t) = 0$ on B .

Then condition (i) is satisfied if $\alpha \geq 0$. Now the condition (ii) is

$$T\psi_0 = T_1(\alpha) - T_2(0) \leq \alpha$$

while condition (iii) is

$$T\varphi_0 = T_1(0) - T_2(\alpha) \geq 0.$$

Let $z^+(t) = \sum_{s=1}^{N-1} G(t, s)p^+(s)$ and $z^-(t) = \sum_{s=1}^{N-1} G(t, s)p^-(s)$. Then the conditions become

$$\lambda[z^+(t)f(\alpha) - z^-(t)f(0)] \leq \alpha \tag{3.4}$$

and

$$\lambda[z^+(t)f(0) - z^-(t)f(\alpha)] \geq 0. \tag{3.5}$$

We consider (3.5) first, and define $z(t) = z^+(t) - (1 + \mu)z^-(t)$. Then z is a solution of

$$\begin{cases} -\Delta^2 z(t-1) = p^+(t) - (1 + \mu)p^-(t), & t \in [1, N-1]_{\mathbb{Z}}, \\ \Delta z(0) = z(N) = 0. \end{cases}$$

Then

$$\Delta z(t) = -\sum_{s=1}^t [p^+(s) - (1 + \mu)p^-(s)] \leq 0$$

by (H3).

Thus z is decreasing and therefore is nonnegative. Then

$$z^+(t) \geq (1 + \mu)z^-(t), \quad t \in [1, N-1]_{\mathbb{Z}}.$$

So (3.5) is satisfied if

$$f(\alpha) \leq (1 + \mu)f(0). \tag{3.6}$$

We select such an α and argue that (3.4) can now be satisfied for small λ . To prove this we will give an explicit estimate. Now as above

$$z^+(t) - z^-(t) = \sum_{s=1}^{N-1} G(t, s)p(s), \quad t \in [1, N-1]_{\mathbb{Z}}, \tag{3.7}$$

and the right-side of (3.7) is a decreasing function, therefore

$$z^+(t) \leq z^-(t) + \sum_{s=1}^{N-1} G(1, s)p(s).$$

We note that

$$\sum_{s=1}^{N-1} (N-s)p(s) = \beta.$$

Hence

$$\begin{aligned} f(\alpha)z^+(t) - f(0)z^-(t) &\leq [f(\alpha) - f(0)]z^-(t) + f(\alpha)\beta \\ &\leq [f(\alpha) - f(0)]z^-(1) + f(\alpha)\beta \end{aligned}$$

since z^- is decreasing (as above for z).

This is turn is dominated by (3.6)

$$\mu f(0)z^-(1) + (1 + \mu)f(0)\beta.$$

Therefore (3.4) is satisfied if

$$\lambda \leq \frac{\alpha}{f(0)[\mu z^-(1) + (1 + \mu)\beta]},$$

where

$$f(0)[\mu z^-(1) + (1 + \mu)\beta] = f(0)\left[\mu \sum_{s=1}^{N-1} G(1, s)p^-(s) + (1 + \mu) \sum_{s=1}^{N-1} (N - s)p(s)\right].$$

□

4. Existence of nonnegative solution for systems

In this part, we still use the iterative method to study the existence of nonnegative solutions for the system (1.5) under the previous framework.

We will make use of the following assumptions:

- (F1) $p, q : [1, N - 1]_{\mathbb{Z}} \rightarrow \mathbb{R}$ are sign-changing functions;
- (F2) $f, g : [0, +\infty) \rightarrow [0, +\infty)$ are continuous and nondecreasing functions;
- (F3) there is an $\mu_1 > 0$ so that

$$\sum_{s=1}^{N-1} G(t, s)p^+(s) \geq (1 + \mu_1) \sum_{s=1}^{N-1} G(t, s)p^-(s), \quad t \in [1, N - 1]_{\mathbb{Z}};$$

- (F4) there is an $\mu_2 > 0$ so that

$$\sum_{s=1}^{N-1} G(t, s)q^+(s) \geq (1 + \mu_2) \sum_{s=1}^{N-1} G(t, s)q^-(s), \quad t \in [1, N - 1]_{\mathbb{Z}}.$$

Let $\Omega_1 = \{t \in [1, N - 1]_{\mathbb{Z}} | p(t) \geq 0, q(t) \geq 0\}$ and $\Omega_2 = \{t \in [1, N - 1]_{\mathbb{Z}} | p(t) < 0, q(t) < 0\}$. Let $\bar{T} : X \times X \rightarrow X \times X$ defined by

$$\bar{T}(u, v)(t) = \left(\lambda \sum_{s=1}^{N-1} G(t, s)p(s)f(v(s)), \lambda \sum_{s=1}^{N-1} G(t, s)q(s)g(u(s)) \right).$$

We note that the operator \bar{T} can be written as follows:

$$\begin{aligned} \bar{T}(\bar{\varphi}, \bar{\psi})(t) &= \left(\lambda \sum_{s \in \Omega_1} G(t, s)p^+(s)f(\bar{\psi}(s)) - \lambda \sum_{s \in \Omega_2} G(t, s)p^-(s)f(\bar{\psi}(s)), \right. \\ &\quad \left. \lambda \sum_{s \in \Omega_1} G(t, s)q^+(s)g(\bar{\varphi}(s)) - \lambda \sum_{s \in \Omega_2} G(t, s)q^-(s)g(\bar{\varphi}(s)) \right). \end{aligned}$$

For convenience we write

$$\bar{T}(\bar{\varphi}, \bar{\psi})(t) = (F_1\bar{\psi}(t) - F_2\bar{\psi}(t), G_1\bar{\varphi}(t) - G_2\bar{\varphi}(t)), \tag{4.1}$$

where F_1, F_2, G_1, G_2 are monotone by (F2).

Lemma 4.1 *Assume that (F1)–(F2) hold. Suppose we have bounded functions $\Phi_0 = (\varphi_0^{(1)}, \varphi_0^{(2)})$, $\Psi_0 = (\psi_0^{(1)}, \psi_0^{(2)})$ such that they satisfy*

(I)

$$(0, 0) \leq \Phi_0 \leq \Psi_0 \quad \text{on } \Omega_1, \quad (0, 0) \leq \Psi_0 \leq \Phi_0 \quad \text{on } \Omega_2;$$

(II)

$$\bar{T}\Psi_0 \leq \Psi_0 \quad \text{on } \Omega_1, \quad \bar{T}\Psi_0 \leq \Phi_0 \quad \text{on } \Omega_2;$$

(III)

$$\bar{T}\Phi_0 \geq \Phi_0 \quad \text{on } \Omega_1, \quad \bar{T}\Phi_0 \geq \Psi_0 \quad \text{on } \Omega_2.$$

Define

(IV)

$$\Phi_1(t) = \begin{cases} \bar{T}\Phi_0(t) & \text{on } \Omega_1, \\ \bar{T}\Psi_0(t) & \text{on } \Omega_2; \end{cases} \quad \Psi_1(t) = \begin{cases} \bar{T}\Psi_0(t) & \text{on } \Omega_1, \\ \bar{T}\Phi_0(t) & \text{on } \Omega_2. \end{cases}$$

Then Φ_1, Ψ_1 also satisfy (I), (II), (III).

Proof Note that (I) implies that

$$\begin{aligned} \bar{T}\Phi_0 &= \bar{T}(\varphi_0^{(1)}, \varphi_0^{(2)}) \\ &= (F_1\varphi_0^{(2)} - F_2\varphi_0^{(2)}, G_1\varphi_0^{(1)} - G_2\varphi_0^{(1)}) \\ &\leq (F_1\psi_0^{(2)} - F_2\psi_0^{(2)}, G_1\psi_0^{(1)} - G_2\psi_0^{(1)}) \\ &= \bar{T}(\psi_0^{(1)}, \psi_0^{(2)}) \\ &= \bar{T}\Psi_0. \end{aligned}$$

This implies that Φ_1, Ψ_1 satisfy (I).

From (II) and (III), we have

$$((F_1 - F_2)\psi_0^{(2)}, (G_1 - G_2)\psi_0^{(1)}) \leq (\psi_0^{(1)}, \psi_0^{(2)}) \quad \text{on } \Omega_1. \tag{4.2}$$

$$((F_1 - F_2)\psi_0^{(2)}, (G_1 - G_2)\psi_0^{(1)}) \leq (\varphi_0^{(1)}, \varphi_0^{(2)}) \quad \text{on } \Omega_2. \tag{4.3}$$

$$((F_1 - F_2)\varphi_0^{(2)}, (G_1 - G_2)\varphi_0^{(1)}) \geq (\varphi_0^{(1)}, \varphi_0^{(2)}) \quad \text{on } \Omega_1. \tag{4.4}$$

$$((F_1 - F_2)\varphi_0^{(2)}, (G_1 - G_2)\varphi_0^{(1)}) \geq (\psi_0^{(1)}, \psi_0^{(2)}) \quad \text{on } \Omega_2. \tag{4.5}$$

From the definition of Φ_1 and Ψ_1 , we have

$$(\varphi_1^{(1)}, \varphi_1^{(2)}) = \Phi_1 = \begin{cases} ((F_1 - F_2)\varphi_0^{(2)}, (G_1 - G_2)\varphi_0^{(1)}) & \text{on } \Omega_1, \\ ((F_1 - F_2)\psi_0^{(2)}, (G_1 - G_2)\psi_0^{(1)}) & \text{on } \Omega_2. \end{cases} \tag{4.6}$$

$$(\psi_1^{(1)}, \psi_1^{(2)}) = \Psi_1 = \begin{cases} ((F_1 - F_2)\psi_0^{(2)}, (G_1 - G_2)\psi_0^{(1)}) & \text{on } \Omega_1, \\ ((F_1 - F_2)\varphi_0^{(2)}, (G_1 - G_2)\varphi_0^{(1)}) & \text{on } \Omega_2. \end{cases} \quad (4.7)$$

Then we claim that $\bar{T}\Psi_1 \leq \Psi_1$ on Ω_1 and $\bar{T}\Psi_1 \leq \Phi_1$ on Ω_2 . By (4.2–4.7), we have

$$\begin{aligned} \bar{T}\Psi_1 &= ((F_1 - F_2)\psi_1^{(2)}, (G_1 - G_2)\psi_1^{(1)}) \\ &= (F_1(G_1 - G_2)\psi_0^{(1)} - F_2(G_1 - G_2)\varphi_0^{(1)}, G_1(F_1 - F_2)\psi_0^{(2)} - G_2(F_1 - F_2)\varphi_0^{(2)}) \\ &\leq (F_1\psi_0^{(2)} - F_2\varphi_0^{(2)}, G_1\psi_0^{(1)} - G_2\varphi_0^{(1)}) \\ &= ((F_1 - F_2)\psi_0^{(2)}, (G_1 - G_2)\psi_0^{(1)}) \\ &= \Psi_1 \text{ on } \Omega_1. \end{aligned}$$

Also, $\bar{T}\Psi_1 \leq \Phi_1$ on Ω_2 .

Now we claim condition (III), $\bar{T}\Phi_1 \geq \Phi_1$ on Ω_1 and $\bar{T}\Phi_1 \geq \Psi_1$ on Ω_2 .

$$\begin{aligned} \bar{T}\Phi_1 &= ((F_1 - F_2)\varphi_1^{(2)}, (G_1 - G_2)\varphi_1^{(1)}) \\ &= (F_1(G_1 - G_2)\varphi_0^{(1)} - F_2(G_1 - G_2)\psi_0^{(1)}, G_1(F_1 - F_2)\varphi_0^{(2)} - G_2(F_1 - F_2)\psi_0^{(2)}) \\ &\geq (F_1\varphi_0^{(2)} - F_2\psi_0^{(2)}, G_1\varphi_0^{(1)} - G_2\psi_0^{(1)}) \\ &= ((F_1 - F_2)\varphi_0^{(2)}, (G_1 - G_2)\varphi_0^{(1)}) \\ &= \Phi_1 \text{ on } \Omega_1. \end{aligned}$$

Also, $\bar{T}\Phi_1 \geq \Psi_1$ on Ω_2 . This completes the proof of the Lemma 4.1. □

Theorem 4.2 Assume (F1) and (F2) hold, and suppose there are functions Φ_0, Ψ_0 that satisfy (I), (II) and (III). Then the system(1.5) has a solution.

Proof We define

$$(\varphi_{n+1}^{(1)}, \varphi_{n+1}^{(2)}) = \Phi_{n+1} = \begin{cases} \bar{T}\Phi_n & \text{on } \Omega_1, \\ \bar{T}\Psi_n & \text{on } \Omega_2; \end{cases} \quad (\psi_{n+1}^{(1)}, \psi_{n+1}^{(2)}) = \Psi_{n+1} = \begin{cases} \bar{T}\Psi_n & \text{on } \Omega_1, \\ \bar{T}\Phi_n & \text{on } \Omega_2. \end{cases}$$

These could be written as follows:

$$(\varphi_{n+1}^{(1)}, \varphi_{n+1}^{(2)}) = \begin{cases} ((F_1 - F_2)\varphi_n^{(2)}, (G_1 - G_2)\varphi_n^{(1)}) & \text{on } \Omega_1, \\ ((F_1 - F_2)\psi_n^{(2)}, (G_1 - G_2)\psi_n^{(1)}) & \text{on } \Omega_2, \end{cases} \quad (4.8)$$

$$(\psi_{n+1}^{(1)}, \psi_{n+1}^{(2)}) = \begin{cases} ((F_1 - F_2)\psi_n^{(2)}, (G_1 - G_2)\psi_n^{(1)}) & \text{on } \Omega_1, \\ ((F_1 - F_2)\varphi_n^{(2)}, (G_1 - G_2)\varphi_n^{(1)}) & \text{on } \Omega_2. \end{cases} \quad (4.9)$$

By the Lemma 4.1 and induction, (Φ_n, Ψ_n) satisfies (I), (II) and (III). Hence, we can get the following inequality easily. i.e.

$$(0, 0) \leq \bar{T}\Phi_n \leq \bar{T}\Phi_{n+1} \leq \bar{T}\Psi_{n+1} \leq \bar{T}\Psi_n \leq \bar{T}\Psi_0 \leq \Psi_0.$$

Hence,

$$(0, 0) \leq ((F_1 - F_2)\varphi_n^{(2)}, (G_1 - G_2)\varphi_n^{(1)}) \leq ((F_1 - F_2)\psi_{n+1}^{(2)}, (G_1 - G_2)\psi_{n+1}^{(1)}) \leq (\psi_0^{(1)}, \psi_0^{(2)}).$$

Thus $\bar{T}\Phi_n \uparrow \bar{\Phi}$ and $\bar{T}\Psi_n \downarrow \bar{\Psi}$ pointwise, i.e.

$$((F_1 - F_2)\varphi_n^{(2)}, (G_1 - G_2)\varphi_n^{(1)}) \rightarrow (\bar{\varphi}^{(1)}, \bar{\varphi}^{(2)}), \quad ((F_1 - F_2)\psi_n^{(2)}, (G_1 - G_2)\psi_n^{(1)}) \rightarrow (\bar{\psi}^{(1)}, \bar{\psi}^{(2)}).$$

Also, $\bar{\Phi} \leq \bar{\Psi}$.

Since

$$\bar{T}\Phi_{n+1} = ((F_1 - F_2)\varphi_{n+1}^{(2)}, (G_1 - G_2)\varphi_{n+1}^{(1)})$$

by (4.8), we have

$$\bar{T}\Phi_{n+1} = (F_1(G_1 - G_2)\varphi_n^{(1)} - F_2(G_1 - G_2)\psi_n^{(1)}, G_1(F_1 - F_2)\varphi_n^{(2)} - G_2(F_1 - F_2)\psi_n^{(2)}).$$

Then, we can easily have

$$F_1(G_1 - G_2)\varphi_n^{(1)} \rightarrow F_1\bar{\varphi}^{(2)},$$

similarly, we get

$$F_2(G_1 - G_2)\psi_n^{(1)} \rightarrow F_2\bar{\psi}^{(2)}, \quad G_1(F_1 - F_2)\varphi_n^{(2)} \rightarrow G_1\bar{\varphi}^{(1)}, \quad G_2(F_1 - F_2)\psi_n^{(2)} \rightarrow G_2\bar{\psi}^{(1)},$$

this implies that

$$(\bar{\varphi}^{(1)}, \bar{\varphi}^{(2)}) = (F_1\bar{\varphi}^{(2)} - F_2\bar{\psi}^{(2)}, G_1\bar{\varphi}^{(1)} - G_2\bar{\psi}^{(1)}).$$

By the similar arguments and using the definition of $\bar{T}\Psi_{n+1}$,

$$\bar{T}\Psi_{n+1} = ((F_1 - F_2)\psi_{n+1}^{(2)}, (G_1 - G_2)\psi_{n+1}^{(1)}),$$

hence,

$$\bar{T}\Psi_{n+1} = (F_1(G_1 - G_2)\psi_n^{(1)} - F_2(G_1 - G_2)\varphi_n^{(1)}, G_1(F_1 - F_2)\psi_n^{(2)} - G_2(F_1 - F_2)\varphi_n^{(2)}),$$

we get

$$(\bar{\psi}^{(1)}, \bar{\psi}^{(2)}) = (F_1\bar{\psi}^{(2)} - F_2\bar{\varphi}^{(2)}, G_1\bar{\psi}^{(1)} - G_2\bar{\varphi}^{(1)}).$$

Now consider the convex set defined by

$$\bar{K} = \{(u, v) \in X \times X | (\bar{\varphi}^{(1)}(i), \bar{\varphi}^{(2)}(i)) \leq (u(i), v(i)) \leq (\bar{\psi}^{(1)}(i), \bar{\psi}^{(2)}(i))\}.$$

$\bar{T}(u, v) = (F_1v - F_2v, G_1u - G_2u)$, where

$$F_1v - F_2v \leq F_1\bar{\psi}^{(2)} - F_2\bar{\varphi}^{(2)} = \bar{\psi}^{(1)},$$

$$F_1v - F_2v \geq F_1\bar{\varphi}^{(2)} - F_2\bar{\psi}^{(2)} = \bar{\varphi}^{(1)},$$

$$G_1v - G_2v \leq G_1\bar{\psi}^{(1)} - G_2\bar{\varphi}^{(1)} = \bar{\psi}^{(2)},$$

$$G_1v - G_2v \geq G_1\bar{\varphi}^{(1)} - G_2\bar{\psi}^{(1)} = \bar{\varphi}^{(2)},$$

then

$$(\bar{\varphi}^{(1)}, \bar{\varphi}^{(2)}) \leq (F_1v - F_2v, G_1u - G_2u) \leq (\bar{\psi}^{(1)}, \bar{\psi}^{(2)}).$$

This implies that $\bar{T}(\bar{K}) \subset \bar{K}$. Because X is a finite-dimensional space, it is easy to show \bar{T} restricted to \bar{K} is a compact operator. Thus we know that \bar{T} has a fixed point by Schauder's fixed point theorem. \square

Theorem 4.3 *If $f(0) > 0$, $g(0) > 0$, Let (F1)–(F4) hold, then $\exists \lambda^* > 0$ depending on f, g, p, q, μ_1, μ_2 such that system (1.5) has a nonnegative solution for $0 \leq \lambda \leq \lambda^*$.*

Proof We will seek Φ_0 and Ψ_0 so that (I), (II), and (III) are satisfied in Lemma 4.1.

Let

$$\Phi_0 = \begin{cases} (0, 0) & \text{on } \Omega_1, \\ (\alpha_1, \alpha_2) & \text{on } \Omega_2; \end{cases} \quad \Psi_1(t) = \begin{cases} (\alpha_1, \alpha_2) & \text{on } \Omega_1, \\ (0, 0) & \text{on } \Omega_2. \end{cases}$$

Then (I) is satisfied if $\alpha_1, \alpha_2 \geq (0, 0)$. Now condition (II) is

$$\bar{T}\Psi_0 = (F_1(\alpha_2) - F_2(0), G_1(\alpha_1) - G_2(0)) \leq (\alpha_1, \alpha_2),$$

while (III) is

$$\bar{T}\Phi_0 = (F_1(0) - F_2(\alpha_2), G_1(0) - G_2(\alpha_1)) \geq (0, 0),$$

Let

$$w^\pm(t) = \sum_{s=1}^{N-1} G(t, s)p^\pm(s), \quad W^\pm(t) = \sum_{s=1}^{N-1} G(t, s)q^\pm(s).$$

Then, the conditions become

$$\lambda[w^+(t)f(\alpha_2) - w^-(t)f(0)] \leq \alpha_1, \tag{4.10}$$

$$\lambda[W^+(t)g(\alpha_1) - W^-(t)g(0)] \leq \alpha_2, \tag{4.11}$$

$$\lambda[w^+(t)f(0) - w^-(t)f(\alpha_2)] \geq 0, \tag{4.12}$$

$$\lambda[W^+(t)g(0) - W^-(t)g(\alpha_1)] \geq 0. \tag{4.13}$$

Define $\gamma(t) = w^+(t) - (1 + \mu_1)w^-(t)$ and $\Gamma(t) = W^+(t) - (1 + \mu_2)W^-(t)$. By (F3) and (F4), we have

$$w^+(t) \geq (1 + \mu_1)w^-(t), \quad W^+(t) \geq (1 + \mu_2)W^-(t).$$

So (4.12) and (4.13) are satisfied if

$$f(\alpha_2) \leq (1 + \mu_1)f(0), \quad g(\alpha_1) \leq (1 + \mu_2)g(0), \tag{4.14}$$

respectively. We will select α_1, α_2 and claim that (4.10) and (4.11) can be satisfied. Since we have

$$w^+(t) - w^-(t) = \sum_{s=1}^{N-1} G(t, s)p(s), \quad W^+(t) - W^-(t) = \sum_{s=1}^{N-1} G(t, s)q(s),$$

so w, W solve the boundary value problems

$$\begin{cases} -\Delta^2 w(t-1) = p(t), & t \in [1, N-1]_{\mathbb{Z}}, \\ \Delta w(0) = w(N) = 0, \end{cases} \quad \begin{cases} -\Delta^2 W(t-1) = q(t), & t \in [1, N-1]_{\mathbb{Z}}, \\ \Delta W(0) = W(N) = 0, \end{cases}$$

respectively. From this we have

$$w^+(t) \leq w^-(t) + \beta_1, \quad \text{where} \left| \sum_{s=1}^{N-1} G(t, s)p(s) \right| \leq \beta_1,$$

$$W^+(t) \leq W^-(t) + \beta_2, \quad \text{where} \left| \sum_{s=1}^{N-1} G(t, s)q(s) \right| \leq \beta_2.$$

Hence

$$\begin{aligned} f(\alpha_2)w^+(t) - f(0)w^-(t) &\leq (1 + \mu_1)f(0)w^-(t) + \beta_1(1 + \mu_1)f(0) - f(0)w^-(t) \\ &= \mu_1 f(0)w^-(t) + \beta_1(1 + \mu_1)f(0) \\ &\leq \mu_1 f(0)\beta_1 + \beta_1(1 + \mu_1)f(0) \\ &= (1 + 2\mu_1)f(0)\beta_1. \end{aligned}$$

Therefore,

$$\lambda[f(\alpha_2)w^+(t) - f(0)w^-(t)] \leq \lambda[(1 + 2\mu_1)f(0)\beta_1].$$

If we choose

$$\lambda \leq \frac{\alpha_1}{(1 + 2\mu_1)f(0)\beta_1} = \lambda_0,$$

then (4.10) is satisfied. Similarly, we have

$$g(\alpha_1)W^+(t) - g(0)W^-(t) \leq (1 + 2\mu_2)g(0)\beta_2.$$

So,

$$\lambda[g(\alpha_1)W^+(t) - g(0)W^-(t)] \leq \lambda[(1 + 2\mu_2)g(0)\beta_2].$$

If we choose

$$\lambda \leq \frac{\alpha_2}{(1 + 2\mu_2)g(0)\beta_2} = \lambda^0,$$

then (4.11) is satisfied. With the choice of $0 \leq \lambda \leq \lambda^* = \min\{\lambda_0, \lambda^0\}$, then (4.10), (4.11) are satisfied. \square

In fact, we also obtain the existence of nonnegative solutions for $n \times n$ system

$$\begin{cases} -\Delta^2 u_1(t-1) = \lambda p_1(t) f_1(u_2(t)), & t \in [1, N-1]_{\mathbb{Z}}, \\ -\Delta^2 u_2(t-1) = \lambda p_2(t) f_2(u_3(t)), & t \in [1, N-1]_{\mathbb{Z}}, \\ \vdots \\ -\Delta^2 u_{n-1}(t-1) = \lambda p_{n-1}(t) f_{n-1}(u_n(t)), & t \in [1, N-1]_{\mathbb{Z}}, \\ -\Delta^2 u_n(t-1) = \lambda p_n(t) f_n(u_1(t)), & t \in [1, N-1]_{\mathbb{Z}}, \\ \Delta u_1(0) = u_1(N) = 0, \\ \Delta u_2(0) = u_2(N) = 0. \\ \vdots \\ \Delta u_{n-1}(0) = u_{n-1}(N) = 0, \\ \Delta u_n(0) = u_n(N) = 0. \end{cases} \quad (4.15)$$

We assume that

(G1) $f_i : [0, +\infty) \rightarrow [0, +\infty)$ are continuous and nondecreasing functions, $i = 1, 2, \dots, n$;

(G2) $p_i : [1, N - 1]_{\mathbb{Z}} \rightarrow \mathbb{R}$ are sign-changing functions, $i = 1, 2, \dots, n$;

(G3) there is an $\mu_1 > 0$ so that

$$\sum_{s=1}^{N-1} G(t, s)p_1^+(s) \geq (1 + \mu_1) \sum_{s=1}^{N-1} G(t, s)p_1^-(s), \quad t \in [1, N - 1]_{\mathbb{Z}};$$

(G4) there is an $\mu_2 > 0$ so that

$$\sum_{s=1}^{N-1} G(t, s)p_2^+(s) \geq (1 + \mu_2) \sum_{s=1}^{N-1} G(t, s)p_2^-(s), \quad t \in [1, N - 1]_{\mathbb{Z}};$$

⋮

(G(n+1)) there is an $\mu_{n-1} > 0$ so that

$$\sum_{s=1}^{N-1} G(t, s)p_{n-1}^+(s) \geq (1 + \mu_{n-1}) \sum_{s=1}^{N-1} G(t, s)p_{n-1}^-(s), \quad t \in [1, N - 1]_{\mathbb{Z}};$$

(G(n+2)) there is an $\mu_n > 0$ so that

$$\sum_{s=1}^{N-1} G(t, s)p_n^+(s) \geq (1 + \mu_n) \sum_{s=1}^{N-1} G(t, s)p_n^-(s), \quad t \in [1, N - 1]_{\mathbb{Z}}.$$

Corollary 4.4 *If $f_i(0) > 0$. Let (G1)–(G(n+2)) hold, then the problem (4.15) has a nonnegative solution for $0 \leq \lambda \leq \bar{\lambda}$, where $\bar{\lambda} > 0$ depending on $f_i, p_i, \mu_i, i = 1, 2, \dots, n$.*

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Conflict of interest

The authors have no conflicts of interest to declare.

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