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New oscillation criteria for differential equations with sublinear and superlinear neutral terms

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Abstract: The aim of this article is to establish some new oscillation criteria for the differential equation of even-order of the form

$$(r(l)(y^{(n-1)}(l))^{\alpha})' + f(l, x(\tau(l))) = 0,$$

where $y(l) = x(l) + p(l)x^{\beta}(\sigma_1(l)) + h(l)x^{\delta}(\sigma_2(l))$. By using Riccati transformations, we present new conditions for oscillation of the studied equation. Furthermore, two illustrative examples showing applicability of the new results are included.

Key words: Sublinear and superlinear neutral terms, even-order differential equations, oscillation criteria

1. Introduction

In this work, we study the oscillatory properties of solutions of the even-order nonlinear differential equation with sublinear and superlinear neutral terms of the form

$$\left(r(l)\left(\left(x(l) + p(l)x^{\beta}(\sigma_1(l)) + h(l)x^{\delta}(\sigma_2(l))\right)^{(n-1)}\right)^{\alpha}\right)' + f(l, x(\tau(l))) = 0, \tag{1.1}$$

where $l \geq l_0$, n is an even natural number. Through the work, we assume the following

(B1) α, β, and δ are ratios of odd natural numbers with $0 < \beta < 1$ and $\delta \geq 1$;

(B2) $r \in C([l_0, \infty), \mathbb{R}^+)$, $r'(l) \geq 0$, and

$$S(l, l_0) := \int_{l_0}^{l} \frac{1}{r^{1/\alpha}(\xi)} d\xi \to \infty \text{ as } l \to \infty;$$

(B3) p, $h \in C[l_0, \infty)$, $p(l) \geq 0$, and $h(l) \geq 0$;

(B4) τ, σ_1, $\sigma_2 \in C([l_0, \infty), \mathbb{R})$, $\tau(l) \leq l$, $\tau' > 0$, $\sigma_1(l) \leq l$, $\sigma_2(l) \leq l$, and $\lim_{l\to\infty} \tau(l) = \lim_{l\to\infty} \sigma_1(l) = \lim_{l\to\infty} \sigma_2(l) = \infty$;

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(B5) $f \in C([l_0,\infty) \times \mathbb{R}, \mathbb{R})$ and there exists a function $q \in C([l_0,\infty), [0,\infty))$ such that $|f(l,x)| \geq q(l)|x|^{\gamma}$ where γ is a ratios of odd natural numbers.

To facilitate calculations, we will denote the corresponding function of the solution x by

$$y := x + p \cdot (x^{\beta} \circ \sigma_1) + h \cdot (x^{\delta} \circ \sigma_2).$$

By a solution of (1.1), we mean a function $x \in C([l_x,\infty))$, $l_x \geq l_0$, with $y,\ r(l)(y'(l))^{\alpha} \in C^1([l_x,\infty))$, and it satisfies (1.1) on $[l_x,\infty)$. We focus in our study on the solutions that satisfy $\sup\{|x(l)| : l \geq l_0\} > 0$, for every $l \geq l_x$. Such a solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. The equation itself is termed oscillatory if all its solutions oscillate.

Differential equations with neutral delay have many applications including population dynamics, automatic control, mixing liquids, and vibrating masses attached to an elastic bar; see Hale [6]. In recent decades, there has been an increasing interest in studying the oscillation theory of solutions of differential equations of different orders, see for example, [1–3, 8–16, 18, 19]. Most of these papers studied the neutral differential equations with corresponding function of the form

$$z := x + p \cdot (x \circ \sigma).$$

Graef et al. [4] related the oscillatory properties of solutions of even-order differential equations with unbounded neutral term of the form

$$z^{(n)}(l) + \int_a^b q(l,\xi)\, x^{\alpha}(g(l,\xi))\, \mathrm{d}\xi = 0,$$

where $\sigma(l) \geq l$, $g(l,\xi) \leq l$, and τ is strictly increasing. Graef et al. [5] studied the oscillation of even-order sublinear neutral differential equation

$$\left(x(l) + p(l)\, x^{\beta}(\sigma(l))\right)^{(n)} + q(l)\, x^{\alpha}(\tau(l)) = 0,$$

where $\sigma(l) \leq l$.

The purpose of the article is to study the oscillatory properties of solutions of (1.1). By using Riccati transformations, we present new oscillation conditions for (1.1). Our results extend and complement the previous related results in [4, 5]. Examples are provided to illustrate the importance of the new results.

2. Some preliminary lemmas

Next, we state some preliminary lemmas, which will be necessary in the proofs of our main results.

Lemma 2.1 *[17] Let $f \in C^n([l_0,\infty),(0,\infty))$ and $f^{(n)}(l)$ is of one sign for all large l. Then, there are a $l_x \geq l_0$ and a $\eta \in [0,n]$ is an integer, with $n+\eta$ even for $f^{(n)}(l) \geq 0$, or $n+\eta$ odd for $f^{(n)}(l) \leq 0$ such that*

$$\eta > 0 \textit{ implies } f^{(k)}(l) > 0 \textit{ for } l \geq l_x,\ k = 0,1,...,\eta-1,$$

and

$$\eta \leq n-1 \textit{ implies } (-1)^{\eta+k} f^{(k)}(l) > 0 \textit{ for } l \geq l_x,\ k = \eta, \eta+1, ..., n-1.$$

Lemma 2.2 *[7] If $A \geq 0,\ B \geq 0$ and $0 < \kappa < 1,$ then*

$$A^{\kappa} - \kappa A B^{\kappa-1} - (1-\kappa) B^{\kappa} \leq 0.$$

Moreover, the equality is satisfied if and only if $A = B$.

Lemma 2.3 *[17] Assume that y is a positive function and differentiable n times on $[l_1, \infty)$. If $y^{(n)}(l) \leq 0$ and $y^{(n)}(l) \neq 0$ on any interval $[l_*, \infty)$, $l_* \geq l_0$ and $y^{(n-1)}(l) \geq 0$ for all $l \geq l_y \geq l_1,$ then there exist a constant $\lambda \in (0,1)$ and a positive constant N such that*

$$y'(\lambda l) \geq N l^{n-2} \left| y^{(n-1)}(l) \right|,$$

for $l \geq l_y$.

Lemma 2.4 *[20] Assume that K and E are real, $E > 0$. Then*

$$Kw - Ew^{(\alpha+1)/\alpha} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{K^{\alpha+1}}{E^{\alpha}}.$$

3. Main results

Now, we present the main theorems which give oscillation criteria for solutions of (1.1). To facilitate calculations, we adopt the following notations:

$$\varphi(l) \quad : \quad = \epsilon^{\gamma} \int_l^{\infty} q(u)\, \Omega(u)\, \mathrm{d}u,$$

$$\varpi(l) \quad : \quad = \alpha \lambda N \tau^{n-2}(l)\, \tau'(l)\, r^{-1/\alpha}(l)$$

and

$$\Omega(l) = \begin{cases} k_1^{\gamma-\alpha} & \text{if } \gamma \geq \alpha; \\ k_3^{\gamma-\alpha} \left(l^{n-2}\right)^{\gamma-\alpha} S^{\gamma-\alpha}(l, l_1) & \text{if } \gamma < \alpha, \end{cases}$$

where $\epsilon,\ \lambda \in (0,1)$, $N,\ k_1$, and k_3 are positive real constants.

Theorem 3.1 *Assume that*

$$\lim_{l\to\infty} h(l) \left(l^{n-2} S(l, l_0)\right)^{\delta-1} = \lim_{l\to\infty} p(l) = 0. \tag{3.1}$$

If

$$\liminf_{l\to\infty} \frac{1}{\varphi(l)} \int_l^{\infty} \varpi(\xi)\, \varphi^{(\alpha+1)/\alpha}(\xi)\, \mathrm{d}\xi > \frac{\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}}, \tag{3.2}$$

for some $\epsilon \in (0,1)$, $k_1, k_2 > 0$ and for all $\lambda \in (0,1)$, $N > 0$, then (1.1) is oscillatory.

Proof Assume that x is a nonoscillatory solution of equation (1.1). Hence, there exists a $l_1 \geq l_0$ such that $x(l) > 0,\ x(\tau(l)) > 0,\ x(\sigma_1(l)) > 0,$ and $x(\sigma_2(l)) > 0$ for $l \geq l_1$. From (1.1), it follows that

$$(r(l)\, (y^{(n-1)})^{\alpha})' \leq -q(l)\, x^{\gamma}(\tau(l))), \tag{3.3}$$

for $l \geq l_1$. Using Lemma 2.1 and taking into account the fact that $r'(l) \geq 0$, we get that there exists a $l_2 \geq l_1$ such that

$$y'(l) > 0,\ y^{(n-1)}(l) > 0,\ \left(r(l)\left(y^{(n-1)}(l)\right)^{\alpha}\right)' \leq 0,\ \text{and } y^{(n)}(l) \leq 0, \tag{3.4}$$

for $l \geq l_2$. Since $x(l) \leq y(l)$, we have, from the definition of y, that

$$\begin{aligned} x(l) &= y(l) - p(l)\, x^{\beta}(\sigma_1(l)) - h(l)\, x^{\delta}(\sigma_2(l)) \\ &\geq y(l) - p(l)\, y^{\beta}(l) - h(l)\, y^{\delta}(l) \\ &= y(l) - h(l)\frac{y(l)}{y^{1-\delta}(l)} - p(l)\left[y^{\beta}(l) - y(l)\right] - p(l)\, y(l). \end{aligned} \tag{3.5}$$

Using Lemma 2.2 with $\kappa = \beta,\ A = y$ and $B = \beta^{1/(1-\beta)}$, we obtain that

$$y^{\beta}(l) - y(l) \leq (1-\beta)\,\beta^{\beta/(1-\beta)}. \tag{3.6}$$

Combining (3.6) and (3.5), we arrive at

$$x(l) \geq y(l)\left[1 - \frac{h(l)}{y^{1-\delta}(l)} - \frac{p(l)(1-\beta)\,\beta^{\beta/(1-\beta)}}{y(l)} - p(l)\right]. \tag{3.7}$$

Since $y(l) > 0$ and $y'(l) > 0$ on $[l_2, \infty)$, there exists a $k_1 > 0$ such that

$$y(l) \geq k_1,\ \text{for } l \geq l_2, \tag{3.8}$$

so

$$y^{\gamma-\alpha}(l) \geq k_1^{\gamma-\alpha},\ \text{for } \gamma \geq \alpha. \tag{3.9}$$

Since $\left(r(l)\left(y^{(n-1)}(l)\right)^{\alpha}\right)' \leq 0$, there exist a $k_2 > 0$ and $l_3 \geq l_2$ such that

$$r(l)\left(y^{(n-1)}(l)\right)^{\alpha} \leq k_2,\ \text{for } l \geq l_3. \tag{3.10}$$

Integrating (3.10) from l_3 to l for a total of $n-1$ times, we have

$$y(l) \leq k_3 l^{n-2} S(l, l_3),\ \text{for } l \geq l_3. \tag{3.11}$$

Thus,

$$y^{\gamma-\alpha}(l) \geq k_3^{\gamma-\alpha}\left(l^{n-2}\right)^{\gamma-\alpha} S^{\gamma-\alpha}(l, l_3),\ \text{when } \gamma < \alpha. \tag{3.12}$$

Therefore, combining (3.9) and (3.12), we arrive at

$$y^{\gamma-\alpha}(l) \geq \Omega(l), \tag{3.13}$$

In view of (3.8) and (3.11), inequality (3.7) becomes

$$\begin{aligned} x(l) &\geq \left[1 - h(l)\left(k_3 l^{n-2} S(l, l_3)\right)^{\delta-1} - p(l)\left(\frac{(1-\beta)\,\beta^{\beta/(1-\beta)}}{k_1} + 1\right)\right] y(l) \\ &\geq \left[1 - k_4\left(h(l)\left(l^{n-2} S(l, l_3)\right)^{\delta-1} + p(l)\right)\right] y(l), \end{aligned} \tag{3.14}$$

where

$$k_4=\max\left(k_3^{\delta-1},\frac{(1-\beta)\,\beta^{\beta/(1-\beta)}}{k_1}+1\right).$$

From (3.1), for any $\epsilon\in(0,1)$ there exists $l_\epsilon\geq l_3$ such that

$$x(l)\geq\epsilon y(l)\ \text{ for } l\geq l_\epsilon,$$

which, with $\lim_{l\to\infty}\tau(l)=\infty$, gives

$$x(\tau(l))\geq\epsilon y(\tau(l))\ \text{ for } l\geq l_4\geq l_\epsilon. \tag{3.15}$$

Taking into account $y^{(n-1)}(l)\,y^{(n)}(l)\leq 0$, and using Lemma 2.3, we have that there exist $\lambda\in(0,1)$ and $N>0$ such that

$$y'(\lambda l)\geq Nl^{n-2}y^{(n-1)}(l). \tag{3.16}$$

Now, we set

$$\Psi(l)=\frac{r(l)\left(y^{(n-1)}(l)\right)^{\alpha}}{y^{\alpha}(\lambda\tau(l))}. \tag{3.17}$$

By differentiating Ψ and and using (3.3), we arrive at

$$\begin{aligned}\Psi'(l) &= \frac{\left(r(l)\left(y^{(n-1)}(l)\right)^{\alpha}\right)'}{y^{\alpha}(\lambda\tau(l))}-\frac{\alpha r(l)\left(y^{(n-1)}(l)\right)^{\alpha}y'(\lambda\tau(l))\,\lambda\tau'(l)}{y^{\alpha+1}(\lambda\tau(l))}\\ &\leq -\frac{q(l)\,x^{\gamma}(\tau(l))}{y^{\alpha}(\lambda\tau(l))}-\frac{\alpha r(l)\left(y^{(n-1)}(l)\right)^{\alpha}y'(\lambda\tau(l))\,\lambda\tau'(l)}{y^{\alpha+1}(\lambda\tau(l))},\end{aligned}$$

which with (3.15) gives

$$\Psi'(l)\leq-\frac{q(l)\,\epsilon^{\gamma}y^{\gamma}(\tau(l))}{y^{\alpha}(\lambda\tau(l))}-\frac{\alpha r(l)\left(y^{(n-1)}(l)\right)^{\alpha}y'(\lambda\tau(l))\,\lambda\tau'(l)}{y^{\alpha+1}(\lambda\tau(l))}. \tag{3.18}$$

From (3.13) and (3.16), (3.18) becomes

$$\begin{aligned}\Psi'(l) &\leq -q(l)\,\epsilon^{\gamma}y^{\gamma-\alpha}(\tau(l))-\frac{\alpha N\tau^{n-2}(l)\,\lambda\tau'(l)\,r(l)\left(y^{(n-1)}(l)\right)^{\alpha+1}}{y^{\alpha+1}(\lambda\tau(l))}\\ &\leq -q(l)\,\epsilon^{\gamma}\Omega(l)-\frac{\alpha N\tau^{n-2}(l)\,\lambda\tau'(l)}{r^{1/\alpha}(l)}\Psi^{(\alpha+1)/\alpha}(l).\end{aligned} \tag{3.19}$$

Integrating (3.19) from l to ∞, and using the facts $\Psi>0$ and $\Psi'<0$, we get

$$-\Psi(l)\leq-\int_l^{\infty}q(\xi)\,\epsilon^{\gamma}\Omega(\xi)\,\mathrm{d}\xi-\int_l^{\infty}\frac{\alpha N\tau^{n-2}(\xi)\,\lambda\tau'(\xi)}{r^{1/\alpha}(\xi)}\Psi^{(\alpha+1)/\alpha}(\xi)\,\mathrm{d}\xi.$$

Furthermore, we may write

$$\frac{\Psi(l)}{\varphi(l)}\geq 1+\frac{1}{\varphi(l)}\int_l^{\infty}\frac{\alpha N\tau^{n-2}(\xi)\,\lambda\tau'(\xi)}{r^{1/\alpha}(\xi)}\varphi(\xi)^{(\alpha+1)/\alpha}\left(\frac{\Psi(\xi)}{\varphi(\xi)}\right)^{(\alpha+1)/\alpha}\mathrm{d}\xi. \tag{3.20}$$

If we set $\kappa = \inf_{l \geq l} \Psi(l) / \varphi(l)$, then obviously $\kappa \geq 1$. Hence, it follows from (3.2) and (3.20) that

$$\kappa \geq 1 + \alpha \left(\frac{\kappa}{\alpha + 1} \right)^{1+1/\alpha}.$$

Or, equivalent

$$\frac{\kappa}{\alpha + 1} \geq \frac{1}{\alpha + 1} + \frac{\alpha}{\alpha + 1} \left(\frac{\kappa}{\alpha + 1} \right)^{1+1/\alpha},$$

this contradicts with the acceptable value for $\kappa \geq 1$ and $\alpha > 0$. Therefore, the proof is complete. □

Corollary 3.2 *Assume that (3.1) holds. If*

$$\int_{l_0}^{\infty} q(l) \, \mathrm{d}l = \infty, \tag{3.21}$$

then (1.1) is oscillatory.

Proof Assume that x is a nonoscillatory solution of equation (1.1). Proceeding as in the proof of Theorem 3.1, we arrive at (3.19) for $l \geq l_1$. It is easy to see that $\left(\alpha N \tau^{n-2}(l) \lambda \tau'(l) \right) \Psi^{(\alpha+1)/\alpha}(l) / r^{1/\alpha}(l) > 0$. Hence, (3.19) reduces to

$$\Psi'(l) \leq -q(l) \epsilon^{\gamma} \Omega(l).$$

Integrating this inequality from l_1 to l, and using (3.21), we get that $\Psi(l) \to -\infty$ as $l \to \infty$. However, this contradicts the positivity of Ψ. Therefore, the proof is complete. □

Define a sequence of functions $\{\upsilon_n(l)\}_{n=0}^{\infty}$ by and

$$\begin{aligned} \upsilon_0(l) \quad &: \quad = \varphi(l) \\ \upsilon_n(l) \quad &: \quad = \int_l^{\infty} \varpi(\xi) \upsilon_{n-1}^{(\alpha+1)/\alpha}(\xi) \, \mathrm{d}\xi + \upsilon_0(l), \ n = 1, 2, 3, ... \end{aligned} \tag{3.22}$$

We see that by induction $\upsilon_n(l) \leq \upsilon_{n+1}(l)$, $n = 1, 2, 3, ...$

Lemma 3.3 *Assume that x is an eventually positive solution of (1.1), $\upsilon_n(l)$ and $\Psi(l)$ are defined as in (3.22) and (3.17), respectively. Then $\upsilon_n(l) \leq \Psi(l)$, there exists a function $\upsilon \in C([l_0, \infty), (0, \infty))$ such that* $\lim_{l \to \infty} \upsilon_n(l) = \upsilon(l)$ *and*

$$\upsilon(l) = \int_l^{\infty} \varpi(\xi) \upsilon^{(\alpha+1)/\alpha}(\xi) \, \mathrm{d}\xi + \upsilon_0(l). \tag{3.23}$$

Proof Assume that x is an eventually positive solution of (1.1). Proceeding as in the proof of Theorem 3.1, we arrive at (3.19). Integrating (3.19) from l to ζ, we get

$$\Psi(\zeta) - \Psi(l) \leq -\int_l^{\zeta} \epsilon^{\gamma} q(\xi) \Omega(\xi) \, \mathrm{d}\xi - \int_l^{\zeta} \varpi(\xi) \Psi^{(\alpha+1)/\alpha}(\xi) \, \mathrm{d}\xi, \tag{3.24}$$

so

$$\Psi(\zeta) - \Psi(l) \geq \int_{l}^{\zeta} \varpi(\xi)\,\Psi^{(\alpha+1)/\alpha}(\xi)\,\mathrm{d}\xi. \tag{3.25}$$

If

$$\int_{l}^{\infty} \varpi(\xi)\,\Psi^{(\alpha+1)/\alpha}(\xi)\,\mathrm{d}\xi = \infty. \tag{3.26}$$

then $\lim_{l\to\infty}\Psi(l) = \infty$, which contradicts the fact that $\Psi'(l) < 0$. Therefore,

$$\int_{l}^{\infty} \varpi(\xi)\,\Psi^{(\alpha+1)/\alpha}(\xi)\,\mathrm{d}\xi < \infty. \tag{3.27}$$

From (3.24) and (3.27), we have

$$\Psi(l) \geq \varphi(l) + \int_{l}^{\infty} \varpi(\xi)\,\Psi^{(\alpha+1)/\alpha}(\xi)\,\mathrm{d}\xi = \upsilon_0(l) + \int_{l}^{\infty} \varpi(\xi)\,\Psi^{(\alpha+1)/\alpha}(\xi)\,\mathrm{d}\xi,$$

that is

$$\Psi(l) \geq \varphi(l) = \upsilon_0(l).$$

Next, by induction, we have that $\Psi(l) \geq \upsilon_n(l)$ for $l \geq l_0$, $n = 1,2,3,\ldots$. Since the sequence $\{\upsilon_n(l)\}_{n=0}^{\infty}$ monotone increasing and bounded above, we get that $\upsilon_n(l)$ converges to $\upsilon(l)$. Letting $n\to\infty$ in (3.22) and using Lebesgue's monotone convergence theorem, we arrive at (3.23). Hence, the proof is complete. □

Theorem 3.4 *Let $\upsilon_n(l)$ be defined as in (3.22). If there exist a $l_1 \geq l_0$ and $n \geq 0$ such that*

$$\int_{l_1}^{\infty} q(l)\,\Omega(l)\exp\left(\int_{l_1}^{l} \varpi(\xi)\,\upsilon_n^{1/\alpha}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}l = \infty, \tag{3.28}$$

for some $\epsilon \in (0,1)$, $k_1, k_2 > 0$ and for all $\lambda \in (0,1)$, $N > 0$, then (1.1) is oscillatory.

Proof Assume that x is an eventually positive solution of (1.1). Proceeding as in the proof of Theorem 3.1, we arrive at (3.4). Using Lemma 3.3, we have that (3.23) holds. Thus,

$$\upsilon'(l) = -\varpi(l)\,\upsilon^{(\alpha+1)/\alpha}(l) - \epsilon^{\gamma} q(l)\,\Omega(l).$$

It follows from $\upsilon_n(l) \leq \upsilon(l)$ that

$$\upsilon'(l) \leq -\varpi(l)\,\upsilon_n^{1/\alpha}(l)\,\upsilon(l) - \epsilon^{\gamma} q(l)\,\Omega(l).$$

This implies, for $l \geq l_1$,

$$\upsilon(l) \leq \exp\left(-\int_{l_1}^{l} \varpi(\xi)\,\upsilon_n^{1/\alpha}(\xi)\,\mathrm{d}\xi\right)\left(\upsilon(l_1) - \int_{l_1}^{l} \epsilon^{\gamma} q(\xi)\,\Omega(\xi)\exp\left(\int_{l_1}^{\xi} \varpi(u)\,\upsilon_n^{1/\alpha}(u)\,\mathrm{d}u\right)\mathrm{d}\xi\right);$$

thus,

$$\int_{l_1}^{l} \epsilon^{\gamma} q(\xi)\,\Omega(\xi)\exp\left(\int_{l_1}^{\xi} \varpi(u)\,\upsilon_n^{1/\alpha}(u)\,\mathrm{d}u\right)\mathrm{d}\xi \leq \upsilon(l_1) < \infty,$$

which contradicts (3.28). Therefore, the proof is complete. □

Theorem 3.5 *Assume that (3.1) holds. If there exist a function* $\psi \in C^1([l_0,\infty),\mathbb{R}^+)$ *such that*

$$\limsup_{l\to\infty}\int_{l_0}^{l}\left(\psi(\xi)\,q(\xi)\,\epsilon^{\gamma}\Omega(\xi)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\psi'(\xi))^{\alpha+1}}{\varpi^{\alpha}(\xi)}\right)\mathrm{d}\xi=\infty. \tag{3.29}$$

for some $\epsilon\in(0,1)$, $k_1,k_2>0$ *and for all* $\lambda\in(0,1)$, $N>0$, *then (1.1) is oscillatory.*

Proof Assume that x is an eventually positive solution of (1.1). Proceeding as in the proof of Theorem 3.1, we arrive at (3.19). Then,

$$q(l)\,\epsilon^{\gamma}\Omega(l)\le-\Psi'(l)-\varpi(l)\,\Psi^{(\alpha+1)/\alpha}(l). \tag{3.30}$$

Multiplying inequality (3.30) by $\psi(\xi)$ and integrating from l_1 to l, we have

$$\begin{aligned}\int_{l_1}^{l}\psi(\xi)\,q(\xi)\,\epsilon^{\gamma}\Omega(\xi)\,\mathrm{d}\xi &\le -\int_{l_1}^{l}\psi(\xi)\,\varpi(\xi)\,\Psi^{(\alpha+1)/\alpha}(\xi)\,\mathrm{d}\xi-\int_{l_1}^{l}\psi(\xi)\,\Psi'(\xi)\,\mathrm{d}\xi\\ &\le -\psi(l)\,\Psi(l)+\psi(l_1)\,\Psi(l_1)+\int_{l_1}^{l}\psi'(\xi)\,\Psi(\xi)\,\mathrm{d}\xi-\int_{l_1}^{l}\psi(\xi)\,\varpi(\xi)\,\Psi^{(\alpha+1)/\alpha}(\xi)\,\mathrm{d}\xi.\end{aligned}$$

Using Lemma 2.4 with $K=\psi'(l)$, $E=\psi(l)\,\varpi(l)$ and $w=\Psi(l)$, we have

$$\int_{l_1}^{l}\left(\psi(\xi)\,q(\xi)\,\epsilon^{\gamma}\Omega(\xi)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\psi'(\xi))^{\alpha+1}}{\varpi^{\alpha}(\xi)}\right)\mathrm{d}\xi\le\psi(l_1)\,\Psi(l_1)<\infty.$$

Taking the lim sup on both sides of the above inequality, we arrive at a contradiction with (3.29). Therefore, the proof is complete. □

Corollary 3.6 *Assume that (3.1) holds. If there exist a function* $\psi\in C^1([l_0,\infty),R^+)$ *such that*

$$\limsup_{l\to\infty}\int_{l_0}^{l}\psi(\xi)\,q(\xi)\,\epsilon^{\gamma}\Omega(\xi)\,\mathrm{d}\xi=\infty$$

and

$$\limsup_{l\to\infty}\int_{l_0}^{l}\frac{(\psi'(\xi))^{\alpha+1}}{\varpi^{\alpha}(\xi)}\mathrm{d}\xi<\infty,$$

then (1.1) is oscillatory.

4. Examples

In this section, we will show some applications of our main results.

Example 4.1 *Let us consider the following equation:*

$$\left(\left(\left(x(l)+\frac{1}{l}x^{1/3}\left(\frac{l}{5}\right)+\frac{1}{1+l^9}x^3\left(\frac{l}{3}\right)\right)'''\right)^5\right)'+\frac{q_0}{l^2}x^5\left(\frac{l}{4}\right)=0, \tag{4.1}$$

where $l \geq 1$. *Here,* $r(l) = 1,\ n = 4,\ p(l) = 1/l,\ h(l) = 1/\left(1 + l^9\right),\ q(l) = q_0/l^2,\ q_0 > 0,\ \tau(l) = l/4,$ $\sigma_1(l) = l/5,\ \sigma_2(l) = l/3,\ \alpha = 5,\ \gamma = 5,\ 0 < \beta = 1/3 < 1$ *and* $\delta = 3 \geq 1$. *By simple calculations, one can deduce that*

$$\lim_{l\to\infty} h(l)\left(l^{n-2}S(l,l_0)\right)^{\delta-1} = 0$$

and

$$\lim_{l\to\infty} p(l) = 0.$$

Now, by choosing $\psi(l) = l$, *we have that*

$$\limsup_{l\to\infty} \int_{l_0}^{l} \left(\psi(\xi)\, q(\xi)\, \epsilon^{\gamma}\Omega(\xi) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\psi'(\xi))^{\alpha+1}}{\varpi^{\alpha}(\xi)}\right) \mathrm{d}\xi$$

$$= \limsup_{l\to\infty} \int_{l_0}^{l} \left(\xi \frac{q_0}{\xi^2}\epsilon^5 - \frac{1}{6^6} \frac{1}{\left(N\xi\left(\xi/4\right)^2 \lambda \frac{1}{4}\right)^5}\right) \mathrm{d}\xi = \infty$$

Then, using Theorem 3.5, equation (4.1) is oscillatory.

Example 4.2 *Let us consider the following equation:*

$$\left(l\left(\left(x(l) + \frac{1}{l^2}x^{1/5}\left(\frac{l}{\sqrt{10}}\right) + \frac{1}{1+l^{20}}x^{21/3}\left(\frac{l}{\sqrt{2}}\right)\right)'''\right)^3\right)' + \frac{1}{l^5}x^7\left(\frac{l}{\sqrt{5}}\right) = 0, \tag{4.2}$$

where $l \geq 1$. *Here,* $r(l) = l,\ n = 4,\ p(l) = 1/l^2,\ h(l) = 1/\left(1 + l^{20}\right),\ q(l) = 1/l^5,\ \tau(l) = l/\sqrt{5},\ \sigma_1(l) = l/\sqrt{10},$ $\sigma_2(l) = l/\sqrt{2},\ \alpha = 3,\ \gamma = 7,\ 0 < \beta = 1/5 < 1$ *and* $\delta = 21/3 \geq 1$. *By simple calculations, one can deduce that*

$$\lim_{l\to\infty} h(l)\left(l^{n-2}S(l,l_0)\right)^{\delta-1} = 0$$

and

$$\lim_{l\to\infty} p(l) = 0.$$

Furthermore, we choose $\psi(l) = l^5$, *it is easy to verify that*

$$\limsup_{l\to\infty} \int_{l_0}^{l} \psi(\xi)\, q(\xi)\, \epsilon^{\gamma}\Omega(\xi)\, \mathrm{d}\xi = \limsup_{l\to\infty} \int_{l_0}^{l} \epsilon_2 \mathrm{d}\xi = \infty$$

and

$$\limsup_{l\to\infty} \int_{l_0}^{l} \frac{(\psi'(\xi))^{\alpha+1}}{\varpi^{\alpha}(\xi)} \mathrm{d}\xi = \limsup_{l\to\infty} \int_{l_0}^{l} \frac{\xi\left(5\xi^4\right)^4}{\left(N\xi^5\left(\xi/\sqrt{5}\right)^2 \lambda\left(1/\sqrt{5}\right)\right)^3} \mathrm{d}\xi < \infty,$$

where $\epsilon_2 = \epsilon^7 k_1^4$. *Hence, by Corollary 3.6, the equation (4.2) is oscillatory.*

Conflict interests

The authors declare that they have no competing interests.

References

[1] Agarwal RP, Bohner M, Li T, Zhang C. A new approach in the study of oscillatory behavior of even-order neutral delay differential equations. Applied Mathematics and Computation 2013; 225: 787-794. doi: 10.1016/j.amc.2013.09.037

[2] Bazighifan O, Grace SR, Alzabut J, Ozbekler A. New results for oscillatory properties of neutral differential equationswith a p-Laplacian like operator. Miskolc Mathematical Notes Vol. 21 2020; 2: 631-640. doi: 10.18514/MMN.2020.3322

[3] Bohner M, Grace SR, Jadlovska I. Oscillation criteria for second-order neutral delay differential equations. Electronic Journal of Qualitative Theory of Differential Equations 2017; 60: 1-12. doi: 10.14232/ejqtde.2017.1.60

[4] Graef J, Grace S, Tunc E. Oscillation criteria for even-order differential equations with unbounded neutral coefficients and distributed deviating arguments. Functional Differential Equations 2018; 45, 143-153.

[5] Graef J, Grace S, Tunc E. Oscillatory behavior of even-order nonlinear differential equations with a sublinear neutral term. Opuscula Mathematica 2019; 39 (1): 39-47. doi: 10.7494/OpMath.2019.39.1.39

[6] Hale JK. Functional differential equations. In: Hsieh PF, Stoddart AWJ (editors). Analytic Theory of Differential Equations. Lecture Notes in Mathematics, Vol 183. Berlin, Germany: Springer-Verlag, 1971.

[7] Hardy GH, Littlewood IE, Polya G. Inequalities. Reprint of the 1952 edition. Cambridge, UK: Cambridge University Press, 1988.

[8] Li T, Rogovchenko YV. Asymptotic behavior of higher-order quasilinear neutral differential equations. Hindawi Publishing Corporation. Abstract and Applied Analysis 2014; 2014: 1-11. doi: 10.1155/2014/395368

[9] Li T, Rogovchenko YV. Oscillation criteria for even-order neutral differential equations. Applied Mathematics Letters 2016; 61: 35-41. doi: 10.1016/j.aml.2016.04.012

[10] Moaaz O, Awrejcewicz J, Muhib A. Establishing new criteria for oscillation of odd-order nonlinear differential equations. Mathematics 2020; 8 (6): 1-15. doi: 10.3390/math8060937

[11] Moaaz O, Baleanu D, Muhib A. New aspects for non-existence of kneser solutions of neutral differential equations with odd-order. Mathematics 2020; 8 (4): 1-11. doi: 10.3390/math8040494

[12] Moaaz O, Cesarano C, Muhib A. Some new oscillation results for fourth-order neutral differential equations. European Journal of Pure and Applied Mathematics 2020; 13 (2): 185-199.

[13] Moaaz O, Dassios I, Muhsin W, Muhib A. Oscillation theory for non-linear neutral delay differential equations of third order. Applied Sciences 2020; 10 (14): 1-16. doi: 10.3390/app10144855

[14] Moaaz O, El-Nabulsi RA, Bazighifan O, Muhib A. New comparison theorems for the even-order neutral delay differential equation. Symmetry 2020; 12 (5): 1-11. doi: 10.3390/sym12050764

[15] Moaaz O, Park Ch, Muhib A, Bazighifan O. Oscillation criteria for a class of even-order neutral delay differential equations. Journal of Applied Mathematics and Computing 2020; 63: 607-617. doi: 10.1007/s12190-020-01331-w

[16] Muhib A, Abdeljawad T, Moaaz O, Elabbasy EM. Oscillatory properties of odd-order delay differential equations with distribution deviating arguments. Applied Sciences 2020; 10 (17): 1-10. doi: 10.3390/app10175952

[17] Philos CG. A new criterion for the oscillatory and asymptotic behavior of delay differential equations. Bulletin de L'academie Polonaise Des Sciences Série des sciences mathématiques 1981; 39 61-64.

[18] Vidhyaa KS, Graef JR, Thandapani E. New oscillation results for third-order half-linear neutral differential equations. Mathematics 2020; 8 (3): 1-9. doi: 10.3390/math8030325

[19] Xing G, Li T, Zhang C. Oscillation of higher-order quasi-linear neutral differential equations. Advances in Difference Equations 2011; (45): 1-10.

[20] Zhang C, Agarwal RP, Bohner M, Li T. New results for oscillatory behavior of even-order half-linear delay differential equations. Applied Mathematics Letters 2013; 26: 179-183. doi: 10.1016/j.aml.2012.08.004