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Korovkin type approximation via triangular $A-$statistical convergence on an infinite interval

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Abstract: In the present paper, using the triangular $A-$statistical convergence for double sequences, which is an interesting convergence method, we prove a Korovkin-type approximation theorem for positive linear operators on the space of all real-valued continuous functions on $[0,\infty)\times[0,\infty)$ with the property that have a finite limit at the infinity. Moreover, we present the rate of convergence via modulus of continuity. Finally, we give some further developments.

Key words: Triangular $A-$statistical convergence, positive linear operator, the Korovkin type theorem, Szasz-Mirakyan operator

1. Introduction and preliminaries

The Korovkin theory [10, 17] deals with an approximation to a function f by means of a sequence $(L_j(f))$ of positive linear operators, which is mainly based on the classical limit $\lim_j L_j(f)=f$. In 1960, first of all, Korovkin [17] determined the sufficient conditions for the uniform convergence of $L_j(f)$ to a function f by the test function $1,\ u,\ u^2$. Many authors have researched these conditions for various operators defined on different spaces (see i.e. [1, 14]). Obviously, using the statistical convergence in approximation theory, which is known as statistical approximation, brings along many advantages. We should note that the statistical approximation was initiated by Gadjiev and Orhan (see [13]) and improved by Anastassiou and Duman (see [2]). These types results have a crucial role in the approximation. Especially, the matrix summability methods of Cesàro type are strong enough to correct the absence of convergence of various sequences of linear operators such as Hermite-Fejér [7]. As it is well known, these types of operators do not converge at points of simple discontinuity. Therefore, such facts present why we need summability methods in the approximation theory (see i.e. [3, 4, 21, 22]). Furthermore, presently, with the help of the concept of statistical convergence, which is a regular (nonmatrix) summability transformation, various statistical approximation results have been proved [12, 16]. Then, it was shown that these new versions of Korovkin type theorems extended the classical one. Recently, Bardaro et al. [5] introduced an approximation theorem with the help of the notion of new statistical convergence called *triangular $A-$statistical convergence. Let* $C_*(D)$ be a Banach space of all real-valued continuous functions on $D:=[0,\infty)\times[0,\infty)$ with the property that $\lim_{(u,v)\to(\infty,\infty)} f(u,v)$ exists and is finite. Moreover, it is endowed with the supremum norm $\|f\|=\sup_{(u,v)\in D}|f(u,v)|$ for $f\in C_*(D)$. In the present work,

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using this convergence method and test functions $1,\ e^{-u},\ e^{-v}$, and $e^{-2u}+e^{-2v}$, we provide a Korovkin-type approximation theorem for positive linear operators on the space $C_*(D)$. We construct two examples such that in the first one, our new approximation theorem works but its statistical case does not work and in the second one, statistical case works but our result does not work. We also present a rate of convergence by means of the modulus of continuity. Finally, we give some further developments.

We will now mention some basic definitions and notations used in the paper.

A double sequence $u=(u_{i,j})$, $i,j\in\mathbb{N}$, is said to be convergent in Pringsheim's sense if, for every $\varepsilon>0$, there exists $J=J(\varepsilon)\in\mathbb{N}$ such that $|u_{i,j}-L|<\varepsilon$ whenever $i,j>J$. Then, L is the Pringsheim limit of u (see [19]). In this case, we say that $u=(u_{i,j})$ is "$P-$convergent to L" and is denoted by $P-\lim u=L$. Moreover, if there exists a positive number M such that $|u_{i,j}|\leq M$ for all $(i,j)\in\mathbb{N}^2=\mathbb{N}\times\mathbb{N}$, then $u=(u_{i,j})$ is said to be bounded. As it is known, different from a convergent single sequence, a convergent double sequence need not be bounded.

We denote the set of all $P-$convergent double sequences by c^2.

Now let $A=(a_{k,l,i,j})$, $k,l,i,j\in\mathbb{N}$, be a four-dimensional summability matrix. For a given double sequence $u=(u_{i,j})$, the $A-$transform of u, denoted by $Au:=((Au)_{k,l})$, is given by

$$(Au)_{k,l}=\sum_{(i,j)\in\mathbb{N}^2}a_{k,l,i,j}u_{i,j},$$

provided the double series converges in Pringsheim's sense for every $(k,l)\in\mathbb{N}^2$. As it is known, $A=(a_{k,l,i,j})$ is said to be $RH-$regular if it maps every bounded $P-$convergent sequence into a $P-$convergent sequence with the same $P-$limit. The well-known characterization of regularity for four-dimensional matrices is known as Robison-Hamilton conditions, or briefly, $RH-$regularity (see, [15, 20]):

(i) $P-\lim\limits_{k,l}a_{k,l,i,j}=0$ for each i and j,

(ii) $P-\lim\limits_{k,l}\sum\limits_{i,j=1,1}^{\infty,\infty}a_{k,l,i,j}=1$,

(iii) $P-\lim\limits_{k,l}\sum\limits_{i=1}^{\infty}|a_{k,l,i,j}|=0$ for each $j\in\mathbb{N}$,

(iv) $P-\lim\limits_{k,l}\sum\limits_{j=1}^{\infty}|a_{k,l,i,j}|=0$ for each $i\in\mathbb{N}$,

(v) $\sum\limits_{i,j=1,1}^{\infty,\infty}|a_{k,l,i,j}|$ is $P-$convergent for every $(k,l)\in\mathbb{N}^2$,

(vi) There exits finite positive integers M and N such that $\sum\limits_{i,j>N}|a_{k,l,i,j}|<M$ holds for every $(k,l)\in\mathbb{N}^2$.

Now, let $A=(a_{k,l,i,j})$ be a nonnegative $RH-$regular summability matrix, and let $K\subset\mathbb{N}^2$. Then, a real double sequence $u=(u_{i,j})$ is said to be $A-$*statistically convergent* to a number L if, for every $\varepsilon>0$,

$$P-\lim_{k,l}\sum_{(i,j)\in K(\varepsilon)}a_{k,l,i,j}=0,$$

where

$$K(\varepsilon):=\{(i,j)\in\mathbb{N}^2:|u_{i,j}-L|\geq\varepsilon\}.$$

In this case we write $st_A^2-\lim_{i,j} u_{i,j}=L$ ([18]). Observe that, a $P-$convergent double sequence is $A-$statistically convergent to the same value but the converse is not always true.

Let $C(1,1)=(c_{k,l,i,j})$ be a double Cesàro matrix defined by $c_{k,l,i,j}=\frac{1}{kl}$ if $1\leq i\leq k,\ 1\leq j\leq l$, and $c_{k,l,i,j}=0$ otherwise. Note that if we take $A=C(1,1)$, then $C(1,1)-$statistical convergence coincides with the concept of statistical convergence for double sequence. Moreover, if the matrix A is replaced by the identity matrix for four-dimensional matrices, then $A-$statistical convergence reduces to the Pringsheim convergence.

We denote the set of all $A-$statistically convergent double sequences by st_A^2.

Consider the double sequence $u=(u_{i,j})$ and suppose that it is neither $A-$statistical convergent nor convergent in the Pringsheim's sense. For answering the question if there is any kind of statistical convergence which is different from $A-$statistical convergence and Pringsheim convergence, Bardaro et al. ([5]) used two-dimensional regular matrices for double sequences. We now recall this interesting convergence method:

Let $A=(a_{i,j})$ be a two-dimensional matrix transformation. For a double sequence $u=(u_{i,j})$ of real numbers, put

$$(Au)_i:=\sum_{j=1}^{\infty}a_{i,j}u_{i,j},$$

if the series is convergent.

A two-dimensional matrix transformation is regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known characterization of regularity for two dimensional matrix $A=(a_{i,j})$ is known as Silverman Toeplitz ([8]) conditions:

(i) $\|A\|=\sup_{i\to\infty}\sum_{j=1}^{\infty}|a_{i,j}|<\infty$,

(ii) $\lim_{i\to\infty}a_{i,j}=0$ for each $j\in\mathbb{N}$,

(iii) $\lim_{i\to\infty}\sum_{j=1}^{\infty}a_{i,j}=1$.

Definition 1.1 *[5] Let $A=(a_{i,j})$ be a nonnegative regular summability matrix, $K\subset\mathbb{N}^2$ be a nonempty set, and for every $i\in\mathbb{N}$, consider $K_i=\{j\in\mathbb{N}:(i,j)\in K,\ j\leq i\}$. Let $|K_i|$ be the cardinality of K_i. We say that the triangular $A-$ density of K is defined by*

$$\delta_A^T(K):=\lim_i\sum_{j\in K_i}a_{i,j}$$

provided that the limit on the right-hand side exists.

Definition 1.2 *[5] Let $A=(a_{i,j})$ be a nonnegative regular summability matrix. The double sequence $u=(u_{i,j})$ is triangular $A-$ statistically convergent to L iff for every $\varepsilon>0$*

$$\lim_i\sum_{j\in K_i(\varepsilon)}a_{i,j}=0,$$

where $K_i(\varepsilon) = \{\ j \in \mathbb{N} : j \leq i,\ \ |u_{i,j} - L| \geq \varepsilon\}$ *and this is denoted by* $st_A^T - \lim_i u_{i,j} = L$.

Note that if $A = C_1$, that is Cesàro matrix, then triangular C_1-statistical convergence coincides with the notion of triangular statistical convergence.

The set of all triangular $A-$statistically convergent sequences denoted by st_A^T.

Now, in the view of above definition, we compare triangular statistical convergence to statistical convergence. First, we present below an example to show that a sequence is triangular $A-$statistically convergent whenever it is not statistically convergent.

Example 1.3 *Let* $A = C_1$ *and the double sequence* $u = (u_{i,j})$ *is given by*

$$u_{i,j} = \begin{cases} 1, & i = j = n^2, \\ \frac{n}{4(n+1)}, & i = 2n,\ j = 2(n-1), \\ \frac{n}{5(n+1)}, & i = 2n+1,\ j = 2n+3, \\ 0, & \textit{otherwise}, \end{cases} \quad n \in \mathbb{N}.$$

For every $\varepsilon \in \left(0, \frac{1}{10}\right]$,

$$\frac{1}{i}\left|\{\ j \in \mathbb{N} : j \leq i,\ \ |u_{i,j}| \geq \varepsilon\}\right| = \begin{cases} 1, & i = 1, \\ \frac{2}{(2n)^2}, & i = (2n)^2, \\ \frac{1}{(2n+1)^2}, & i = (2n+1)^2, \\ \frac{1}{2n}, & i = 2n \textit{ and } i \textit{ is not square}, \\ 0, & \textit{otherwise}, \end{cases} \quad n \in \mathbb{N},$$

then clearly,

$$\lim_i \frac{1}{i}\left|\{\ j \in \mathbb{N} : j \leq i,\ \ |u_{i,j}| \geq \varepsilon\}\right| = 0.$$

Hence, we obtain $st_{C_1}^T - \lim_i u_{i,j} = 0$. *Nevertheless,* $u = (u_{i,j})$ *is not Pringsheim's and* $C(1,1)-$*statistically convergent.*

From Example 1.3, we can say that the concept of triangular $A-$statistical convergence does not characterize as given below:

A double sequence $u = (u_{i,j})$ is triangular $A-$statistically convergent to L if and only if there exists a set $K \subset \mathbb{N}^2$ such that the triangular $A-$density of K is 1 and

$$P - \lim_{i,j\to\infty \text{ and } (i,j)\in K} u_{i,j} = L. \tag{1.1}$$

By (1.1) we mean that for every $\varepsilon > 0$ there exists an integer N such that

$$|u_{i,j} - L| \leq \varepsilon \text{ if } i, j \geq N \text{ and } (i,j) \in K.$$

The following example shows that a sequence is $A-$statistically convergent whenever it is not triangular $A-$statistically convergent.

Example 1.4 *Take $A = C(1,1)$ and the double sequence $u = (u_{i,j})$ is given by*

$$u_{i,j} = \begin{cases} \sqrt{ij}, & \text{if } i \text{ and } j \text{ are squares,} \\ \frac{1}{ij}, & \text{otherwise.} \end{cases}$$

It is easy to see that $st^2_{C(1,1)} - \lim_{i,j} u_{i,j} = 0$. Nevertheless, u is not Pringsheim's and triangular statistically convergent.

Example 1.5 *Let $A = C_1$ and the double sequence $u = (u_{i,j})$ is given by*

$$u_{i,j} = \begin{cases} 1, & i = j = n^2, \\ 0, & \text{otherwise.} \end{cases} \quad n \in \mathbb{N}\,.$$

Similarly, $st^T_{C_1} - \lim_{i} u_{i,j} = 0$ and $st^2_{C(1,1)} - \lim_{i,j} u_{i,j} = 0$.

Example 1.6 *Take $A = C_1$ and the double sequence $u = (u_{i,j})$ is given by*

$$u_{i,j} = \begin{cases} 1, & i = j = n^2, \\ \frac{n}{2n+1}, & i = 2n+1,\ j = 2n-1, \\ \frac{n}{4n+2}, & i = 2n,\ j = 2\,(n+1)\,, \\ n, & i = n^2,\ j = n^2+1, \\ 0, & \text{otherwise.} \end{cases} \quad n \in \mathbb{N}\,.$$

For every $\varepsilon \in \left(0, \frac{1}{6}\right]$,

$$\frac{1}{i}\left|\{\, j \in \mathbb{N} : j \le i,\ \ |u_{i,j}| \ge \varepsilon\}\right| = \begin{cases} 1, & i = 1, \\ \frac{2}{(2n+1)^2}, & i = (2n+1)^2\,, \\ \frac{1}{(2n)^2}, & i = (2n)^2\,, \\ \frac{1}{2n+1}, & i = 2n+1 \text{ and } i \text{ is not square,} \\ 0, & \text{otherwise.} \end{cases} \qquad n \in \mathbb{N}\,,$$

Then, it is seen to be

$$\lim_{i} \frac{1}{i}\left|\{\, j \in \mathbb{N} : j \le i,\ \ |u_{i,j}| \ge \varepsilon\}\right| = 0.$$

Hence, we obtain $st^T_{C_1} - \lim_{i} u_{i,j} = 0$. Neither $u = (u_{i,j})$ is Pringsheim's and $C\,(1,1)-$statistically convergent nor bounded.

Remark 1.7 *[5] i) Statistical convergence and triangular statistical convergence are incompatible; i.e. $st^2_A \nsubseteq st^T_A$ and $st^T_A \nsubseteq st^2_A$.*

ii) $P-$convergent double sequence is $A-$statistically convergent and triangular $A-$statistically convergent to the same value but the inverse implications are not true, i.e. $st^2_A \nsubseteq c^2$ and $st^T_A \nsubseteq c^2$.

2. A Korovkin-type theorem

Boyanov and Veselinov [9] have results for single sequences, where they proved the Korovkin theorem on $C_*[0,\infty)$, which is the Banach space of all real-valued continuous functions on $[0,\infty)$ with the property that $\lim_{u\to\infty} f(u)$ exists and is finite, endowed with the supremum norm $\|f\| = \sup_{u\in[0,\infty)} |f(u)|$ for $f \in C_*[0,\infty)$, by using the test function $1,\ e^{-u},\ e^{-2u}$. In this section, using the concept of triangular $A-$statistical convergence for double sequences and test functions $1,\ e^{-u},\ e^{-v}$, and $e^{-2u}+e^{-2v}$, we provide a Korovkin-type approximation for positive linear operators on the space $C_*(D)$.

Now, we remind Korovkin's theorem for double sequences of positive linear operators given by Demirci and Karakuş [11] and then, we give the Korovkin theorem in the triangular statistical sense.

Let L be a linear operator from $C_*(D)$ into itself. Then, as usual, we say that L is positive linear operator provided that $f \geq 0$ implies $L(f) \geq 0$. Moreover, we denote the value of $L(f)$ at a point $(u,v) \in D$ by $L(f;u,v)$.

We note that a function which is continuous on D and has a finite limit to infinity is uniformly continuous. Hence, Theorem 2.2. in [11] can be given as follow:

Theorem 2.1 *[11] Let $A = (a_{k,l,i,j})$ be a nonnegative $RH-$regular summability matrix. Let $(L_{i,j})$ be a double sequence of positive linear operators acting from $C_*(D)$ into itself. Then, for all $f \in C_*(D)$*

$$st_A^2 - \lim_{i,j} \|L_{i,j}(f) - f\| = 0$$

if and only if the following statements hold:

a) $st_A^2 - \lim_{i,j} \|L_{i,j}(1) - 1\| = 0,$

b) $st_A^2 - \lim_{i,j} \|L_{i,j}(e^{-s}) - e^{-u}\| = 0,$

c) $st_A^2 - \lim_{i,j} \|L_{i,j}(e^{-t}) - e^{-v}\| = 0,$

d) $st_A^2 - \lim_{i,j} \left\|L_{i,j}\left(e^{-2s}+e^{-2t}\right) - \left(e^{-2u}+e^{-2v}\right)\right\| = 0.$

Before giving our main theorem, we prove the following lemma:

Lemma 2.2 *Let $(L_{i,j})$ be a double sequence of positive linear operators acting from $C_*(D)$ into itself. Then, for every $\varepsilon > 0$, there exists a $\delta > 0$ and $M > 0$ such that for every $L_{i,j} : C_*(D) \to C_*(D)$, $f \in C_*(D)$, and $(u,v) \in D$, the following inequality holds*

$$\begin{aligned} & |L_{i,j}(f;u,v) - f(u,v)| \\ \leq\ & \varepsilon + \left(\varepsilon + M + \frac{4M}{\delta^2}\right) |L_{i,j}(1;u,v) - 1| \\ & + \frac{4M}{\delta^2} \left|L_{i,j}\left(e^{-s};u,v\right) - e^{-u}\right| + \frac{4M}{\delta^2} \left|L_{i,j}\left(e^{-t};u,v\right) - e^{-v}\right| \\ & + \frac{2M}{\delta^2} \left|L_{i,j}\left(e^{-2s}+e^{-2t};u,v\right) - \left(e^{-2u}+e^{-2v}\right)\right|. \end{aligned}$$

Proof Following the same lines as in the proof of Theorem 2.2 in [11], the proof is obtained. □

Now, we give the following, which is the main result in this section.

Theorem 2.3 *Let* $A=(a_{i,j})$ *be a nonnegative regular summability matrix and* $(L_{i,j})$ *be a double sequence of positive linear operators acting from* $C_*(D)$ *into itself. Then, for all* $f\in C_*(D)$

$$st_A^T-\lim_i \|L_{i,j}(f)-f\|=0$$

iff the following statements hold:

a) $st_A^T-\lim_i \|L_{i,j}(1)-1\|=0,$

b) $st_A^T-\lim_i \|L_{i,j}(e^{-s})-e^{-u}\|=0,$

c) $st_A^T-\lim_i \|L_{i,j}(e^{-t})-e^{-v}\|=0,$

d) $st_A^T-\lim_i \left\|L_{i,j}\left(e^{-2s}+e^{-2t}\right)-\left(e^{-2u}+e^{-2v}\right)\right\|=0.$

Proof Since the necessity is obvious, then it is enough to prove sufficiency. Suppose that the conditions (a), (b), (c), and (d) are satisfied. Let $f\in C_*(D)$. Then, from Lemma 2.2, for every $\varepsilon>0$, there exists a $\delta>0$ and $M>0$ such that for every $(i,j)\in\mathbb{N}^2$ and $(u,v)\in D$, the following inequality holds

$$\begin{aligned}
&|L_{i,j}(f;u,v)-f(u,v)|\\
\leq\ &\varepsilon+\left(\varepsilon+M+\frac{4M}{\delta^2}\right)|L_{i,j}(1;u,v)-1|\\
&+\frac{4M}{\delta^2}\left|L_{i,j}\left(e^{-s};u,v\right)-e^{-u}\right|+\frac{4M}{\delta^2}\left|L_{i,j}\left(e^{-t};u,v\right)-e^{-v}\right|\\
&+\frac{2M}{\delta^2}\left|L_{i,j}\left(e^{-2s}+e^{-2t};u,v\right)-\left(e^{-2u}+e^{-2v}\right)\right|.
\end{aligned}$$

Then taking the supremum over $(u,v)\in D$, we get

$$\begin{aligned}
&\|L_{i,j}(f)-f\| \qquad (2.1)\\
\leq\ &\varepsilon+K\left\{\|L_{i,j}(1)-1\|+\left\|L_{i,j}\left(e^{-s}\right)-e^{-u}\right\|\right.\\
&\left.+\left\|L_{i,j}\left(e^{-t}\right)-e^{-v}\right\|+\left\|L_{i,j}\left(e^{-2s}+e^{-2t}\right)-\left(e^{-2u}+e^{-2v}\right)\right\|\right\}
\end{aligned}$$

where $K:=\max\left\{\varepsilon+M+\frac{4M}{\delta^2},\frac{4M}{\delta^2},\frac{2M}{\delta^2}\right\}$. For a given $r>0$ choose $\varepsilon>0$ such that $\varepsilon<r$. Define the following

sets:

$$
\begin{aligned}
S_i &: = \{ \, j \in \mathbb{N} : j \leq i, \ \|L_{i,j}(f) - f\| \geq r \} , \\
S_i^1 &: = \left\{ \, j \in \mathbb{N} : j \leq i, \ \|L_{i,j}(1) - 1\| \geq \frac{r-\varepsilon}{4K} \right\} , \\
S_i^2 &: = \left\{ \, j \in \mathbb{N} : j \leq i, \ \left\|L_{i,j}\left(e^{-s}\right) - e^{-u}\right\| \geq \frac{r-\varepsilon}{4K} \right\} , \\
S_i^3 &: = \left\{ \, j \in \mathbb{N} : j \leq i, \ \left\|L_{i,j}\left(e^{-t}\right) - e^{-v}\right\| \geq \frac{r-\varepsilon}{4K} \right\} , \\
S_i^4 &: = \left\{ \, j \in \mathbb{N} : j \leq i, \ \left\|L_{i,j}\left(e^{-2s}+e^{-2t}\right) - \left(e^{-2u}+e^{-2v}\right)\right\| \geq \frac{r-\varepsilon}{4K} \right\} .
\end{aligned}
$$

In the view of (2.1), it can be easily seen that $S_i \subset S_i^1 \cup S_i^2 \cup S_i^3 \cup S_i^4$. Thus, for each $j \in \mathbb{N}$, we may write

$$
\sum_{j \in S_i} a_{i,j} \leq \sum_{j \in S_i^1} a_{i,j} + \sum_{j \in S_i^2} a_{i,j} + \sum_{j \in S_i^3} a_{i,j} + \sum_{j \in S_i^4} a_{i,j} . \tag{2.2}
$$

From (2.2), using (a), (b), (c), and (d), we get that

$$
\lim_i \sum_{j \in S_i} a_{i,j} = 0
$$

whence the result. □

Now, we give the following examples of double sequences of positive linear operators.

Example 2.4 *Consider the following Szasz–Mirakyan operators (see [23])*

$$
S_{i,j}(f;u,v) = e^{-iu} e^{-jv} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} f\left(\frac{l}{i}, \frac{k}{j}\right) \frac{(iu)^l}{l!} \frac{(jv)^k}{k!}, \tag{2.3}
$$

where $(u,v) \in D$, $f \in C_*(D)$. *Moreover, observe that*

$$
\begin{aligned}
S_{i,j}(1;u,v) &= 1, \\
S_{i,j}\left(e^{-s};u,v\right) &= e^{-iu\left(1-e^{-\frac{1}{i}}\right)}, \\
S_{i,j}\left(e^{-t};u,v\right) &= e^{-jv\left(1-e^{-\frac{1}{j}}\right)}, \\
S_{i,j}\left(e^{-2s}+e^{-2t};u,v\right) &= e^{-iu\left(1-e^{-\frac{2}{i}}\right)} + e^{-jv\left(1-e^{-\frac{2}{j}}\right)}.
\end{aligned}
$$

Now we take $A = C_1$ *and define a double sequence* $(\alpha_{i,j})$ *by*

$$
\alpha_{i,j} = \begin{cases} 1, & i = j = n^2, \\ \frac{n}{4(n+1)}, & i = 2n, \ j = 2(n-1), \\ \frac{n}{3(n+1)}, & i = 2n+1, \ j = 2n+3, \\ 0, & \text{otherwise.} \end{cases} \quad n \in \mathbb{N} \tag{2.4}
$$

It is clear that

$$st_{C_1}^{T}-\lim_{i}\alpha_{i,j}=0. \tag{2.5}$$

Now in the view of (2.3) and (2.4), we define the following positive linear operators on $C_*(D)$ *as follows:*

$$L_{i,j}(f;u,v)=(1+\alpha_{i,j})\,S_{i,j}(f;u,v)\,. \tag{2.6}$$

Therefore, by the Theorem 2.3 and (2.5), we see that

$$st_{C_1}^{T}-\lim_{i}\|L_{i,j}(f)-f\|=0.$$

Moreover, since $(\alpha_{i,j})$ *is not* $P-$ *convergent and statistical convergent, we can say that the Korovkin theorems in the Pringsheim's and statistical sense do not work for our operators defined by (2.6).*

Example 2.5 *Let Szasz–Mirakyan operators be the same as in Example 2.4. Now take* $A=C(1,1)$ *and define a double sequence* $(\beta_{i,j})$ *by*

$$\beta_{i,j}=\begin{cases}\sqrt{ij}, & \text{if } i \text{ and } j \text{ are squares,}\\ \frac{-1}{ij} & \text{otherwise.}\end{cases} \tag{2.7}$$

It is clear that

$$st_{C(1,1)}^{2}-\lim_{i,j}\beta_{i,j}=0. \tag{2.8}$$

Now in the view of (2.3) and (2.7), we define the following positive linear operators on $C_*(D)$ *as follows:*

$$L_{i,j}(f;u,v)=(1+\beta_{i,j})\,S_{i,j}(f;u,v)\,. \tag{2.9}$$

Therefore, by the Theorem 2.1 and (2.8), we see that

$$st_{C(1,1)}^{2}-\lim_{i,j}\|L_{i,j}(f)-f\ \|=0.$$

Moreover, since $(\beta_{i,j})$ *is not triangular statistical convergent, we can say that the Theorem 2.3 does not work for operators defined by (2.9).*

3. The rate of approximation

In this section, we study the rate of convergence of a sequence of positive linear operators defined from $C_*(D)$ into itself with the help of modulus of continuity. The earlier results proved in [5]. Now, we begin by recalling the following:

Definition 3.1 *[5] Let* $A=(a_{i,j})$ *be a nonnegative regular summability matrix and let* (α_i) *be a positive nonincreasing sequence. Then, a double sequence* $u=(u_{i,j})$ *is said to be triangular* $A-$ *statistically convergent to a number* L *with the rate of* $o_k(\alpha_i)$ *iff for every* $\varepsilon>0$,

$$\lim_{i}\sum_{j\in K_i(\varepsilon)}a_{i,j}=0,$$

where

$$K_i(\varepsilon) := \{\ j \in \mathbb{N} : |u_{i,j} - L| \geq \varepsilon\ \alpha_j\}.$$

In this case, one can say that

$$u_{i,j} - L = st_A^T - o_k(\alpha_i) \quad as \quad i \to \infty.$$

Definition 3.2 *[5] Let $A = (a_{i,j})$ and (α_i) be the same as in Definition 3.1. Then, a double sequence $u = (u_{i,j})$ is said to be triangular $A-$statistically bounded with the rate of $O_k(\alpha_i)$ iff for every $\varepsilon > 0$,*

$$\lim_i \sum_{j \in K_i(\varepsilon)} a_{i,j} = 0,$$

where

$$K_i(\varepsilon) := \{j \in \mathbb{N} : |u_{i,j}| \geq \varepsilon\ \alpha_j\}.$$

In this case, one can say that

$$u_{i,j} = st_A^T - O_k(\alpha_i) \quad as \quad i \to \infty.$$

Lemma 3.3 *[5] Let $(u_{i,j})$ and $(v_{i,j})$ be double sequences. Suppose that $A = (a_{i,j})$ is a nonnegative regular summability matrix. Moreover, let (α_i) and (β_i) be positive nonincreasing sequences. If $u_{i,j} - L_1 = st_A^T - o_k(\alpha_i)$ and $v_{i,j} - L_2 = st_A^T - o_k(\beta_i)$, then we have*

(i) $(u_{i,j} - L_1) \mp (v_{i,j} - L_2) = st_A^T - o_k(\gamma_i)$ *as* $i \to \infty$, *where* $\gamma_i := \max\{\alpha_i, \beta_i\}$ *for each* $i \in \mathbb{N}$,

(ii) $\lambda(u_{i,j} - L_1) = st_A^T - o_k(\alpha_i)$ *as* $i \to \infty$ *for any real number* λ.

Furthermore, similar conclusions hold with the symbol "o_k" replaced by "O_k".

Recall that for a continuous function $f \in C(I)$ defined on an interval $I \subset \mathbb{R}^2$, the function $\omega : C(I) \times [0, \infty) \to \mathbb{R} \cup \{\infty\}$, defined as follows:

$$\omega(f;\delta) := \sup\left\{|f(s,t) - f(u,v)| : (s,t), (u,v) \in D, \sqrt{(s-u)^2 + (t-v)^2} \leq \delta\right\}$$

is called its usual modulus of continuity.

To estimate the rate of convergence of a double sequence of positive linear operators defined from $C_*(D)$ into itself, we make use of the following modulus of continuity:

$$\omega^*(f;\delta) := \sup\left\{|f(s,t) - f(u,v)| : (s,t), (u,v) \in D, \sqrt{(e^{-s} - e^{-u})^2 + (e^{-t} - e^{-v})^2} \leq \delta\right\}.$$

where $f \in C_*(D)$ and $\delta > 0$. In fact, this modulus can be given the usual modulus of continuity by the relation:

$$\omega^*(f;\delta) = \omega(f^*;\delta)$$

where f^* is the continuous function defined on $[0,1] \times [0,1]$ by

$$f^*(u,v) = \begin{cases} f(-\ln u, -\ln v), & (u,v) \in (0,1] \times (0,1], \\ \lim_{(s,t)\to(\infty,\infty)} f(s,t), & u = 0 \text{ or } v = 0. \end{cases}$$

With simple calculation, it can be seen that for any $\delta > 0$

$$|f(s,t) - f(u,v)| \leq \left(1 + \frac{(e^{-s} - e^{-u})^2 + (e^{-t} - e^{-v})^2}{\delta^2}\right)\omega^*(f;\delta)$$

for all $f \in C_*(D)$.

Now we can give the following result.

Theorem 3.4 *Let $A = (a_{i,j})$ be a nonnegative regular summability matrix method and $(L_{i,j})$ be a double sequence of positive linear operators from $C_*(D)$ into itself. Suppose that the following conditions hold:*

(i) $\|L_{i,j}(1) - 1\| = st_A^T - o_k(\alpha_i)$ *as* $i \to \infty$,

(ii) $\omega^*(f;\delta_{i,j}) = st_A^T - o_k(\beta_i)$ *as* $i \to \infty$, *where* $\delta_{i,j} := \sqrt{\|L_{i,j}(\varphi_{(u,v)})\|}$ *with* $\varphi_{(u,v)}(s,t) = (e^{-s} - e^{-u})^2 + (e^{-t} - e^{-v})^2$. *Then, for any* $f \in C_*(D)$

$$\|L_{i,j}(f) - f\| = st_A^T - o(\gamma_i), \text{ as } i \to \infty,$$

where $\gamma_i := \max\{\alpha_i, \beta_i\}$ *for each* $i \in \mathbb{N}$. *We can say that the similar results hold when the symbol "o_k" is replaced by "O_k".*

Proof Let $f \in C_*(D)$ and $(u,v) \in D$ be fixed. Using linearity and positivity of the $L_{i,j}$, we have, for any $(i,j) \in \mathbb{N}^2$,

$$\begin{aligned}
& |L_{i,j}(f;u,v) - f(u,v)| \\
= \; & |L_{i,j}(f(s,t) - f(u,v);u,v) - f(u,v)(L_{i,j}(1;u,v) - 1)| \\
\leq \; & L_{i,j}(|f(s,t) - f(u,v)|;u,v) + M|L_{i,j}(1;u,v) - 1| \\
\leq \; & L_{i,j}\left(\left(1 + \frac{(e^{-s} - e^{-u})^2 + (e^{-t} - e^{-v})^2}{\delta^2}\right)\omega^*(f;\delta);u,v\right) \\
& + M|L_{i,j}(1;u,v) - 1| \\
\leq \; & \omega^*(f;\delta)L_{i,j}\left(\left(1 + \frac{(e^{-s} - e^{-u})^2 + (e^{-t} - e^{-v})^2}{\delta^2}\right);u,v\right) \\
& + M|L_{i,j}(1;u,v) - 1| \\
\leq \; & \omega^*(f;\delta)|L_{i,j}(1;u,v) - 1| + \frac{\omega^*(f;\delta)}{\delta^2}L_{i,j}(\varphi_{(u,v)};u,v) + \omega^*(f;\delta) \\
& + M|L_{i,j}(1;u,v) - 1|,
\end{aligned}$$

where $M := \|f\|$. Thus, taking supremum over $(u,v) \in D$ on the both sides of the above inequality, we get, for any $\delta > 0$,

$$\begin{aligned}
\|L_{i,j}(f) - f\| \leq \; & \omega^*(f;\delta)\|L_{i,j}(1) - 1\| + \frac{\omega^*(f;\delta)}{\delta^2}\left\|L_{i,j}\left(\varphi_{(u,v)}\right)\right\| \\
& + \omega^*(f;\delta) + M\|L_{i,j}(1) - 1\|
\end{aligned}$$

Now if $\delta := \delta_{i,j} := \sqrt{\left\|L_{i,j}(\varphi_{(u,v)})\right\|}$, then we can write

$$\left\|L_{i,j}\left(f\right)-f\right\| \leq \omega^{*}\left(f;\delta\right)\left\|L_{i,j}\left(1\right)-1\right\|+2\omega^{*}\left(f;\delta\right)+M\left\|L_{i,j}\left(1\right)-1\right\|$$

and hence

$$\left\|L_{i,j}\left(f\right)-f\right\| \leq N\left\{\omega^{*}\left(f;\delta\right)\left\|L_{i,j}\left(1\right)-1\right\|+\omega^{*}\left(f;\delta\right)+\left\|L_{i,j}\left(1\right)-1\right\|\right\} \tag{3.1}$$

where $N=\max\left\{2,M\right\}$. Now, for a given $r>0$, we define the following sets:

$$\begin{aligned}
U_i &: =\left\{j\in\mathbb{N}: j\leq i, \left\|L_{i,j}\left(f\right)-f\ \right\|\geq r\gamma_i\right\},\\
U_i^1 &: =\left\{j\in\mathbb{N}: j\leq i, \omega^{*}\left(f;\delta\right)\left\|L_{i,j}\left(1\right)-1\right\|\geq \frac{r}{3N}\beta_i\alpha_i\right\},\\
U_i^2 &: =\left\{j\in\mathbb{N}: j\leq i, \omega^{*}\left(f;\delta\right)\geq \frac{r}{3N}\beta_i\right\},\\
U_i^3 &: =\left\{j\in\mathbb{N}: j\leq i, \left\|L_{i,j}\left(1\right)-1\right\|\geq \frac{r}{3N}\alpha_i\right\}.
\end{aligned}$$

In the view of (3.1) that

$$U_i\subset U_i^1\cup U_i^2\cup U_i^3.$$

Moreover, define the sets:

$$\begin{aligned}
U_i^4 &: =\left\{j\in\mathbb{N}: j\leq i, \omega^{*}\left(f;\delta\right)\geq \sqrt{\frac{r}{3N}\beta_i}\right\},\\
U_i^5 &: =\left\{j\in\mathbb{N}: j\leq i, \left\|L_{i,j}\left(1\right)-1\right\|\geq \sqrt{\frac{r}{3N}\alpha_i}\right\}.
\end{aligned}$$

Then, observe that $U_i^1\subset U_i^4\cup U_i^5$. Therefore, we have $U_i\subset U_i^2\cup U_i^3\cup U_i^4\cup U_i^5$. Now, since $\gamma_i:=\max\left\{\alpha_i,\beta_i\right\}$ for each $i\in\mathbb{N}$, we get

$$\begin{aligned}
\sum_{j\in U_i} a_{i,j} &\leq \sum_{j\in U_i^2} a_{i,j}+\sum_{j\in U_i^3} a_{i,j}\\
&\quad +\sum_{j\in U_i^4} a_{i,j}+\sum_{j\in U_i^5} a_{i,j}.
\end{aligned}$$

Letting $i\to\infty$ and from (i) and (ii), we have

$$\lim_i \sum_{j\in U_i} a_{i,j}=0$$

whence the result. □

4. Possible further developments

Here we recall the notion of $A-$statistical type convergence, includes the triangular $A-$statistical convergence as a particular case, given by Bardaro et al. ([5, 6]).

Let $A=(a_{i,j})$ be a nonnegative regular summability matrix, $\Psi:\mathbb{N}\times\mathbb{N}\to\mathbb{R}$ a fixed function. A sequence $u=(u_{i,j})$ is said to be $\Psi-A-$*statistically convergent* to a real number L provided that for every $\varepsilon>0$

$$\lim_{i}\sum_{j\in K_i(\varepsilon)}a_{i,j}=0,$$

where $K_i(\varepsilon)=\{j\in\mathbb{N}:\Psi(i,j)\geq 0,\ |u_{i,j}-L|\geq\varepsilon\}$ (See [5, 6]).

Note that, if $\Psi(i,j)=i-j$ then, the triangular $A-$statistical convergence is the special case of $\Psi-A-$statistical convergence. If $\Psi(i,j)=\beta(i)-j$, where $\beta:\mathbb{N}\to\mathbb{N}$ is a suitable increasing function, then we get other interesting results. Hence, all the theory given in the previous sections can be carried on also in the setting of $\Psi-A-$statistical convergence.

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