Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 634 – 646
© TÜBİTAK
doi:10.3906/mat-2004-68

Order compact and unbounded order compact operators

Nazife ERKURŞUN ÖZCAN[1], Niyazi Anıl GEZER[2], Şaziye Ece ÖZDEMİR[3], İrem Mesude URGANCI[4,*]

[1]Department of Mathematics, Faculty of Science, Hacettepe University, Ankara, Turkey
[2]Department of Mathematics, Faculty of Science and Art, Middle East Technical University, Ankara, Turkey
[3]Department of Mathematics, Faculty of Science and Art, Pamukkale University, Denizli, Turkey
[4]Department of Mathematics, Faculty of Science, Gazi University, Ankara, Turkey

Received: 18.04.2020 • **Accepted/Published Online:** 27.12.2020 • **Final Version:** 26.03.2021

Abstract: We investigate properties of order compact, unbounded order compact and relatively uniformly compact operators acting on vector lattices. An operator is said to be order compact if it maps an arbitrary order bounded net to a net with an order convergent subnet. Analogously, an operator is said to be unbounded order compact if it maps an arbitrary order bounded net to a net with *uo*-convergent subnet. After exposing the relationships between order compact, unbounded order compact, semicompact and *GAM*-compact operators; we study those operators mapping an arbitrary order bounded net to a net with a relatively uniformly convergent subnet. By using the nontopological concepts of order and unbounded order convergences, we derive new results related to these classes of operators.

Key words: Compact operator, unbounded order convergence, vector lattice

1. Introduction

Banach lattices can be equipped with various canonical convergence structures such as order, relatively uniform, unbounded order and unbounded norm convergences. Although some of these convergences are not topological, they share the common property that the underlying order structure plays a dominant role in deriving properties related to operators acting on these lattices.

The notion of unbounded order convergence was initially introduced in [13] under the name individual convergence, and, "uo-convergence" was proposed firstly in [7]. Recently in [4, 6, 8–11, 17], see also the references therein, further properties of various types of unbounded convergences are investigated.

In the present paper, we study compactness properties of operators between vector lattices by utilizing various nontopological convergences. In addition to the fact that results obtained in the settings of vector lattices will shed light on the case of operators on lattice-normed spaces, see [3, 5, 14]; the diversity of various types of convergences on vector lattices allows one to derive further results in more general settings.

The structure of the present paper is as follows. In Section 2, we derive several properties of order compact operators. We expose the relationships between order compact, order bounded, order continuous and *GAM*-compact operators. In particular, as a consequence of our results we deduce that order compactness of an operator is a natural and conceptual notion to study. In Section 3, we introduce the analogous notion of *uo*-compact operators. One of the impetuses for studying *uo*-compact operators comes from the relationships

*Correspondence: irem.mesude@hotmail.com

2010 *AMS Mathematics Subject Classification:* Primary 47B07, 46B42 ; Secondary 46A40

between compact, order compact and uo-compact operators. In Section 4, we define the notion of relatively uniformly compact operators and study some of its properties.

Throughout the paper, all vector lattices are Archimedean. Unexplained terminology about vector lattices can be found in [1, 2, 16, 18].

2. Order compact operators

Definition 2.1 *A net x_α in a vector lattice E is order convergent to an element $x \in E$, in symbols $x_\alpha \xrightarrow{o} x$, whenever there exists a net y_α in E (with the same index set) satisfying $|x_\alpha - x| \leq y_\alpha \downarrow 0$, see [2, 12, 16, 18].*

Example 2.2 *In some cases, order convergence agrees with pointwise convergence. A sequence space is a linear subspace λ of the vector space s of all real sequences $x = (x_n)$ such that λ contains c_{00}, the space of those sequences which vanish for all but finitely many indices. Sequence spaces are assumed by their canonical positive cones. If an order bounded net x^α in λ order converges to $x \in \lambda$ then $\lim_\alpha x_n^\alpha = x_n$ for each n.*

Definition 2.3 *Let E and F be two vector lattices.*

i. *An operator $T : E \to F$ is called order continuous, see [2, Chapter 1.4], $Tx_\alpha \xrightarrow{o} 0$ in F whenever $x_\alpha \xrightarrow{o} 0$ in E. If this condition holds for sequences then T is called sequentially order continuous (abbreviated as σ-order continuous).*

ii. *An operator $T : E \to F$ is called order bounded if it maps order bounded sets in E to order bounded sets in F.*

Various results on the properties of order continuous, σ-order continuous and order bounded operators can be found in [2, Chapter 1.4] and [18, Chapter 12].

Example 2.4 *Some vector lattices formed by integrable real valued functions over a σ-finite measure space can be used as the underlying space on which the class of σ-order continuous operators and the class of order continuous operators do not coincide. Indeed, [2, Example 1.55] shows that a positive σ-order continuous operator may not be order continuous even in the case that the underlying space upon which this operator acts is the vector lattice of Lebesgue integrable real valued functions on $[0, 1]$. Since σ-order continuity of this operator follows as a corollary of the Lebesgue dominated convergence theorem, this example provides a motivation to study compactness properties for operators where compactness is related to nontopological convergences such as order and unbounded order convergence.*

Definition 2.5 *Let E and F be two vector lattices.*

i. *An operator $T : E \to F$ is called order compact if for any order bounded net x_α in E there exists a subnet x_{α_β} and $y \in F$ such that $Tx_{\alpha_\beta} \xrightarrow{o} y$ in F.*

ii. *An operator $T : E \to F$ is called sequentially order compact if for any order bounded sequence x_n in E there exists a subsequence x_{n_k} and $y \in F$ such that $Tx_{n_k} \xrightarrow{o} y$ in F.*

Example 2.6 *An example given in [5] shows that an order continuous operator need not be sequentially order compact. Consider the identity operator $I : L_1[0, 1] \to L_1[0, 1]$. It is clear that the identity operator I is* *both order bounded and order continuous. We denote by r_n the nth Rademacher function. We recall that $r_n : C[0,1] \to \mathbb{R}$ where $r_n(t) = sgn(\sin(2^n \pi t))$ for $t \in [0,1]$ and that $|r_n| = 1$ for each $n \geq 1$. Assume there exists r_{n_k} such that $r_{n_k} \xrightarrow{o} r$ for some r. We have $\int_0^1 r_{n_k} r_{n_m} d\mu = 0$ for every $m > k$. On the other hand, we have $r_{n_k} r_{n_m} \xrightarrow{o} r_{n_k} r$. It follows that $\int_0^1 r_{n_k} r d\mu \to \int_0^1 r^2 d\mu = 0$. Hence, the identity operator I is not sequentially order compact.*

Example 2.7 *A sequentially order compact operator need not be order compact. We denote by c the vector lattice of all convergent sequences, see Example 2.2. Let f_n be an order bounded positive sequence in c. Hence, there is g such that $0 \leq f_n \leq g$ for all n. Denote by $f_n(m)$ the mth coordinate of f_n for $m \geq 1$. For any n, there is $a_n \geq 0$ such that for every $\epsilon > 0$ the set $\{m \colon |f_n(m) - a_n| \geq \epsilon\}$ is finite. Because the sequence a_n can be chosen as bounded, there is a subsequence a_{n_k} and $a \geq 0$ such that $a_{n_k} \to a$ as $k \to \infty$. For each $l, k \in \mathbb{N}$, consider $A_{l,n_k} = \{m \colon |f_{n_k}(m) - a_{n_k}| \geq 1/l\}$ and $A = \bigcup_{l=1}^{\infty} \bigcup_{k=1}^{\infty} A_{l,n_k}$. If $h = a\chi_{\mathbb{N}\setminus A}$ then $f_{n_k} \xrightarrow{o} h$ because order convergence in c agrees with pointwise convergence. Hence, the identity operator $I \colon c \to c$ is sequentially order compact. On the other hand, let Λ be the set of all finite subsets of $\mathbb{N}$. The set Λ is partially ordered with respect to inclusion. For each $\alpha \in \Lambda$ we put $f_\alpha = \chi_{\mathbb{N}\setminus\alpha}$. It is clear that $f_\alpha \in c$ and $f_\alpha(m) \leq 1$ for all m. For every subnet f_{α_β} we have $f_{\alpha_\beta}(m) \not\to 1$. Hence, f_{α_β} does not converge in order. Therefore, $I \colon c \to c$ is not order compact.*

Example 2.8 *An order compact operator between two vector lattices need not be sequentially order compact. An example related to this fact can be found in [5, Example 7].*

Lemma 2.9 *Suppose that $T \colon E \to F$ is an operator between two vector lattices E and F. If T is order compact then it is order bounded.*

Proof Details of the proof can be found in [5, Theorem 2]. It uses the fact that an order convergent net has an order bounded tail. □

In view of Lemma 2.9, if $T \colon E \to F$ is an order compact operator, and, F is an order complete vector lattice then for every order bounded net x_α in E there exists some $y \in F$ such that

$$\inf_{\gamma} \sup_{\beta \geq \gamma} T x_{\alpha_\beta} = \sup_{\gamma} \inf_{\beta \geq \gamma} T x_{\alpha_\beta} = y$$

where both β and γ run through the same index set.

We remark the obvious that order completeness of vector lattices can be dropped in items (i) and (ii) of Proposition 2.10, see below. In view of the fact that monotonic nets of operators play an inevitable role in studying the vector lattices form by these operators, items (iii) and (iv) of Proposition 2.10 abstract two situations where monotonic nets of order compact operators appear.

Proposition 2.10 *Suppose that $T \colon E \to F$, $L \colon F \to G$ and $R \colon G \to E$ where E, F and G are order complete vector lattices.*

i. *If T is order compact and L is order continuous then $L \circ T$ is order compact.*

ii. *If T is order compact and R is order bounded then $T \circ R$ is order compact.*

iii. *If T is a positive, order continuous and order compact operator and $R_\alpha \downarrow 0$ is a decreasing net of order bounded operators then $T \circ R_\alpha \downarrow 0$ is a decreasing net of order compact operators.*

iv. *If T is positive, order continuous and order compact and L_α is a net of positive order continuous operators satisfying $L_\alpha \uparrow L$ for some L then $L_\alpha \circ T \uparrow L \circ T$ and $L \circ T$ is order compact.*

Proof $(i.)$ Let x_α be an order bounded net in E. Because T is order compact, there exists a subnet x_{α_β} and $y \in F$ such that $Tx_{\alpha_\beta} \xrightarrow{o} y$. As L is order continuous, it follows that $L(Tx_{\alpha_\beta}) \xrightarrow{o} Lx$. Hence, $L \circ T$ is order compact.

$(ii.)$ Let x_α be an order bounded net in G. Because R is an order bounded operator, the net Rx_α is order bounded in E. As T is order compact, there exists a subnet x_{α_β} and $y \in F$ such that $T(R(x_{\alpha_\beta})) \xrightarrow{o} y$. This shows that $T \circ R$ is order compact.

$(iii.)$ By [2, Theorem 1.18], a net R_α of order bounded operators from G into E satisfies $R_\alpha \downarrow 0$ if and only if $R_\alpha(x) \downarrow 0$ in E for each $x \in G^+$. As $T \colon E \to F$ is both order continuous and order compact, $T \circ R_\alpha$ is order bounded and order compact for each α, see also Lemma 2.9. It follows that for each $x \in G^+$, we have $T(R_\alpha(x)) \xrightarrow{o} 0$ because T is order continuous. Since T is a positive operator we have $T \circ R_\alpha \downarrow 0$.

$(iv.)$ As T and L_α are order continuous, the operator $L_\alpha \circ T$ is order continuous for each α. It follows from $L_\alpha \uparrow L$ that $L_\alpha(Tx) \uparrow LT(x)$ for every $x \in E^+$. Hence, $L_\alpha \circ T \uparrow L \circ T$. It follows from [2, Theorem 1.57] that the operator $L \circ T$ is both order continuous and order compact. □

Proposition 2.11 *Suppose that E is an order complete vector lattice. If $T \colon E \to E$ is sequentially order compact, and, a strictly positive order continuous functional, see [11, Section 2.2], is left fixed by the adjoint operator T^* then there exists a Banach lattice F containing E as a norm dense ideal such that $T \colon E \to F$ is sequentially order compact.*

Proof Let x_n be an order bounded sequence in E. As the operator $T \colon E \to E$ is order compact there exists a subsequence x_{n_k} and $y \in E$ such that $Tx_{n_k} \xrightarrow{o} y$ in E. Denote by x^* the strictly positive order continuous functional which is left fixed by T^*. It follows from order continuity of both x^* and lattice operations that $x^*(|Tx_{n_k}|) \to x^*(|y|)$. Let F be the norm completion of E with respect to lattice norm $\|x\|_F = x^*(|x|)$ for $x \in E$. Hence, $Tx_{n_k} \xrightarrow{\|\cdot\|_F} y$ in F. As F is a Banach lattice there exists a subsequence $x_{n_{k_m}}$ such that $\mathrm{T}x_{n_{k_m}} \xrightarrow{o} y$ in F, see [16, Theorem VII.2.1]. Hence, the operator $T \colon E \to F$ is sequentially order compact. □

Definition 2.12 *An operator $T : E \to F$ from a vector lattice E into a Banach space F is said to be GAM-compact, see [3, 14], if for every order bounded set A in E, the set $T(A)$ is relatively compact in F.*

It follows that if E is an AM-space with a strong norm unit and F is a Banach lattice then a GAM-compact operator $T \colon E \to F$ is compact. Following result can be used to produce examples of sequentially order compact operators.

Theorem 2.13 *Suppose that $T : E \to F$ is an operator where E is a vector lattice and F is a Banach lattice. If T is GAM-compact then T is sequentially order compact.*

Proof Let x_n be an order bounded sequence in E. Since T is GAM-compact there is a subsequence x_{n_k} and $y \in F$ such that $\|Tx_{n_k} - y\|_F \to 0$. As F is a Banach lattice, there exists a subsequence $x_{n_{k_m}}$ such that $\mathrm{T}x_{n_{k_m}} \xrightarrow{o} y$ in F, see [16, Theorem VII.2.1]. Hence, the operator T is sequentially order compact. □

Following example shows that converse of Theorem 2.13 is not correct.

Example 2.14 *The identity operator $I\colon \ell_\infty \to \ell_\infty$ is sequentially order compact. The standard basis $\{e_n\colon n \geq 1\}$ is order bounded but not relatively compact in the norm topology of ℓ_∞. Hence, the operator $I\colon \ell_\infty \to \ell_\infty$ is not GAM-compact.*

Proposition 2.15 *Suppose that E is a Banach lattice, and, F is a σ-order continuous Banach lattice. If an operator $T\colon E \to F$ is sequentially order compact then T is bounded.*

Proof If $T\colon E \to F$ is not bounded then there exists a sequence x_n in E such that $\|x_n\| \leq 2^{-n}$ and $\|Tx_n\| \to \infty$. We observe that this sequence is order bounded. Since T is sequentially order compact, there exists a subsequence x_{n_k} and $y \in F$ such that $Tx_{n_k} \xrightarrow{o} y$ in F. Because F is a σ-order continuous Banach lattice, $Tx_{n_k} \xrightarrow{\|\cdot\|} y$. This is a contradiction. □

Corollary 2.16 *Suppose that E and F are σ-order continuous Banach lattices. If an operator $T\colon E \to F$ is sequentially order compact then T is sequentially order to norm continuous.*

Proof Let x_n be an order convergent sequence in E. Since E is σ-order continuous, the sequence x_n is norm convergent in E. As the operator $T\colon E \to F$ is bounded, Tx_n is norm convergent in F, see Proposition 2.15. Hence, $T\colon E \to F$ is sequentially order to norm continuous. □

Remark 2.17 *We remark that the conclusions of both Proposition 2.15 and Corollary 2.16 are still correct when the Banach lattices E and F are order continuous.*

It is known that if we restrict order convergence on a vector lattice F onto a subspace K of F then the resulting convergence on the subspace K differs from the original order convergence on F. When we compare this perspective of unbounded order convergence and that of order convergence, we see that regular sublattices hold a crucial role in preserving the order convergence.

Proposition 2.18 *Suppose that E and F are vector lattices and that K is regular sublattice of F. If $T : E \to K$ is an order compact operator then $T : E \to F$ is also an order compact operator. In particular, this is the case if K is an order dense vector sublattice of F.*

Proof Let x_α be an order bounded net in E. Since $T\colon E \to K$ is order compact there exists a subnet x_{α_β} and $y \in K$ such that $Tx_{\alpha_\beta} \xrightarrow{o} y \in K$. Because K is regular sublattice of F, we have $Tx_{\alpha_\beta} \xrightarrow{o} y$ where order convergence is considered in F, see [1, Theorem 1.20]. Therefore $T : E \to F$ is order compact. □

Lemma 2.19 *Let K be a regular order complete sublattice of F. Suppose that $y_\alpha \xrightarrow{o} y$ in F for some order bounded net y_α in K and some vector $y \in F$. Then $y \in K$ and $y_\alpha \xrightarrow{o} y$ in K.*

Proof Proof can be found in [10, Lemma 2.11]. □

Proposition 2.20 *Suppose that E and F are vector lattices. If an operator $T : E \to F$ is order compact and the range $R(T)$ is a subspace of a regular, majorizing and order complete sublattice K of F then $T : E \to K$ is order compact.*

Proof Let x_α be an order bounded net in E. Because $T\colon E \to F$ is order compact, there exists a subnet x_{α_β} and $y \in F$ such that $Tx_{\alpha_\beta} \xrightarrow{o} y$ in F. By Lemma 2.9, the operator $T\colon E \to F$ is order bounded. As x_α is order bounded in E, the subnet Tx_{α_β} is also order bounded in F. Since K is majorizing and the range $R(T)$ is a subspace of K, the subnet Tx_{α_β} is order bounded in K. It follows from Lemma 2.19 that $Tx_{\alpha_\beta} \xrightarrow{o} y$ in K. □

Proposition 2.21 *Suppose that E and F are vector lattices. If an operator $T\colon E \to F$ is both sequentially order compact and order continuous, and, the range $R(T)$ is a subspace of a regular, majorizing and order σ-complete vector sublattice K of F then $T\colon E \to K$ is sequentially order compact.*

Proof Let x_n be an order bounded sequence in E. Because $T\colon E \to F$ is sequentially order compact, there exists a subsequence x_{n_k} and $y \in F$ such that $Tx_{n_k} \xrightarrow{o} y$ in F. Because $T\colon E \to F$ is order continuous, the operator T is order bounded. As x_n is order bounded in E, the subsequence Tx_{n_k} is also order bounded in F. Since K is majorizing, the subsequence Tx_{n_k} is order bounded in K. It follows from [4, Lemma 27] that $Tx_{n_k} \xrightarrow{o} y$ in K. □

In Proposition 2.20, the conditions on K cannot be dropped. We give the details in the next example.

Example 2.22 *The space c_0 is a regular sublattice of ℓ_∞. Consider the operator $T : C[0,1] \to c_0$ defined by*

$$Tf = (f(1) - f(0), f(\frac{1}{2}) - f(0), f(\frac{1}{3}) - f(0), \dots)$$

for $f \in C[0,1]$. We claim that T is not order bounded. To see this, assume by way of contradiction that there exists some vector $u = (u_1, u_2, ...) \in c_0$ satisfying $| Tf | \leq u$ for all $f \in [0, \mathbb{1}]$, where $\mathbb{1}$ denotes the constant function one. For each n pick some $f_n \in [0, \mathbb{1}]$ with $f_n(0) = 0$ and $f_n(\frac{1}{n}) = 1$, and note that $1 = |f_n(\frac{1}{n}) - f_n(0)| \leq u_n$ holds. This shows that $u \notin c_0$, which is a contradiction. Hence, T is not order bounded, as claimed. In this case $T\colon C[0,1] \to c_0$ is not order compact. However, $T : C[0,1] \to l_\infty$ is order compact. Indeed, for every order bounded net f_α in $C[0,1]$, there exists a subnet f_{α_β} and $u \in \ell_\infty$ such that $Tf_{\alpha_\beta} \xrightarrow{o} u$.

The old nomenclature "normal lattice homomorphism", see [1, 15] for details, can be thought as the analogues of order continuous lattice homomorphism.

Definition 2.23 *An operator $\pi\colon E \to F$ between two vector lattices E and F is called a normal lattice homomorphism if π is a lattice homomorphism such that $x_\alpha \xrightarrow{o} 0$ in E implies $\pi(x_\alpha) \xrightarrow{o} 0$ in F.*

Following result emphasizes the fact that the natural notion of order compact operators have the left ideal property with respect to normal lattice homomorphisms.

Corollary 2.24 *Suppose that $\pi : K \to F$ is a normal lattice homomorphism and that $T : E \to K$ is order compact where E, F and K are vector lattices. Then the operator $\pi \circ T \colon E \to F$ is order compact.*

Proof Since $\pi\colon K \to F$ is a normal lattice homomorphism, π is order continuous. Let x_α be an order bounded net in E. Since $T\colon E \to K$ is order compact, there exist a subnet x_{α_β} and $y \in K$ such that $Tx_{\alpha_\beta} \xrightarrow{o} y \in K$. Because π is normal lattice homomorphism, $\pi T x_{\alpha_\beta} \xrightarrow{o} \pi y$, and hence, $\pi \circ T$ is order compact, see also Proposition 2.18. □

Proposition 2.25 *Let $T\colon E \to F$, $S\colon F/B \to G$, and $\pi\colon F \to F/B$ where E, F and G are vector lattices, B is a band of F, and π denotes the canonical quotient operator. If T is order compact and S is order continuous then the operator $S\pi T$ is order compact. Same conclusion also holds if T is order continuous and S is order compact.*

Proof By [12, Theorem 18.13] and [15, Theorem D], the canonical quotient operator $\pi\colon F \to F/B$ is a normal lattice homomorphism. If $T\colon E \to F$ is order compact and $S\colon F/B \to G$ is order continuous then the operator $S\pi$ is order continuous, and hence, $S\pi T$ is order compact. If $T\colon E \to F$ is order continuous and $S\colon F/B \to G$ is order compact then the operator πT is order continuous, and hence, $S\pi T$ is order compact. Therefore, in both cases the operator $S\pi T$ is order compact. □

3. Uo-compact operators

Definition 3.1 *A net x_α in E unbounded order convergent to $x \in E$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for all $u \in E^+$, see [4, 6–11]. In this case, we say that x_α uo-converges to x, and, we write $x_\alpha \xrightarrow{uo} x$.*

Example 3.2 *The notion of unbounded order convergence in vector lattices is a generalization of almost everywhere convergence. Let (Ω, Σ, μ) be a σ-finite measure space. A sequence x_n in $L^p(\Omega)$ order converges to $x \in L^p(\Omega)$ $(1 \le p \le \infty)$ if and only if x_n converges to x almost everywhere and there exists some $z \in L^p(\Omega)$ such that $|x_n| \le z$ for all n. In the case $p < \infty$, x_n is unbounded order convergent to x if and only if x_n converges almost everywhere to x. In the cases of c_0 and ℓ_p with $1 \le p \le \infty$, uo-convergence of nets agrees with pointwise convergence, see [11].*

Definition 3.3 *Let E and F be two vector lattices. An operator $T\colon E \to F$ is said to be uo-compact, if for any order bounded net x_α in E, there is a subnet x_{α_β} and $y \in F$ such that $Tx_{\alpha_\beta} \xrightarrow{uo} y$ in F.*

Proposition 3.4 *Suppose that $T\colon E \to F$ is an operator between vector lattices E and F. If T is order compact then it is uo-compact.*

Proof Let x_α be an order bounded net in E. Since T is order compact, there exists a subnet x_{α_β} and $y \in F$ such that $Tx_{\alpha_\beta} \xrightarrow{o} y$ in F. It follows that $Tx_{\alpha_\beta} \xrightarrow{uo} y$ in F. This shows that the operator $T\colon E \to F$ is *uo*-compact. □

In view of Proposition 3.4, main examples of *uo*-compact operators are provided by order compact operators.

We recall that an operator $T\colon E \to F$ from a vector lattice E into a normed lattice F is called semicompact, see [18, Chapter 18] and [3], if T maps order bounded subsets of E into almost order bounded subsets of F.

Proposition 3.5 *Suppose that $T\colon E \to F$ is an operator where E is a Banach lattice and F is an order continuous Banach lattice. If the operator T is both semicompact and sequentially uo-compact then T is sequentially order compact.*

Proof Let x_n be an order bounded sequence in E. Since $T\colon E \to F$ is sequentially uo-compact, there is a subsequence x_{n_k} and $y \in F$ such that $Tx_{n_k} \xrightarrow{uo} y$ in F. As T is semicompact, the sequence Tx_{n_k} is almost order bounded in F. It follows that $Tx_{n_k} \xrightarrow{\|\cdot\|} y$, see [11, Prop.3.7]. Hence, there is a further subsequence $x_{n_{k_m}}$, see [16, Theorem VII.2.1], such that $Tx_{n_{k_m}} \xrightarrow{o} y$. This shows that $T\colon E \to F$ is sequentially order compact. □

Proposition 3.6 *Let E be a Banach lattice and F be a σ-order continuous normed lattice. If an operator $T : E \to F$ is sequentially uo-compact and order bounded then $T : E \to F$ is GAM-compact.*

Proof Let x_n be an order bounded sequence in E. Since T is sequentially uo-compact, there exists a subsequence x_{n_k} and some $y \in F$ such that $Tx_{n_k} \xrightarrow{uo} y$ in F. By the σ-order continuity of the lattice norm of F, we have $Tx_{n_k} \xrightarrow{un} y$ in F, see [8, Proposition 2.5]. Moreover, since T is order bounded, the sequence Tx_n is order bounded in F. In particular, the subsequence Tx_{n_k} is almost order bounded in F. It follows that $Tx_{n_k} \xrightarrow{\|\cdot\|} y$, see [8, Lemma 2.9]. Therefore, the operator $T\colon E \to F$ is GAM-compact. □

Properties of uo-convergence on vector lattices can be used to obtain results on uo-continuous operators. An operator $T\colon E \to F$ between two vector lattices is said to be uo-continuous if $x_\alpha \xrightarrow{uo} 0$ implies $Tx_\alpha \xrightarrow{uo} 0$.

Proposition 3.7 *Suppose that $R\colon E \to F$, $T\colon F \to G$, $L\colon G \to E$ where E, F and G are vector lattices.*

i. *If R is order bounded and T is uo-compact then $T \circ R$ is uo-compact.*

ii. *If L is uo-continuous and T is uo-compact then $L \circ T$ is uo-compact.*

iii. *If T is a positive, order continuous and uo-compact operator and $R_\alpha \downarrow 0$ is a decreasing net of order bounded operators then $T \circ R_\alpha \downarrow 0$ is a decreasing net of uo-compact operators.*

Proof ($i.$) Let x_α be an order bounded net in E. Since R is order bounded, Rx_α is order bounded in F. As T is uo-compact there exists a subnet Rx_{α_β} and $y \in G$ such that $TRx_{\alpha_\beta} \xrightarrow{uo} y$ in G. Hence, the operator $T \circ R$ is uo-compact.

($ii.$) Let x_α be an order bounded net in F. Since T is uo-compact there is a subnet x_{α_β} and $y \in G$ such that $Tx_{\alpha_\beta} \to y$ in G. As L is uo-continuous $L(Tx_{\alpha_\beta}) \xrightarrow{uo} L(y)$. This shows that the operator $L \circ T$ is uo-compact.

($iii.$) By [2, Theorem 1.18], a net R_α of order bounded operators from E into F satisfies $R_\alpha \downarrow 0$ if and only if $R_\alpha(x) \downarrow 0$ in F for each $x \in E^+$. As $T\colon F \to G$ is both order continuous and uo-compact, $T \circ R_\alpha$ is order bounded and uo-compact for each α, see also Lemma 2.9. It follows that for each $x \in E^+$, we have $T(R_\alpha(x)) \xrightarrow{o} 0$ because T is order continuous. Since T is a positive operator we have $T \circ R_\alpha \downarrow 0$. □

Following result can be used to produce examples of order compact operators, also see Theorem 2.13.

Theorem 3.8 *Let E be an AM-space with a strong norm unit, and, F a σ-order continuous Banach lattice. If an operator $T : E \to F$ is sequentially order compact if and only if it is compact.*

Proof Let x_n be a norm bounded sequence in E. Because E is an AM-space with a strong norm unit, the sequence x_n is order bounded. As $T\colon E \to F$ is sequentially order compact, there exists a subsequence x_{n_k} and $y \in F$ such that $Tx_{n_k} \xrightarrow{o} y$. Since F is σ-order continuous Banach lattice, $Tx_{n_k} \xrightarrow{\|\cdot\|} y$ in F. Hence, $T\colon E \to F$ is compact. Conversely, let x_n be an order bounded sequence in E. Because order bounded sets are in particular norm bounded, the sequence x_n is norm bounded. As $T\colon E \to F$ is compact there exists a subsequence x_{n_k} and $y \in F$ such that $Tx_{n_k} \xrightarrow{\|\cdot\|} y$. It follows from [16, Theorem VII.2.1] that there exists a further subsequence $x_{n_{k_m}}$ such that $Tx_{n_{k_m}} \xrightarrow{o} y$ in F. Hence, the operator $T\colon E \to F$ is sequentially order compact. □

Theorem 3.9 *Let F be a vector lattice, and K be a sublattice of F. Following statements are equivalent:*

i. *K is regular sublattice of F,*

ii. *For any net y_α in K, $y_\alpha \xrightarrow{uo} 0$ in K implies $y_\alpha \xrightarrow{uo} 0$ in F.*

iii. *For any net y_α in K, $y_\alpha \xrightarrow{uo} 0$ in K implies $y_\alpha \xrightarrow{uo} 0$ in F.*

Proof See [10, Theorem 3.2]. □

Theorem 3.10 *Suppose that E and F are vector lattices and that K is a regular sublattice of F. If $T : E \to K$ is a uo-compact operator then $T : E \to F$ is again a uo-compact operator. Conversely, if an operator $T : E \to F$ is uo-compact and the range $R(T)$ is a subspace of a uo-closed regular sublattice K of F then $T : E \to K$ is uo-compact.*

Proof Suppose that $T\colon E \to K$ is a uo-compact operator. Let x_α be an order bounded net in E. There exists a subnet x_{α_β} and $y \in K$ such that $Tx_{\alpha_\beta} \xrightarrow{uo} y$ in K. Since K is regular, by Theorem 3.9, $Tx_{\alpha_\beta} \xrightarrow{uo} y$ in F. This shows that the operator $T : E \to F$ is uo-compact.

For the second statement, let x_α be an order bounded net in E. Since $T\colon E \to F$ is uo-compact, there exists a subnet x_{α_β} and $y \in F$ such that $Tx_{\alpha_\beta} \xrightarrow{uo} y$ in F. Because the range of T is a subspace of the uo-closed subspace K of F, we have $y \in K$. It follows from $Tx_{\alpha_\beta} \xrightarrow{uo} y$ in F that $Tx_{\alpha_\beta} - y \xrightarrow{uo} 0$ in F. By Theorem 3.9, $T(x_{\alpha_\beta}) - y \xrightarrow{uo} 0$ in the regular sublattice K. Hence, $T\colon E \to K$ is uo-compact. □

Following fact emphasizes the fact that the natural notion of uo-compact operators have the left ideal property with respect to surjective normal lattice homomorphisms. A discussion related to surjectiveness of vector lattice homomorphisms can be found in [15, p. 293].

Corollary 3.11 *Suppose that $\pi\colon K \to F$ is a surjective normal lattice homomorphism and that $T\colon E \to K$ is uo-compact where E, F and K are vector lattices. Then the operator $\pi \circ T\colon E \to F$ is uo-compact.*

Proof Since $\pi\colon K \to F$ is a normal lattice homomorphism, π is order continuous. Let x_α be an order bounded net in E. Since $T\colon E \to K$ is uo-compact, there exist a subnet x_{α_β} and $y \in K$ such that $Tx_{\alpha_\beta} \xrightarrow{uo} y \in K$.

Given $u \in F^+$, let $v \in K^+$ be such that $\pi(v) = u$. Hence, $|Tx_{\alpha_\beta} - y| \wedge v \xrightarrow{o} 0$. It follows from normality of $\pi \colon K \to F$ that

$$\pi(|Tx_{\alpha_\beta} - y| \wedge v) = |\pi(T(x_{\alpha_\beta})) - \pi(y)| \wedge u \xrightarrow{o} 0$$

where the limit is taken over β. Hence, the operator $\pi \circ T \colon E \to F$ is uo-compact. □

4. Relatively uniformly compact operators

Definition 4.1 *A net x_α in E relatively uniformly convergent to $x \in E$ if there exists an element $u \in E^+$ such that for every $\epsilon > 0$ there exists some α_0 such that $|x_\alpha - x| \leq \epsilon u$ for each $\alpha \geq \alpha_0$, see [3, 12, 16]. In this case, we say that the net x_α is r-convergent to x, and, we write $x_\alpha \xrightarrow{r} x$.*

Example 4.2 *Let (Ω, Σ, μ) be a σ-finite measure space. A sequence x_n in $L^p(\Omega)$ relatively uniformly convergent to some $x \in L^p(\Omega)$ $(1 \leq p \leq \infty)$ if and only if the sequence x_n is order convergent to x.*

Definition 4.3 *An operator $T \colon E \to F$ between two vector lattices is said to be relatively uniformly compact, abbreviated as r-compact, if for every order bounded net x_α in E there exists some subnet x_{α_β} and $y \in F$ such that $Tx_{\alpha_\beta} \xrightarrow{r} y$.*

In view of Definition 4.3, an operator $T \colon E \to F$ is sequentially r-compact if for every order bounded sequence x_n in E there exists some subsequence x_{n_k} and $y \in F$ such that $Tx_{n_k} \xrightarrow{r} y$.

The following result is a special case of [3, Remark 4.i].

Proposition 4.4 *Suppose that E and F are vector lattices. If $T \colon E \to F$ is r-compact then it is order compact.*

Proof Let x_α be an order bounded net in E. As $T \colon E \to F$ is r-compact, there exists a subnet x_{α_β} and $y \in F$ such that $Tx_{\alpha_\beta} \xrightarrow{r} y$. This means that there exists some $u \in F^+$ such that for every $\epsilon > 0$ we have $|Tx_{\alpha_\beta} - y| \leq \epsilon u$. Hence, $Tx_{\alpha_\beta} \xrightarrow{o} y$ in F. □

In the following proposition, GAM-compactness of the operator can be replaced by AM-compactness because neither norm completeness of the underlying spaces nor order boundedness of the operator are required.

Proposition 4.5 *Suppose that E is a vector lattice and F is a Banach lattice. If $T \colon E \to F$ is r-compact then it is GAM-compact.*

Proof Let x_α be an order bounded net in E. As the operator $T \colon E \to F$ is r-compact, there exists a subnet x_{α_β} and $y \in F$ such that $Tx_{\alpha_\beta} \xrightarrow{r} y$. This means that there exists some $u \in F^+$ such that for every $\epsilon > 0$ we have $|Tx_{\alpha_\beta} - y| \leq \epsilon u$. By considering the lattice norm on F, we conclude that $\||Tx_{\alpha_\beta} - y|\| \leq \epsilon \|u\|$. Hence, $T \colon E \to F$ is GAM-compact. □

Proposition 4.6 *Suppose that E and F are Banach lattices. If an operator $T \colon E \to F$ is compact then it is sequentially relatively uniformly compact.*

Proof Let x_n be an order bounded sequence in E. Hence, the sequence is norm bounded. As $T\colon E \to F$ is compact, there exists a subsequence x_{n_k} and $y \in F$ such that $x_{n_k} \xrightarrow{\|\cdot\|} y$ in F. Hence, the sequence x_{n_k} has a further subsequence $x_{n_{k_m}}$ such that $Tx_{n_{k_m}} \xrightarrow{r} y$, see [18, Theorem 105.15]. This shows that the operator $T\colon E \to F$ is relatively uniformly compact. $\square$

We recall from [16, Chapter VI.4] and [12, Theorem 84.3] that order convergence on a vector lattice F is said to be stable if for every $y_n \xrightarrow{o} 0$ there exists $0 \le \lambda_n \uparrow \infty$ such that $\lambda_n y_n \xrightarrow{o} 0$. Order convergence on L^p and ℓ_p for $1 \le p < \infty$ is stable. Almost regular vector lattices, see [16, Chapter VI.4], are among those vector lattices satisfying this stability assumption.

Proposition 4.7 *Suppose that E is a vector lattice and F is an almost regular vector lattice. An operator $T\colon E \to F$ is sequentially r-compact if and only if T is sequentially order compact.*

Proof If F is an almost regular vector lattice then order convergence and relative uniform convergence are equivalent for sequences, see [16, Chapter VI.4]. Therefore, the classes of sequentially r-compact operators and sequentially order compact operators agree. $\square$

Proposition 4.8 *Suppose that $T\colon E \to F$ is an order compact operator where E and F are vector lattices and the order convergence on F is stable. In this case, T is sequentially order continuous.*

Proof Because the order convergence on F is stable, order convergence and relatively uniform convergence on F agree for sequences. By Lemma 2.9, the operator $T\colon E \to F$ is order bounded. By [18, Theorem 84.3], the operator T is sequentially order continuous. $\square$

Corollary 4.9 *Suppose that $T : E \to F$ is an operator where E is a vector lattice and the order convergence on the Banach lattice F is stable. If T is GAM-compact then T is sequentially relatively uniformly compact.*

Proof Let x_n be an order bounded sequence in E. Since T is GAM-compact there is a subsequence x_{n_k} and $y \in F$ such that $\|Tx_{n_k} - y\|_F \to 0$. As F is a Banach lattice there exists a subsequence $x_{n_{k_m}}$ such that $Tx_{n_{k_m}} \xrightarrow{o} y$ in F, see [16, Theorem VII.2.1]. Because the order convergence on F is stable, $Tx_{n_{k_m}} \xrightarrow{r} y$, see [12, Theorem 84.3]. Hence, the operator T is sequentially relatively uniformly compact. $\square$

Proposition 4.10 *Suppose that E is an AM-space with a strong norm unit and F is a Banach lattice. If $T\colon E \to F$ is r-compact then it is compact.*

Proof Let B be a norm bounded subset of E. Since E is an AM-space with a strong norm unit, the set B is order bounded. Let x_α be a net in B. As $T\colon E \to F$ is r-compact, there exists a subnet x_{α_β} and $y \in F$ such that $Tx_{\alpha_\beta} \xrightarrow{r} y$. Hence, $Tx_{\alpha_\beta} \to y$ in norm. It follows that the set $T(B)$ is relatively compact in F. Hence, the operator $T\colon E \to F$ is compact. $\square$

Example 4.11 *There exists semicompact operators which are not relatively uniformly compact. Let K be an infinite Hausdorff compact space. The identity operator $I : C(K) \to C(K)$ is semicompact but not compact. Hence, it is not relatively uniformly compact.*

Example 4.12 *There exists sequentially order compact operators which are not* r*-compact. Because* ℓ_∞ *is an atomic* KB*-space, the identity operator* $I\colon \ell_\infty \to \ell_\infty$ *is sequentially order compact but it is not* r*-compact.*

Acknowledgments
This work was partially supported by The Scientific and Technological Research Council of Turkey (TÜBİTAK–Project 118F204). The authors gratefully acknowledge an anonymous referee for useful comments improved the presentation of this paper.

References

[1] Aliprantis CD, Burkinshaw O. Locally solid Riesz spaces with applications to economics. Mathematical Surveys and Monographs, Vol. 105, Providence, RI, USA: American Mathematical Society, 2003.

[2] Aliprantis CD, Burkinshaw O. Positive operators (reprint of the 1985 original). Dordrecht, Netherlands: Springer, 2006.

[3] Aydın A, Emelyanov EY, Erkurşun Özcan N, Marabeh MAA. Compact-like operators in lattice-normed spaces. Indagationes Mathematicae 2018; 29 (2): 633-656. doi: 10.1016/j.indag.2017.11.002

[4] Azouzi Y. Completeness for vector lattices. Journal of Mathematical Analysis and Applications 2019; 472: 216-230. doi: 10.1016/j.jmaa.2018.11.019

[5] Azouzi Y, Ben Amor MA. On Compact operators between lattice normed spaces. arXiv 2019; 1903.00370v1.

[6] Dabboorasad YAM, Emel'yanov EYu, Marabeh MAA. $u\tau$-convergence in locally solid vector lattices. Positivity 2018; 22: 1065-1080. doi: 10.1007/s11117-018-0559-4

[7] DeMarr R. Partially ordered linear spaces and locally convex linear topological spaces. Illinois Journal of Mathematics 1964; 8: 601-606. doi: 10.1215/ijm/1256059459

[8] Deng Y, O'Brien M, Troitsky VG. Unbounded norm convergence in Banach lattices. Positivity 2017; 21: 963-974. doi: 10.1007/s11117-016-0446-9

[9] Erkurşun-Özcan N, Gezer NA. Unbounded asymptotic equivalences of operator nets with applications. Positivity 2019; 23. doi: 10.1007/s11117-018-0640-z

[10] Gao N, Troitsky VG, Xanthos F. Uo-convergence and its applications to Cesaro means in Banach lattices. Israel Journal of Mathematics 2017; 220: 649-689. doi: 10.1007/s11856-017-1530-y

[11] Gao N, Xanthos F. Unbounded order convergence and application to martingales without probability. Journal of Mathematical Analysis and Applications 2014; 415: 931-947. doi: 10.1016/j.jmaa.2014.01.078

[12] Luxemburg WAJ, Zaanen AC. Riesz Spaces I. Amsterdam, Netherlands: North-Holland Publishing Company, 1971.

[13] Nakano H. Ergodic theorems in semi-ordered linear spaces. Annals of Mathematics 1948; 49 (2): 538-556. doi: 10.2307/1969044

[14] Pliev M. Narrow operators on lattice-normed spaces. Central European Journal of Mathematics 2001; 9 (6): 1276-1287. doi: 10.2478/s11533-011-0090-3

[15] Tucker CT. Homomorphisms of Riesz spaces. Pacific Journal of Mathematics 1974; 55 (1).

[16] Vulikh BZ. Introduction to the theory of partially ordered spaces. Groningen, Netherlands: Wolters-Noordhoff Scientific Publications, 1967.

[17] Wickstead AW. Weak and unbounded order convergence in Banach lattices. Journal of the Australian Mathematical Society 1977; 24 (3): 312-319. doi: 10.1017/S1446788700020346

[18] Zaanen AC. Riesz Spaces II. Vol. 30 of North-Holland Mathematical Library. Amsterdam, Netherlands: North-Holland Publishing Company, 1983.