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Completeness conditions of systems of Bessel functions in weighted L^2-spaces in terms of entire functions

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Dedicated to the memory of Professor Bohdan V. Vynnyts'kyi

Abstract: Let J_ν be the Bessel function of the first kind of index $\nu \geq 1/2$, $p \in \mathbb{R}$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers. Sufficient conditions for the completeness of the system $\{x^{-p-1}\sqrt{x\rho_k}J_\nu(x\rho_k) : k \in \mathbb{N}\}$ in the weighted space $L^2((0;1);x^{2p}dx)$ are found in terms of an entire function with the set of zeros coinciding with the sequence $(\rho_k)_{k\in\mathbb{N}}$.

Key words: Bessel function, entire function, complete system, minimal system, basis, weighted space

1. Introduction

Let $\gamma \in \mathbb{R}$ and $L^2((0;1);t^\gamma dt)$ be the weighted Lebesgue space of all measurable functions $f : (0;1) \to \mathbb{C}$, satisfying

$$\int_0^1 t^\gamma |f(t)|^2\, dt < +\infty.$$

Let

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}, \quad z = x + iy = re^{i\varphi},$$

be the Bessel function of the first kind of index $\nu \in \mathbb{R}$. It is well known that (see [5, p. 94], [7, p. 350]), for $\nu > -1$ the function J_ν has an infinite set $\{\rho_k : k \in \mathbb{Z}\}$ of real roots, among which ρ_k, $k \in \mathbb{N}$, are the positive roots and $\rho_{-k} := -\rho_k$, $k \in \mathbb{N}$, are the negative roots. All roots are simple except, perhaps, the root $\rho_0 = 0$. A system of elements $\{e_k : k \in \mathbb{N}\}$ in a separable Hilbert space H is called complete ([6, p. 131]) if $\overline{\operatorname{span}}\{e_k : k \in \mathbb{N}\} = H$.

Various approximation properties of the systems of Bessel functions has been studied in many papers (see, for instance, [1–5, 7–13]). In particular, it is well known that the system $\{\sqrt{x}J_\nu(x\rho_k) : k \in \mathbb{N}\}$ is an orthogonal basis for the space $L^2(0;1)$ if $\nu > -1$ and $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of positive zeros of J_ν (see [1, 3], [7, pp. 355-357]). From this, it follows that if $\nu > -1$ and $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of positive zeros of J_ν, then the system

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$\{x^{-\nu}J_\nu(x\rho_k) : k\in\mathbb{N}\}$ is complete and minimal in $L^2((0;1);x^{2\nu+1}dx)$. The system $\{\sqrt{x}J_\nu(x\rho_k) : k\in\mathbb{N}\}$ is also complete ([7, pp. 347, 356]) in $L^2(0;1)$ if $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of zeros of the function J'_ν. Besides, from [2] it follows that if $\nu>-1/2$ and $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of distinct positive numbers such that $\rho_k\le\pi(k+\nu/2)$ for all sufficiently large $k\in\mathbb{N}$, then the system $\{\sqrt{x}J_\nu(x\rho_k) : k\in\mathbb{N}\}$ is complete in $L^2(0;1)$.

Basis properties of the above systems of Bessel functions with an arbitrary sequence of complex numbers $(\rho_k)_{k\in\mathbb{N}}$ has been studied in [4, 8–13]. In particular, in [9] the authors obtained the necessary and sufficient conditions for the completeness and minimality of system $\{\sqrt{x\rho_k}J_\nu(x\rho_k) : k\in\mathbb{N}\}$ in the space $L^2(0;1)$ if $\nu\ge-1/2$ and $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of distinct nonzero complex numbers. In [12], it was proven that if the system $\{\sqrt{x\rho_k}J_\nu(x\rho_k) : k\in\mathbb{N}\}$ is complete and minimal in $L^2(0;1)$, then its biorthogonal system is also complete and minimal in $L^2(0;1)$, where $\nu\ge-1/2$ and $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of nonzero complex numbers such that $\rho_k^2\ne\rho_m^2$ for $k\ne m$. In addition, in [13] (see also [10]) the authors found a criterion of unconditional basicity of the system $\{\sqrt{x\rho_k}J_\nu(x\rho_k) : k\in\mathbb{N}\}$ in $L^2(0;1)$, where $\nu\ge-1/2$ and $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of distinct nonzero complex numbers. Moreover, in [11] has been established a criterion for the completeness and minimality of more general system $\{\Theta_{k,\nu,p} : k\in\mathbb{N}\}$, $\Theta_{k,\nu,p}(x):=x^{-p-1}\sqrt{x\rho_k}J_\nu(x\rho_k)$, in the space $L^2((0;1);x^{2p}dx)$, where $\nu\ge1/2$, $p\in\mathbb{R}$ and $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of distinct nonzero complex numbers. Besides, in [4] it was proven that the system $\{\Theta_{k,\nu,p} : k\in\mathbb{N}\}$ is complete in $L^2((0;1);x^{2p}dx)$ if and only if $\sum_{k=1}^\infty 1/|\rho_k|=+\infty$, where $\nu\ge1/2$, $p\in\mathbb{R}$ and $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of distinct nonzero complex numbers such that $|\operatorname{Im}\rho_k|\ge\delta|\rho_k|$ for all $k\in\mathbb{N}$ and some $\delta>0$. Those results are formulated in terms of sequences of zeros of functions from certain classes of entire functions.

The aim of this paper is to prove Theorems 3.1–3.4, where we obtained some other sufficient conditions for the completeness of system $\{\Theta_{k,\nu,p} : k\in\mathbb{N}\}$ in the space $L^2((0;1);x^{2p}dx)$ in terms of entire functions. This complements the results of papers [4] and [8–13].

2. Preliminaries

An entire function G is said to be of exponential type $\sigma\in[0;+\infty)$ ([6, p. 4]) if for any $\varepsilon>0$ there exists a constant $c(\varepsilon)$ such that $|G(z)|\le c(\varepsilon)\exp((\sigma+\varepsilon)|z|)$ for all $z\in\mathbb{C}$. To prove our main results, we need the following auxiliary lemmas.

Lemma 2.1 *([11]) Let $\nu\ge1/2$ and $p\in\mathbb{R}$. An entire function Ω has the representation*

$$\Omega(z)=z^{-\nu}\int_0^1\sqrt{t}J_\nu(tz)t^{p-1}q(t)\,dt \tag{2.1}$$

with some function $q\in L^2((0;1);x^{2p}dx)$ if and only if it is an even entire function of exponential type $\sigma\le1$ such that $z^{-\nu+1/2}(z^{2\nu}\Omega(z))'\in L^2(0;+\infty)$. In this case,

$$q(t)=t^{-p}\int_0^{+\infty}\sqrt{tz}J_{\nu-1}(tz)z^{-\nu+1/2}(z^{2\nu}\Omega(z))'\,dz.$$

Let $\widetilde{E}_{p,2}$ be the class of the entire functions Ω that can be represented in the form (2.1), and let $E_{p,2}$ be the class of even entire functions Ω of exponential type $\sigma\le1$ such that $z^{-\nu+1/2}(z^{2\nu}\Omega(z))'\in L^2(0;+\infty)$. In view of Lemma 2.1, we remark that $\widetilde{E}_{p,2}=E_{p,2}$.

Lemma 2.2 *([11]) Let $\nu \geq 1/2$, $p \in \mathbb{R}$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of nonzero complex numbers such that $\rho_k^2 \neq \rho_n^2$ for $k \neq n$. For a system $\{\Theta_{k,\nu,p} : k \in \mathbb{N}\}$ to be incomplete in the space $L^2((0;1); x^{2p}dx)$ it is necessary and sufficient that a sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $\rho_{-k} := -\rho_k$, $k \in \mathbb{N}$, is a subsequence of zeros of some nonzero entire function $\Omega \in E_{p,2}$.*

Lemma 2.3 *([11]) Let $\nu \geq 1/2$, $p \in \mathbb{R}$ and an entire function Ω be defined by the formula (2.1). Then for all $z = x + iy = re^{i\varphi} \in \mathbb{C}$, we have (here and so on by C_j we denote positive constants)*

$$|\Omega(z)| \leq C_1(1 + |z|)^{-\nu} \exp(|\operatorname{Im} z|).$$

Let $n(t)$ be the number of points of the sequence $(\rho_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ satisfying the inequality $|\rho_k| \leq t$, i.e. $n(t) := \sum_{|\rho_k|\leq t} 1$, and let

$$N(r) := \int_0^r \frac{n(t)}{t}\, dt, \quad r > 0.$$

Lemma 2.4 *([4]) Let $\nu \geq 1/2$, $p \in \mathbb{R}$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers. If*

$$\limsup_{r\to+\infty} \left(N(r) - \frac{2r}{\pi} + \nu \log(1 + r) \right) = +\infty,$$

then the system $\{\Theta_{k,\nu,p} : k \in \mathbb{N}\}$ is complete in $L^2((0;1); x^{2p}dx)$.

3. Main results

Theorem 3.1 *Let $\nu \geq 1/2$, $p \in \mathbb{R}$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers such that $\rho_k^2 \neq \rho_m^2$ for $k \neq m$. Let a sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $\rho_{-k} := -\rho_k$, be a sequence of zeros of some even entire function G of exponential type $\sigma \leq 1$ for which on the rays $\{z : \arg z = \varphi_j\}$, $j \in \{1; 2; 3; 4\}$, $\varphi_1 \in [0; \pi/2)$, $\varphi_2 \in [\pi/2; \pi)$, $\varphi_3 \in (\pi; 3\pi/2]$, $\varphi_4 \in (3\pi/2; 2\pi)$, we have*

$$|G(z)| \geq C_2(1 + |z|)^{-\alpha} \exp(|\operatorname{Im} z|), \tag{3.1}$$

with some $\alpha < \nu$. Then the system $\{\Theta_{k,\nu,p} : k \in \mathbb{N}\}$ is complete in $L^2((0;1); x^{2p}dx)$.

Proof Assume the converse. Then, according to Lemma 2.2, there exists a nonzero even entire function $\Omega \in E_{p,2}$ for which the sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$ is a subsequence of zeros. Let $V(z) = \Omega(z)/G(z)$. Then V is an even entire function of order $\tau \leq 1$, for which by Lemmas 2.1 and 2.3, we obtain

$$|V(z)| \leq C_3(1 + |z|)^{\alpha-\nu}, \quad \arg z = \varphi_j, \quad j \in \{1; 2; 3; 4\}. \tag{3.2}$$

Therefore, according to the Phragmén–Lindelöf theorem (see [6, p. 39]), we get $V(z) \equiv 0$. Hence, $\Omega(z) \equiv 0$. This contradiction proves the theorem. □

Theorem 3.2 *Let $\nu \geq 1/2$, $p \in \mathbb{R}$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers such that $\rho_k^2 \neq \rho_m^2$ for $k \neq m$. Let a sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $\rho_{-k} := -\rho_k$, be a sequence of zeros of some even* *entire function* $G \notin E_{p,2}$ *of exponential type* $\sigma \le 1$ *for which on the rays* $\{z : \arg z = \varphi_j\}$, $j \in \{1;2;3;4\}$, $\varphi_1 \in [0;\pi/2)$, $\varphi_2 \in [\pi/2;\pi)$, $\varphi_3 \in (\pi;3\pi/2]$, $\varphi_4 \in (3\pi/2;2\pi)$, *the inequality* (3.1) *holds with* $\alpha < 5/2$. *Then the system* $\{\Theta_{k,\nu,p} : k \in \mathbb{N}\}$ *is complete in* $L^2((0;1);x^{2p}dx)$.

Proof Assume the converse. Then, according to Lemma 2.2, there exists a nonzero even entire function $\Omega \in E_{p,2}$ for which the sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$ is a subsequence of zeros. Let $V(z) = \Omega(z)/G(z)$. Then V is an even entire function of order $\tau \le 1$, satisfying (3.2) (see the proof of Theorem 3.1). Since $\alpha - \nu \le \alpha - 1/2 < 2$ and V is an even entire function, then, according to the Phragmén–Lindelöf theorem, the function V is a constant. Hence, $\Omega(z) = C_4 G(z)$ and $\Omega \notin E_{p,2}$. Thus, we have a contradiction and the proof of the theorem is completed. □

Theorem 3.3 *Let* $\nu \ge 1/2$, $p \in \mathbb{R}$ *and* $(\rho_k)_{k\in\mathbb{N}}$ *be a sequence of distinct nonzero complex numbers such that* $\rho_k^2 \ne \rho_m^2$ *for* $k \ne m$. *Let a sequence* $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$, *where* $\rho_{-k} := -\rho_k$, *be a sequence of zeros of some even entire function* $F \notin E_{p,2}$ *of exponential type* $\sigma \le 1$ *such that for some* $\alpha < 2$ *and* $h \in \mathbb{R}$

$$|F(x+ih)| \ge \delta |x|^{-\alpha}, \quad \delta > 0, \quad |x| > 1. \tag{3.3}$$

Then the system $\{\Theta_{k,\nu,p} : k \in \mathbb{N}\}$ *is complete in* $L^2((0;1);x^{2p}dx)$.

Proof Let $F \notin E_{p,2}$ and the inequality (3.3) is true. Suppose, to the contrary, that the system $\{\Theta_{k,\nu,p} : k \in \mathbb{N}\}$ is not complete in $L^2((0;1);x^{2p}dx)$. Then, by Lemma 2.2, there exists a nonzero even entire function $\Omega \in E_{p,2}$ which vanishes at the points ρ_k. However, the sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$ is a sequence of zeros of an entire function $F(z) \notin E_{p,2}$ of exponential type $\sigma \le 1$. Therefore, $E(z) = \Omega(z)/F(z)$ is an even entire function of order $\tau \le 1$. Since $\Omega \in E_{p,2}$, then taking into account Lemma 2.3, we obtain

$$|\Omega(x+ih)| \le C_5(1+\sqrt{x^2+h^2})^{-\nu} e^{|h|} \le C_6 < +\infty, \quad x \in \mathbb{R}.$$

Using (3.3), we get

$$|E(x+ih)| \le C_7(1+|x|)^{\alpha}, \quad x \in \mathbb{R}.$$

In view of this, we have that $E(z)$ is a polynomial of degree $\alpha < 2$. Furthermore, since E is an even entire function, then $E(z) = C_8$. Furthermore, $F(z) = C_9\Omega(z)$ and $F(z) \in E_{p,2}$. This contradiction concludes the proof of the theorem. □

Theorem 3.4 *Let* $\nu \ge 1/2$, $p \in \mathbb{R}$ *and* $(\rho_k)_{k\in\mathbb{N}}$ *be a sequence of distinct nonzero complex numbers. Let* $|\rho_k| \le \Delta k + \beta + \alpha_k$ *for* $0 < \Delta \le \pi/2$, $-\Delta < \beta < \Delta(\nu - 1/2)$, *and the sequence* $(\alpha_k)_{k\in\mathbb{N}}$ *such that* $\alpha_k \ge 0$, $\alpha_k = O(1)$ *as* $k \to +\infty$ *and*

$$\sum_{k=1}^{\infty} |\alpha_{k+1} - \alpha_k| < +\infty, \quad \sum_{k=1}^{\infty} \frac{\alpha_k}{k} < +\infty. \tag{3.4}$$

Then the system $\{\Theta_{k,\nu,p} : k \in \mathbb{N}\}$ *is complete in* $L^2((0;1);x^{2p}dx)$.

Proof Let $\mu_k = \Delta k + \beta + \alpha_k$, $k \in \mathbb{N}$, and

$$n_1(t) = \sum_{\mu_k \le t} 1, \quad N_1(r) = \int_0^r \frac{n_1(t)}{t}\,dt, \quad r > 0.$$

Then $n(t) \geq n_1(t)$, $N(r) \geq N_1(r)$ and $n_1(t) = m$ for $\Delta m + \beta + \alpha_m \leq t < \Delta(m+1) + \beta + \alpha_{m+1}$ ($n_1(t) = 0$ on $(0;\mu_1)$). Let $r \in [\mu_s;\mu_{s+1})$. Then $s = \frac{r}{\Delta} + O(1)$ as $r \to +\infty$. Therefore, by analogy with [4, p. 9], we obtain as $r \to +\infty$

$$\begin{aligned} N_1(r) &= \sum_{k=1}^{s-1} \int_{\mu_k}^{\mu_{k+1}} \frac{n_1(t)}{t}\,dt + \int_{\mu_s}^{r} \frac{n_1(t)}{t}\,dt \\ &= \sum_{k=1}^{s-1} k \int_{\mu_k}^{\mu_{k+1}} \frac{dt}{t} + \int_{\mu_s}^{r} \frac{s}{t}\,dt = \sum_{k=1}^{s-1} k \log \frac{\mu_{k+1}}{\mu_k} + s \log \frac{r}{\mu_s} \\ &= \sum_{k=1}^{s-1} k \log \frac{\Delta(k+1)+\beta+\alpha_{k+1}}{\Delta k+\beta+\alpha_k} + s \log \frac{r}{\Delta s+\beta+\alpha_s} \\ &= \sum_{k=1}^{s-1} k \left(\log \frac{\Delta(k+1)+\beta+\alpha_{k+1}}{\Delta k+\beta+\alpha_k} - \log \frac{\Delta(k+1)+\beta}{\Delta k+\beta} \right) + \sum_{k=1}^{s-1} k \log \frac{\Delta(k+1)+\beta}{\Delta k+\beta} + O(1). \end{aligned} \tag{3.5}$$

Furthermore (see [4, p. 9]),

$$\sum_{k=1}^{s-1} k \log \frac{\Delta(k+1)+\beta}{\Delta k+\beta} = \frac{r}{\Delta} - \left(\frac{1}{2} + \frac{\beta}{\Delta}\right) \log r + O(1), \quad r \to +\infty. \tag{3.6}$$

Furthermore, using the Lagrange theorem, we get

$$\begin{aligned} &\log \frac{\Delta(k+1)+\beta+\alpha_{k+1}}{\Delta k+\beta+\alpha_k} - \log \frac{\Delta(k+1)+\beta}{\Delta k+\beta} \\ &= \log\left(1 + \frac{\Delta+\alpha_{k+1}-\alpha_k}{\Delta k+\beta+\alpha_k}\right) - \log\left(1 + \frac{\Delta}{\Delta k+\beta}\right) \\ &= \frac{1}{1+C_k}\left(\frac{\Delta+\alpha_{k+1}-\alpha_k}{\Delta k+\beta+\alpha_k} - \frac{\Delta}{\Delta k+\beta}\right) \\ &= \frac{1}{1+C_k}\left(\frac{(\Delta k+\beta)(\alpha_{k+1}-\alpha_k) - \alpha_k \Delta}{(\Delta k+\beta+\alpha_k)(\Delta k+\beta)}\right), \quad C_k > 0. \end{aligned}$$

Therefore,

$$\left| k \left(\log \frac{\Delta(k+1)+\beta+\alpha_{k+1}}{\Delta k+\beta+\alpha_k} - \log \frac{\Delta(k+1)+\beta}{\Delta k+\beta} \right) \right| \leq C\left(|\alpha_{k+1}-\alpha_k| + \frac{\alpha_k}{k}\right), \quad C > 0.$$

Hence,

$$\begin{aligned} &\left| \sum_{k=1}^{s-1} k \left(\log \frac{\Delta(k+1)+\beta+\alpha_{k+1}}{\Delta k+\beta+\alpha_k} - \log \frac{\Delta(k+1)+\beta}{\Delta k+\beta} \right) \right| \\ &\leq \sum_{k=1}^{s-1} C\left(|\alpha_{k+1}-\alpha_k| + \frac{\alpha_k}{k}\right) \leq \sum_{k=1}^{\infty} C\left(|\alpha_{k+1}-\alpha_k| + \frac{\alpha_k}{k}\right). \end{aligned} \tag{3.7}$$

Thus, combining relations (3.4)–(3.7), we obtain

$$N_1(r) \geq \frac{r}{\Delta} - \left(\frac{1}{2} + \frac{\beta}{\Delta}\right) \log r + O(1), \quad r \to +\infty.$$

In view of this, for $0 < \Delta \le \pi/2$ and $-\Delta < \beta < \Delta(\nu - 1/2)$, we have

$$\limsup_{r\to+\infty}\left(N(r) - \frac{2r}{\pi} + \nu\log(1+r)\right) \ge \limsup_{r\to+\infty}\left(N_1(r) - \frac{2r}{\pi} + \nu\log(1+r)\right)$$
$$\ge \limsup_{r\to+\infty}\left(\frac{r}{\Delta} - \left(\frac{1}{2} + \frac{\beta}{\Delta}\right)\log r - \frac{2r}{\pi} + \nu\log r + O(1)\right) = +\infty.$$

Finally, according to Lemma 2.4, we obtain the required proposition. The proof of theorem is completed. □

Corollary 3.5 *([4]) Let $\nu \ge 1/2$, $p \in \mathbb{R}$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers. If $|\rho_k| \le \Delta k + \beta$ for $0 < \Delta \le \pi/2$, $-\Delta < \beta < \Delta(\nu - 1/2)$ and all sufficiently large $k \in \mathbb{N}$, then the system $\left\{\Theta_{k,\nu,p} : k \in \mathbb{N}\right\}$ is complete in $L^2((0;1);x^{2p}dx)$.*

Indeed, this corollary follows directly from Theorem 3.4, because the sequence $\alpha_k = 0$ satisfies its conditions.

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